REGULARITY FOR GENERATORS OF INVARIANT SUBSPACES OF THE DIRICHLET SHIFT

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Abstract. Let D denote the classical Dirichlet space of analytic functions on the open unit disc whose derivative is square area integrable. For a set $E \subseteq \partial\mathbb{D}$ we write

$$D_E = \{f \in D : \lim_{r\to 1^-} f(re^{it}) = 0 \quad q.e.\},$$

where $q.e.$ stands for "except possibly for e^{it} in a set of logarithmic capacity 0". We show that if E is a Carleson set, then there is a function $f \in D_E$ that is also in the disc algebra and that generates D_E in the sense that $D_E =$ clos $\{pf : p \text{ is a polynomial }\}$.

We also show that if $\varphi \in D$ is an extremal function (i.e. $\langle p\varphi, \varphi\rangle = p(0)$ for every polynomial p), then the limits of $|\varphi(z)|$ exist for every $e^{it} \in \partial\mathbb{D}$ as z approaches e^{it} from within any polynomially tangential approach region.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane and let dA denote Lebesgue measure on $\mathbb{D}$. The Hardy space H^2 is

$$H^2 = \{f : \mathbb{D} \to \mathbb{C} : f \text{ analytic and } \|f\|^2_{H^2} = \sup_{r<1} \int_{|z|=1} |f(rz)|^2 \frac{|dz|}{2\pi} < \infty\}$$

and the classical Dirichlet space on $\mathbb{D}$ is defined by

$$D = \{f : \mathbb{D} \to \mathbb{C} : f \text{ analytic and } \int_{\mathbb{D}} |f'|^2 dA < \infty\}.$$

The norm on D is defined by $\|f\|^2 = \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi}$. For $f \in \mathcal{H}$ for $\mathcal{H} = H^2$ or D we write

$$[f]_{\mathcal{H}} = \text{clos }_{\mathcal{H}}\{pf : p \text{ a polynomial}\},$$

and we say that f is cyclic in $\mathcal{H}$, if $[f]_{\mathcal{H}} = \mathcal{H}$. Usually it will be clear from the context which space we are considering, and we will drop the subscript and write $[f]$ instead of $[f]_{\mathcal{H}}$.

The cyclic functions in H^2 were characterized as a consequence of Beurling's invariant subspace theorem, [4]. They turned out to be the outer functions in H^2. However, it is one of the most intriguing open questions about the Dirichlet space to determine exactly which functions are cyclic in D. It follows easily from the contractive inclusion $D \subseteq H^2$ that cyclic functions in D have to be cyclic in H^2. Thus cyclic functions in D must be outer. Furthermore, in [7] Brown and Shields showed that for cyclic functions f in D the radial zero set

$$Z(f) = \{e^{it} \in \partial\mathbb{D} : \lim_{r\to 1^-} f(re^{it}) = 0\}$$

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has logarithmic capacity zero. The Brown-Shields conjecture stipulates that these two conditions characterize the cyclic functions in D, see [7].

There are many results which support this conjecture, see e.g. [16], [23] and [12]. For further details the book [11] is a good reference. Here we just mention that while the conjecture has been shown to be true for some functions whose radial zero set satisfies certain regularity conditions, in general the conjecture is still open even for the case where one assumes that the function f extends to be continuous on $\overline{\mathbb{D}}$, i.e. $f \in \mathbb{A}(\mathbb{D})$, the disc algebra.

We now discuss the more general problem of finding a description of all invariant subspaces of D. A subspace $\mathcal{M} \subseteq D$ is called invariant, if the function $zf \in \mathcal{M}$, whenever $f \in \mathcal{M}$. It was shown in [20] that every non-zero invariant subspace of D is of the form $\mathcal{M} = [\varphi]$, where φ is extremal for $\mathcal{M}$, i.e. $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$. Furthermore, if $\varphi = Sf$ is the inner-outer factorization of the extremal function for $\mathcal{M}$, then $f \in D$ and $\mathcal{M} = SH^2 \cap [f]$. Thus, the quest for a description of all invariant subspaces splits into the two subquestions of which inner factors may occur for Dirichlet functions, and what are the invariant subspaces generated by outer functions. Beurling showed that for any $f \in D$ the radial limit $\lim_{r\to 1^-} f(re^{it})$ exists for every $e^{it} \in \partial\mathbb{D}$ except perhaps for e^{it} in a set of logarithmic capacity zero (see [3]). We will say a property holds quasi-everywhere (q.e.), if it holds except perhaps on a set of capacity zero. In Section 2 we will review further details about this concept, and also consider other capacities. It follows easily from the modern proofs of Beurling's theorem that for any $E \subseteq \partial\mathbb{D}$ the set

$$D_E = \{f \in D : \lim_{r\to 1^-} f(re^{it}) = 0 \text{ q.e. on } E\}$$

is closed and then it easily follows that it is an invariant subspace of D. Furthermore, if $D_E \neq (0)$, then it is not hard to see that D_E must contain an outer function. This leads to the extended Brown-Shields conjecture:

Conjecture 1.1. *(see Question 11 of [7]) If $f \in D$ is outer, then $[f] = D_E$ for some $E \subseteq \partial\mathbb{D}$.*

Note that if $f \in D_E$, then $Z(f) \supseteq E$ q.e. and hence $[f] \subseteq D_{Z(f)} \subseteq D_E$. Thus, if the extended Brown-Shields conjecture holds, then it must hold with $E = Z(f)$. Of course, if $Z(f)$ has logarithmic capacity 0, then $1 \in D_{Z(f)}$ and hence $D_{Z(f)} = D$. Thus the extended Brown-Shields conjecture implies the Brown-Shields conjecture.

Unfortunately very little is known about the extended Brown-Shields conjecture. Let A^∞ be the algebra of all functions that are infinitely differentiable on the closed unit disc and analytic on the open unit disc. Recall that a closed set $E \subseteq \partial\mathbb{D}$ is called a Carleson set, if $\log \frac{2}{dist(e^{it},E)} \in L^1(\partial\mathbb{D})$. Korenblum showed that if E is a Carleson set, then there is an outer function $f_E \in A^\infty$ such that f_E and all of its derivatives vanish on E, but f_E is nonzero on $\overline{\mathbb{D}} \setminus E$, [17]. Thus, a special case of the extended Brown-Shields conjecture would be to show that $[f_E] = D_E$ for Carleson sets. For algebras of analytic functions on $\mathbb{D}$ that satisfy a Lipschitz condition Korenblum developed a method to show that f_E generates the ideal of all functions vanishing on E (along with an appropriate number of derivatives of f), see [17] and [5], [18], [26],[29], [30].

By use of a result from [12] one easily shows

Theorem 1.2. *If $E \subseteq \partial\mathbb{D}$ is a Carleson set, and if there is a function $g \in D$ such that*

- *$D_E = [g]$,*
- *there are $c, \alpha > 0$ such that*

$$|g(z)| \leq c(dist(z, E))^\alpha \text{ for a.e. } z \in \partial\mathbb{D},$$

then for any outer function $f \in D$ with $Z(f) = \underline{Z}(f) = E$ we have $[f] = D_E$. Here

$$\underline{Z}(f) = \{e^{it} : \liminf_{w \to e^{it}} |f(w)| = 0\}.$$

Proof. If $f \in D$ is an outer function with $Z(f) = \underline{Z}(f) = E$, then $f \in D_E = [g]$. Thus it suffices to show that $g \in [f]$. For $\alpha = 1$ this is an immediate consequence of Theorem 3.1 of [12]. It remains to handle the case $0 < \alpha < 1$. Since $D_E = [g]$ g must be outer. In that case $g^{1/\alpha} \in D$ and we conclude from the above that $[g^{1/\alpha}] \subseteq [f]$. The result now follows from the fact that $[g] = [g^{1/\alpha}]$, see [23]. □

In particular, it follows from the above theorem that if $[g] = D_E$ for some $g \in \text{Lip } \alpha$, then $[f] = D_E$ for all outer functions $f \in D \cap \mathbb{A}(\mathbb{D})$ with $Z(f) = E$. In this paper we will show the following theorem.

Theorem 1.3. *If $E \subseteq \partial\mathbb{D}$ is a Carleson set, then there is $g \in D \cap \mathbb{A}(\mathbb{D})$ such that $[g] = D_E$.*

Our starting point will be the fact that extremal functions are known to satisfy certain extra regularity properties. However, we do not know whether the extremal function for D_E satisfies the conclusion of Theorem 1.3, rather we will modify it appropriately. For all extremal functions we have the following theorem:

Theorem 1.4. *If $\varphi \in D$ is an extremal function, then for every $c > 0$, $\alpha \geq 1$, and $e^{it} \in \partial\mathbb{D}$*

$$\lim_{w \to e^{it}, w \in \Gamma_c^\alpha(e^{it})} |\varphi(w)| = \limsup_{w \to e^{it}, w \in \mathbb{D}} |\varphi(w)|. \tag{1.5}$$

Here

$$\Gamma_c^\alpha(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}|^\alpha \leq c(1 - |z|)\}$$

is a polynomially tangential approach region.

In [24] the above theorem was shown with a radial limit on the left hand side of (1.5). Thus the novelty here is that the limit exist even from within the approach regions $\Gamma_c^\alpha(e^{it})$. Note that if $\alpha = 1$ and $c > 1$, then the region $\Gamma_c(e^{i\theta})$ is simply a non-tangential approach region (Stolz angle). If $c > 0$, then

$$\Gamma_c^2(e^{i\theta}) \approx O_c(e^{i\theta}) := \{z \in \mathbb{D} : |z - e^{i\theta}|^2 \leq c(1 - |z|^2)\}$$

and a short computation shows that $O_c(e^{i\theta})$ is a disc internally tangent to the unit circle $\partial\mathbb{D}$ at $e^{i\theta}$ with center $\dfrac{e^{i\theta}}{c+1}$ and radius $\dfrac{c}{c+1}$. This region is also called an oricylic approach region. If $\alpha > 1$, then $\Gamma_c^\alpha(e^{i\theta})$ is a region contained in the unit disc $\mathbb{D}$ which touches unit circle $\partial\mathbb{D}$ at $e^{i\theta}$ tangentially. As α increases, the degree of tangency also increases. We note that Nagel, Rudin, and Shapiro, [19] showed that for functions in D boundary limits exist a.e. even from within approach regions that make exponential contact with $\partial\mathbb{D}$.

The extended Brown-Shields conjecture is about the invariant subspace structure of the Dirichlet shift. By [21] every invariant subspace is of the form $\mathcal{M} = \varphi D(m_\varphi)$. Here m_φ is a measure associated with the extremal function φ for $\mathcal{M}$ and $D(m_\varphi)$ is a harmonically weighted Dirichlet space. The main tool for the proof of Theorem 1.3 is Theorem 3.2, which holds for all harmonically weighted Dirichlet spaces. Since that is interesting in its own right, we decided to prove this result in greater generality than would be needed for Theorem 1.3. We have the included the necessary background details in Section 2.

We would like to thank the referee for bringing the paper [13] to our attention. In Theorem 1 of that paper the authors establish the special case of our Theorem 3.2, where the sets E and F coincide and thus the F_σ-set F is assumed to be compact.

2. Background on harmonically weighted Dirichlet spaces

We start with the definition of the local Dirichlet integral of an $L^1(\partial\mathbb{D})$-function, see [25], Section 4 for more details. This will be needed for the treatment of capacities for the weighted Dirichlet spaces.

If $f \in L^1(\partial\mathbb{D})$ and $\lambda \in \mathbb{D}$, then $f(\lambda) = P[f](\lambda)$ is the value of the Poisson integral of f at λ and

$$D_\lambda(f) = \int_{|z|=1} |\frac{f(z) - f(\lambda)}{z - \lambda}|^2 \frac{|dz|}{2\pi}. \tag{2.1}$$

If $\lambda \in \partial\mathbb{D}$ and if $P[f]$ has nontangential limit $f(\lambda)$ at λ, then (2.1) also defines $D_\lambda(f)$, while for those $\lambda \in \partial\mathbb{D}$ where the nontangential limit of $P[f]$ does not exist, we set $D_\lambda(f) = \infty$. $D_\lambda(f)$ is called the local Dirichlet integral of f. Clearly, if $D_\lambda(f) < \infty$ for some λ, then $f \in L^2(\partial\mathbb{D})$. We refer the reader to [22] for more information on $D_\lambda(f)$ for analytic functions f. Furthermore, Lemma 4.1 of [25] shows that if $f, g \in H^2$, then $D_\lambda(f + \overline{g}) = D_\lambda(f) + D_\lambda(g)$ for all $|\lambda| \le 1$.

Consider a positive measure μ on $\partial\mathbb{D}$, define

$$D_h(\mu) = \{f \in L^2(\partial\mathbb{D}) : \int_{\lambda\in\partial\mathbb{D}} D_\lambda(f) d\mu(\lambda) < \infty\}$$

with norm

$$\|f\|_\mu^2 = \|f\|_{L^2}^2 + \int_{\lambda\in\partial\mathbb{D}} D_\lambda(f) d\mu(\lambda).$$

We also set $D(\mu) = D_h(\mu) \cap H^2$. If μ is normalized Lebesgue measure on $\partial\mathbb{D}$, then $D(\mu) = D$. In fact, if for any measure μ on $\partial\mathbb{D}$, we let $w_\mu = P[\mu]$ be the Poisson integral of μ, then

$$\int_{|\lambda|=1} D_\lambda(f) d\mu(\lambda) = \int_{\mathbb{D}} |f'|^2 w_\mu \frac{dA}{\pi} \text{ for all } f \in H^2, \tag{2.2}$$

see [22].

We now fix a measure μ that is supported in $\partial\mathbb{D}$. All of the concepts that we are about to define will depend on μ. For ease of presentation we will suppress that dependence in the notation.

The real parts of functions of $D_h(\mu) \subseteq L^2(\partial\mathbb{D})$ form a Dirichlet space in the sense of Beurling-Deny, see [2]. An extensive general potential theory of such spaces has been developed. A good reference for this is [14], Chapter 2. For the case of the spaces $D_h(\mu)$ we recall the definition of capacity and we describe the related concepts that we will need. If $U \subseteq \partial\mathbb{D}$ is an open set, then the capacity is defined by

$$c_\mu(U) = \inf\{\|f\|^2 : f \in D_h(\mu), f \ge 1 \text{ a.e. on } U\}.$$

For arbitrary subsets $A \subseteq \partial\mathbb{D}$ one sets

$$c_\mu(A) = \inf\{c_\mu(U) : A \subseteq U, U \text{ open}\}.$$

This capacity turns out to be a Choquet capacity, [Cho54], and as a consequence one has

$$c_\mu(E) = \sup\{c_\mu(K) : K \subseteq E, K \text{ compact}\}$$

for every Borel set $E \subseteq \partial\mathbb{D}$. One says that a property holds quasi-everywhere (q.e.) if it holds except perhaps on a set of capacity 0. A set $E \subseteq \partial\mathbb{D}$ is called quasi-closed, if for each $\varepsilon > 0$, there is an open set A of capacity $< \varepsilon$ such that $E \setminus A$ is closed. Similarly, a function f is called quasi-continuous, if for any $\varepsilon > 0$ there is an open set A of capacity $< \varepsilon$ such that $f|\partial\mathbb{D} \setminus A$ is continuous. Every $f \in D_h(\mu)$ has a quasi-continuous a.e. representative, and any two quasi-continuous representatives of the same function agree q.e.. We refer the reader to Chapter 2 of [14] for the results mentioned above and further basic results about these capacities and exceptional sets. The paper [15] also contains a nice overview. Furthermore we mention that Chacon [9] showed that the harmonic extensions of $D_h(\mu)$-functions have nontangential limits q.e., and it follows from this that the resulting non-tangential limit function is a quasi-continuous representative.

We will need results about equilibrium potentials and equilibrium measures of Borel sets. Again we refer the reader e.g. to Chapter 2 of [14] for the proofs of these results. Every Borel set E has an equilibrium potential f_E satisfying:

$$\begin{aligned} &0 \le f_E \le 1 \text{ on } \partial\mathbb{D}, \\ &f_E = 1 \text{ q.e. on E, and} \\ &c_\mu(E) = \|f_E\|_\mu^2 \end{aligned}$$

A positive Borel measure σ on $\partial\mathbb{D}$ has finite energy, if there is $C > 0$ such that $\int |g| d\sigma \le C\|g\|_\mu$ for all continuous functions in $D_h(\mu)$. Note that the trigonometric polynomials are dense in $D_h(\mu)$ (see [21] for the density of polynomials in $D(\mu)$ and then use the identity $D_\lambda(f+\overline{g}) = D_\lambda(f)+D_\lambda(g)$). Thus by the Riesz representation theorem any such measure gives rise to a function f_σ such that $\langle g, f_\sigma\rangle_\mu = \int g d\sigma$ for all $g \in D_h(\mu) \cap C(\partial\mathbb{D})$. If σ has finite energy, then sets of capacity 0 have σ-measure 0 and one gets

$$\langle g, f_\sigma\rangle_\mu = \int g d\sigma \text{ for all } g \in D_h(\mu).$$

It turns out that equilibrium potentials come from unique measures of finite energy in the way just described. Such measures are called equilibrium measures. If E is a quasi-closed set, then the equilibrium measure σ_E for E satisfies $f_E = f_{\sigma_E}$, $\sigma(\partial\mathbb{D} \setminus E) = 0$, and hence

$$c_\mu(E) = \|f_E\|_\mu^2 = \langle f_E, f_{\sigma_E}\rangle_\mu = \int f_E d\sigma_E = \sigma_E(E).$$

Let $k_\lambda(z)$ be the reproducing kernel for $D(\mu)$, then one checks that $u_\lambda(z) = 2\,\text{Re}\, k_\lambda(z) - 1$ is the reproducing kernel for $D_h(\mu)$, $f(\lambda) = P[f](\lambda) = \langle f, u_\lambda\rangle_\mu$ for all $f \in D_h(\mu)$. In particular, if σ has finite energy, then

$$f_\sigma(\lambda) = \langle f_\sigma, u_\lambda\rangle_\mu = \int u_\lambda(z) d\sigma(z).$$

In [28] Shimorin showed that $k_\lambda(z)$ is a normalized complete Nevanlinna-Pick kernel, i.e. it is of the form

$$k_\lambda(z) = \frac{1}{1 - \sum_n \overline{b_n(\lambda)} b_n(z)}$$

for some functions $b_n \in D(\mu)$ with $b_n(0) = 0$ for all n. From this it follows easily that $u_\lambda(z) \geq 0$ for all $\lambda, z \in \mathbb{D}$. The following is the analogue of Lemma 3.4.2 of [11].

Lemma 2.3. *Let μ be a measure on $\partial\mathbb{D}$, and let σ be a measure of finite energy that is supported on a compact set F. Set $f(\lambda) = \int u_\lambda d\sigma$ and $g(\lambda) = \int (2k_\lambda(z) - 1) d\sigma(z)$. Then*

(a) $0 \leq f = \text{ Re } g$
(b) $\|g\|_\mu^2 \leq 2\|f\|_\mu^2$
(c) $|g(\lambda)| \leq \frac{4\sigma(F)}{dist(\lambda, F)}$.

In particular, if σ is the equilibrium measure for a set F, then $\|g\|_\mu^2 \leq 2c_\mu(F)$ and $|g(\lambda)| \leq \frac{4c_\mu(F)}{dist(\lambda,F)}$.

Proof. (a) is obvious and we have

$$f = \text{ Re } g = \frac{1}{2}(g - g(0) + 2g(0) + \overline{g - g(0)}).$$

Thus by orthogonality $\|f\|_\mu^2 = \frac{\|g-g(0)\|_\mu^2}{2} + |g(0)|^2 = \frac{\|g\|_\mu^2 + |g(0)|^2}{2}$, and (b) follows.

In order to prove (c) we recall formula (12) from Shimorin's paper [27]: For all $|\lambda| < 1$ and a.e. $z \in \partial\mathbb{D}$ we have

$$|\frac{1}{1 - \overline{\lambda} z}|^2 = \frac{(1 - |\lambda|^2) k_\lambda(\lambda)}{|1 - \overline{\lambda} z|^2} + |k_\lambda(z) - \frac{1}{1 - \overline{\lambda} z}|^2 + \int_{\mathbb{D}} |\frac{k_\lambda(z) - k_\lambda(u)}{z - u}|^2 d\mu(u).$$

This implies that for all $|\lambda| < 1$ and a.e. $z \in \partial\mathbb{D}$ we have

$$|2k_\lambda(z) - 1| \leq \frac{4}{|1 - \overline{\lambda} z|}$$

and hence the inequality holds also for all $|\lambda|, |z| < 1$, and thus for fixed $|\lambda| < 1$ it holds for q.e. $z \in \partial\mathbb{D}$. Hence

$$|g(\lambda)| \leq \int |2k_\lambda(z) - 1| d\sigma(z) \leq \frac{4\sigma(F)}{dist(\lambda, F)}.$$

□

3. The construction

We start with a known lemma.

Lemma 3.1. *If $\log f \in D(\mu)$ and $f \in H^\infty$, then f is cyclic in $D(\mu)$.*

This is Theorem 5.8 of [1]. Since that reference may not be widely available, we have indicated a short proof below.

Proof. Set $g = \log f$. If $\alpha > 0$, then $f^\alpha = e^{\alpha g}$ and hence

$$|(f^\alpha)'(z)| = \alpha |f(z)|^\alpha |g'(z)| \leq \alpha \|f\|_\infty^\alpha |g'(z)|.$$

By (2.2) and dominated convergence this implies that $f^\alpha \to 1$ as $\alpha \to 0^+$. We know by Theorem 9.1.7 of [11] that $[f^\alpha] = [f]$, thus $1 \in [f]$ and f must be cyclic. □

This observation is why part (d) of the following construction is important.

Theorem 3.2. *Let μ be a measure on $\partial\mathbb{D}$. Let $E \subseteq \partial\mathbb{D}$ be closed, and let $F \subseteq E$ be an F_σ-set with $c_\mu(F) = 0$. Then there is an $f \in D(\mu)$ such that*
(a) $|f(z)| \le 1$ for all $z \in \mathbb{D}$,
(b) $\limsup_{z\to w} |f(z)| = 0$ for all $w \in F$,
(c) f is continuous on $\overline{\mathbb{D}} \setminus E$ and $f(w) \ne 0$ for all $w \in \overline{\mathbb{D}} \setminus E$,
(d) $\log f \in D(\mu)$.

Proof. If $D(\mu) = D$ and $E = F$ is a compact set of logarithmic capacity 0, then this theorem reduces to a result of Brown-Cohn [6], which in turn was based on a result that was published in [8], Theorem 4. Our proof follows the ideas of Brown-Cohn as presented in the proof of Theorem 3.4.1 of the book [11]. Let F_n be an increasing sequence of compact sets with $\bigcup_n F_n = F$, and let K_n be an increasing sequence of compact sets such that $K_n \subseteq \text{ int}K_{n+1}$ for each n and $\bigcup_{n=1}^\infty K_n = \partial\mathbb{D} \setminus E$.

Let $\varepsilon_n > 0$ such that $\sum_n \varepsilon_n < \infty$, and for each $n \in \mathbb{N}$ choose an open set $U_n \subseteq \partial\mathbb{D}$ such that $F_n \subseteq U_n \subseteq \partial\mathbb{D} \setminus K_n$, and $c_\mu(U_n) < \varepsilon_n^2$.

Now it is routine to find open sets V_n such that $F_n \subseteq V_n \subseteq \overline{V}_n \subseteq U_n$. For each n let σ_n be the equilibrium measure for $\overline{V}_n$, and let g_n be the analytic function as in Lemma 2.3 such that Re $g_n = f_n =$ the equilibrium potential for $\overline{V}_n$. Then $\|g_n\|_\mu \le \sqrt{2}\varepsilon_n$.

Notice that $f_n(w) = 1$ a.e. on the open set V_n. F_n is a compact subset of V_n, hence there is an r_n such that $f_n(z) \ge 1/2$ for all $|z| \ge r_n$ and $z/|z| \in F_n$.

Now consider the function $f(z) = e^{-\sum_{n=1}^\infty g_n(r_n z)}$. It satisfies (a) since each Re $g_n(r_n z) \ge 0$ in $\mathbb{D}$ and it satisfies (d), because $\|\log f\|_\mu = \|\sum_n g_{n,r_n}\|_\mu \le \sum_n \|g_{n,r_n}\|_\mu \le c\sum_n \varepsilon_n < \infty$.

Next we will verify (b). Let $w \in F$, then there is $N \in \mathbb{N}$ such that $w \in F_n$ for all $n > N$. By Fatou's lemma we have

$$\begin{aligned}\liminf_{z\to w} \sum_{n=1}^\infty f_n(r_n z) &\ge \sum_{n=N+1}^\infty \liminf_{z\to w} f_n(r_n z) \\ &= \sum_{n=N+1}^\infty f_n(r_n w) \\ &\ge \sum_{n\ge N+1} \frac{1}{2} = \infty.\end{aligned}$$

Thus (b) holds.

Finally we show that (c) is satisfied as well. Since each function g_{n,r_n} is continuous on $\overline{\mathbb{D}}$ it suffices to show that $\sum_{n=1}^\infty g_{n,r_n}$ converges uniformly on each set K_N.

Fix $N \in \mathbb{N}$. Note $\overline{V_n} \subseteq U_n \subseteq \partial\mathbb{D} \setminus K_n \subseteq \partial\mathbb{D} \setminus \text{ int}K_n \subseteq \partial\mathbb{D} \setminus \text{ int}K_{N+1}$ for each $n \ge N+1$ and

$$d_N = \inf\{|rz - w| : 0 \le r < 1, z \in K_N, w \in \partial\mathbb{D} \setminus \text{ int}K_{N+1}\} > 0.$$

Thus for $z \in K_N$ and $n \ge N+1$ we have by Lemma 2.3 (c)

$$|g_{n,r_n}(z)| \le \frac{c c_\mu(F_n)}{dist(r_n z, \overline{V_n})} \le \frac{c\varepsilon_n^2}{dist(r_n z, \overline{V_n})} \le \frac{c\varepsilon_n^2}{d_N}.$$

Hence it follows from the Weierstrass M-test that $\sum_n g_{n,r_n}$ converges uniformly on K_N. ☐

We can now prove Theorem 1.3.

Proof. We will use Theorem 3.2 with μ equal to normalized Lebesgue measure. Then $D(\mu) = D$ and the capacity c_μ is equivalent to logarithmic capacity.

Let φ be the extremal function for D_E. The φ is a multiplier of D (see [23]), and by Theorem 5.1 of [24], the radial limit $\lim_{r\to 1} |\varphi(rw)|$ exists for every $w \in \partial\mathbb{D}$, we have

$$\lim_{r\to 1} |\varphi(rw)| = \limsup_{z\to w} |\varphi(z)| \quad \text{for all } w \in \partial\mathbb{D},$$

and $Z(\varphi)$ is a G_δ-set with $Z(\varphi) \subseteq \underline{Z}(\varphi)$.

For a subset $\mathcal{M} \subseteq D$ we set $\underline{Z}(\mathcal{M}) = \bigcap_{h\in\mathcal{M}} \underline{Z}(h)$. Then since E is a Carleson set, it is clear that $\underline{Z}(D_E) \subseteq E$. By Theorem 3.2 and Lemma 3.1(a) of [24] $\underline{Z}(\varphi) = \underline{Z}(D_E) \subseteq E$ and φ extends to be analytic across each complementary arc of $\underline{Z}(D_E)$.

Since $\varphi \in D_E$ it follows that $F = E \setminus Z(\varphi)$ has logarithmic capacity 0. Also note that $F \subseteq E$ is an F_σ-set. Thus let f be the corresponding function given by the previous theorem that satisfies (a)-(d).

We claim that $g = f\varphi$ has the required properties. Indeed, since f is cyclic there will be polynomials p_n such that $p_n f \to 1$ in D. Since φ is a multiplier we have $p_n f\varphi \to \varphi$ and hence $\varphi \in [f\varphi]$ and hence $D_E = [\varphi] \subseteq [f\varphi]$. The other inclusion is obvious.

It is left to show that $f\varphi$ extends to be continuous on $\overline{\mathbb{D}}$ and $Z(fg) = E$.

We need to check that $f\varphi$ is continuous at every point $w \in \partial\mathbb{D}$. If $w \in \partial\mathbb{D} \setminus E$, then f extends to be continuous by construction and φ is analytic in a neighborhood of w, hence the product will extend to be continuous at w. It also follows from the construction that $f\varphi \neq 0$ on $\partial\mathbb{D} \setminus E$. On $E = F \cup Z(\varphi)$ we set $f\varphi$ to be 0. Now if $w \in F$ and $z_n \in \mathbb{D}$ such that $z_n \to w$ then $|f(z_n)\varphi(z_n)| \leq |f(z_n)| \to 0$. The case of $w \in Z(\varphi)$ follows analogously. This implies that $f\varphi$ extends to be continuous on $\overline{\mathbb{D}}$.

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Question 3.3. *In Theorem 1.3, can one drop the hypothesis Carleson set and replace it with E closed such that $D_E \neq (0)$?*

4. Tangential Limits of Extremal functions

In this Section we will prove Theorem 1.4. Let φ be an extremal function in the Dirichlet space D. Then by formula (5.1) in [24], we have the following equation for $w = re^{it} \in \mathbb{D}$,

$$r = \int_0^r \left(\int_{\partial\mathbb{D}} P_{se^{it}}(z) D_z(\varphi) \frac{|dz|}{2\pi} \right) ds + r \int_{\partial\mathbb{D}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi}, \tag{4.1}$$

where $P_w(z) = \dfrac{1-|w|^2}{|z-w|^2}$ denotes the Poisson kernel.

The first integral on the right-hand side of (4.1) is positive and increasing in r, hence its limit exists for each e^{it} and its limit is less than or equal to 1.

By an application of Hölder's inequality to the Poisson integral representation of φ or Theorem 2.12 of [10], we have

$$|\varphi(w)|^2 \leq \int_{\partial\mathbb{D}} P_w(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} \leq 1$$

where the second inequality follows from the equation (4.1) and hence

$$\lim_{r\to 1^-} \int_{\partial\mathbb{D}} P_{re^{it}}(z)|\varphi(z)|^2 \frac{|dz|}{2\pi}$$

exists for every $e^{it} \in \partial\mathbb{D}$.

Since the Dirichlet space is contained in VMOA (see [31]), the radial limit

$$|\varphi|^2(e^{it}) := \lim_{r\to 1^-} |\varphi(re^{it})|^2 = \lim_{r\to 1^-} \int_{\partial\mathbb{D}} P_{re^{it}}(z)|\varphi(z)|^2 \frac{|dz|}{2\pi}$$

exists for each $e^{it} \in \partial\mathbb{D}$ and is less than or equal 1 as well. Also, by dividing equation (4.1) by r, we obtain

$$\int_{\partial\mathbb{D}} P_w(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} = 1 - \int_{\partial\mathbb{D}} \left(\frac{1}{r}\int_0^r P_{se^{it}}(z)ds\right) D_z(\varphi)\frac{|dz|}{2\pi}. \tag{4.2}$$

A direct computation shows

$$\frac{1}{r}\int_0^r P_{se^{it}}(z)ds = 2\mathrm{Re}k_w(z) - 1, \tag{4.3}$$

where $k_w(z) = \dfrac{1}{\overline{w}z}\log\dfrac{1}{1-\overline{w}z}$ is the reproducing kernel for the Dirichlet space D. Therefore, (4.2) is equal to

$$\int_{\partial\mathbb{D}} P_w(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} = 1 - \int_{\partial\mathbb{D}} \big(2\mathrm{Re}k_w(z) - 1\big) D_z(\varphi)\frac{|dz|}{2\pi}. \tag{4.4}$$

Proposition 4.5. *Let φ be an extremal function in the Dirichlet space D. Then*

$$\sup_{0\neq w\in\mathbb{D}} \int_{|z|=1} \frac{1}{|w|}\left|\log\left(\frac{1}{1-\overline{w}z}\right)\right| D_z(\varphi)\frac{|dz|}{2\pi} < \infty.$$

Proof. Let $w \in \mathbb{D}\setminus\{0\}$ and $z \in \partial\mathbb{D}$ be such that $\overline{w}z = re^{i\theta}$. Since the reproducing kernel $k_w(z)$ is small for small w, we will assume that $|w| \geq \dfrac{1}{2}$. Observe that

$$\mathrm{Re}k_w(z) = \frac{\cos(\theta)}{|r|}\log\frac{1}{|1-re^{i\theta}|} + \frac{\sin(\theta)}{|r|}\mathrm{Arg}\left(\frac{1}{1-re^{i\theta}}\right).$$

Set

$$S_w = \left\{z \in \partial\mathbb{D} : \overline{w}z = re^{i\theta}, \left|\theta - \frac{\pi}{2}\right| < \frac{\pi}{4}\right\}.$$

Then there is a constant C that is independent of $|w| \geq 1/2$ such that

$$\frac{1}{|w|}\left|\log\left(\frac{1}{1-\overline{w}z}\right)\right| \leq C$$

for all $z \in \partial\mathbb{D}\setminus S_w$. This implies

$$\sup_{1/2\leq|w|<1} \int_{z\in\partial\mathbb{D}\setminus S_w} \frac{1}{|w|}\left|\log\left(\frac{1}{1-\overline{w}z}\right)\right| D_z(\varphi)\frac{|dz|}{2\pi} < C.$$

Furthermore on the set S_w it is easy to see $|k_w(z)|$ is comparable with the real part of the reproducing kernel $k_w(z)$, i.e. $|k_w(z)| \approx \mathrm{Re} k_w(z)$, where again the constant is independent of w, $1/2 \leq |w| < 1$. Thus equation (4.4) implies that

$$\sup_{1/2 \leq |w| < 1} \int_{z \in S_w} \frac{1}{|w|} \left| \log \left(\frac{1}{1 - \overline{w} z} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} < C.$$

This finishes the proof. □

Now we are ready to prove Theorem 1.4.

Proof. Let $c > 0$, $\alpha \geq 1$, and let φ be an extremal function in D. Without loss of generality, we assume $e^{it} = 1$. We have to show that $|\varphi(w)|^2 \to \limsup_{w \to 1, w \in \mathbb{D}} |\varphi(w)|^2$ as $w = re^{i\theta} \to 1$ in the approach region $\Gamma_c^\alpha(1)$. Theorem 5.1 of [24] states that $\lim_{r \to 1^-} |\varphi(r)|^2 = \limsup_{w \to 1, w \in \mathbb{D}} |\varphi(w)|^2$. Furthermore, the Dirichlet space is contained in VMOA, hence we have $\lim_{z \to 1} P_z[|\varphi|^2] - |\varphi(z)|^2 = 0$ and we may substitute $P_z[|\varphi|^2]$ for $|\varphi(z)|^2$. Thus, it will suffice to prove that

$$\int_{\partial \mathbb{D}} P_{|w|}(z) |\varphi(z)|^2 \frac{|dz|}{2\pi} - \int_{\partial \mathbb{D}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi} \to 0$$

as $w \to 1$, $w \in \Gamma_c^\alpha(1)$. We now apply (4.4) with w and $|w|$ and observe that this is equivalent to proving that

$$U(w) := \int_{\partial \mathbb{D}} \mathrm{Re}(k_{|w|}(z) - k_w(z)) D_z(\varphi) \frac{|dz|}{2\pi} \to 0$$

as $w \to 1$, $w \in \Gamma_c^\alpha(1)$.

Write $w = re^{i\theta}$. Then for all $|z| = 1$

$$k_r(z) - k_{re^{i\theta}}(z) = \frac{1}{rz} \log \left(\frac{1 - zre^{-i\theta}}{1 - rz} \right) + \frac{1}{rz} (1 - e^{i\theta}) \log \left(\frac{1}{1 - zre^{-i\theta}} \right),$$

and hence

$$|\mathrm{Re}(k_r(z) - k_{re^{i\theta}}(z))| \leq \frac{1}{r} \left| \log \left(\frac{1 - zre^{-i\theta}}{1 - rz} \right) \right| + \frac{1}{r} \left| 1 - e^{i\theta} \right| \left| \log \left(\frac{1}{1 - zre^{-i\theta}} \right) \right|.$$

Therefore,

$$U(re^{i\theta}) \leq \int_{|z|=1} \frac{2}{r} \left| \log \left(\frac{1 - zre^{-i\theta}}{1 - rz} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} + \frac{2}{r} \left| 1 - e^{i\theta} \right| \int_{|z|=1} \left| \log \left(\frac{1}{1 - zre^{-i\theta}} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} \tag{4.6}$$

By Proposition 4.5 the second term on the right-hand side of (4.6) will go to zero as $w = re^{i\theta} \to 1$ in $\mathbb{D}$. It is left to show that the first integral in (4.6) goes to zero as well. Note that

$$\int_{|z|=1} \frac{2}{r}\left|\log\left(\frac{z-re^{i\theta}}{z-r}\right)\right| D_z(\varphi)\frac{|dz|}{2\pi} \leq \int_{|z|=1} \frac{2}{r}\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi} + \int_{|z|=1} \frac{2}{r}\left|\operatorname{Arg}\left(\frac{z-re^{i\theta}}{z-r}\right)\right| D_z(\varphi)\frac{|dz|}{2\pi} \tag{4.7}$$

The second integral on the right-hand side of (4.7) will converge to zero as $re^{i\theta} \to 1$ by the Dominated Convergence Theorem (DCT), since $\left|\operatorname{Arg}\left(\frac{z-re^{i\theta}}{z-r}\right)\right|$ is bounded (one can see this from the picture) and will approach zero as $re^{i\theta} \to 1$ in $\Gamma_c^\alpha(1)$. Thus, what we really need to show that is the first integral on the right-hand side of (4.7) converges to zero.

For $w = re^{i\theta} \in \Gamma_c^\alpha(1)$, define

$$R_w = \left\{ z \in \partial\mathbb{D} : \left|\frac{r-w}{z-r}\right| \geq \frac{1}{2} \right\}.$$

Since $\frac{z-w}{z-r} = 1 + \frac{r-w}{z-r}$ we have

$$\log\frac{1}{2} < \log\left|\frac{z-w}{z-r}\right| < \log\frac{3}{2}$$

for all $z \in \partial\mathbb{D} \setminus R_w$. Thus by the dominated convergence theorem we conclude that

$$\int_{z\in\partial\mathbb{D}\setminus R_w} \frac{2}{r}\left|\log\left|\frac{z-w}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi} \to 0$$

as $w \to 1$.

It remains to show that

$$\int_{z\in R_w} \frac{2}{r}\left|\log\left|\frac{z-w}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi} \to 0$$

as $w \to 1$ in $\Gamma_c^\alpha(1)$. By Proposition 4.5 and the generalized dominated convergence theorem it will be enough to show that

$$\frac{\left|\log\left|\frac{z-w}{z-r}\right|\right|}{\log\frac{2}{|z-r|}} \leq 1 + \frac{\log\frac{2}{|z-w|}}{\log\frac{2}{|z-r|}} \tag{4.8}$$

is bounded on R_w.

To this end let $w = re^{i\theta} \in \Gamma_c^\alpha(1)$ and $z \in R_w$. Then

$$\begin{aligned} |z-r| &\leq 2\,|r-w| \\ &\leq 2((1-r) + |1-w|) \\ &\leq 2((1-r) + (c(1-r))^{\frac{1}{\alpha}}) \\ &\leq 2(1+c^{1/\alpha})(1-r)^{1/\alpha} \end{aligned}$$

Hence

$$\log\frac{2}{|z-r|} \geq \frac{1}{\alpha}\log\frac{1}{1-r} - \log(1+c^{1/\alpha})$$

and we conclude that for all r sufficiently close to 1

$$\frac{\log \dfrac{2}{|z-w|}}{\log \dfrac{2}{|z-r|}} \leq \alpha \frac{\log \dfrac{2}{1-r}}{\log \dfrac{1}{1-r} - \alpha \log(1+c^{1/\alpha})}.$$

This last expression has a finite limit as $r \to 1^-$, hence the term in (4.8) is uniformly bounded for $w = re^{i\theta} \in \Gamma_c^\alpha(1)$ and $z \in R_w$. That finishes the proof. $\square$

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