A model for two-isometric operator tuples with finite defect.

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joint work with Carl Sundberg

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September, 2011

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Example

$(M_z, H^2(\partial B_d))$ is a spherical isometry.

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$$\Delta_{n+1} = \left(\sum_{i=1}^{d} T_i^* \Delta_n T_i \right) - \Delta_n$$

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T is an n-isometric operator tuple, if $\Delta_n = 0$.

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Facts:

If T is an n-iso $\Rightarrow$

- $\Delta_k = 0$ for all $k \geqslant n$
- $\Delta_{n-1} \geqslant 0$,
- but for $n \geqslant 3$ it is possible that $\Delta_k \ngeq 0$ for some $1 \leqslant k \leqslant n-2$

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- $\mathcal{H}(k) \subseteq \text{Hol}(\mathbb{B}_d)$ with reproducing kernel,

$$k_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^j},$$

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- If T is an n-isometric tuple, $\mathcal{M} \in \text{Lat}T_i \quad \forall i$,
then $T|\mathcal{M}$ is an n-isometric tuple.

$$\langle \Delta_n x, x \rangle = \sum_{j=1}^{n} (-1)^j \binom{n}{j} \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} \|T^\alpha x\|^2 = 0 \quad \forall x$$

More Examples of two-isometric tuples

- If $d = 1$, then two-isometric operators on finite dimensional spaces must be unitary.
- $d = 2$

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T_2^2 = 0$$

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$$\Delta_2 = T_1^* \Delta_1 T_1 + T_2^* \Delta_1 T_2 - \Delta_1 = {T_2^*}^2 T_2^2 = 0$$

More generally

If $\alpha = (\alpha_1, ..., \alpha_d) \in \partial\mathbb{B}_d$, $V_i : \mathbb{C}^m \to \mathbb{C}^n$, $\sum_{i=1}^{d} \overline{\alpha_i} V_i = 0$, then

$S = (S_1, ..., S_d)$,

$$S_i = \begin{pmatrix} \alpha_i I_n & V_i \\ 0 & \alpha_i I_m \end{pmatrix}$$

defines a two-isometric d-tuple.

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Thm

If T is a two-isometric d-tuple on a finite dimensional space, then

$$T = U \oplus S,$$

where U is spherical unitary and S is a direct sum of operator tuples as above.

From now on let $T = (T_1, ..., T_d)$ be a two-isometric tuple without nontrivial finite dimensional reducing subspace.

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Thus $\sum_{i=1}^d T_i^* \Delta_1 T_i - \Delta_1 = 0 \quad \Leftrightarrow$

$$\sum_{i=1}^d \|DT_i x\|^2 = \|Dx\|^2 \ \ \forall x$$

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"T induces a spherical isometry w.r.t. $\|x\|_* = \|Dx\|$"

Spherical isometries are jointly subnormal (Athavale), hence for all $x_0 \in \mathcal{H}$ there is $\mu \in M_+(\partial B_d)$ with

$$\|Dp(T)x_0\|^2 = \int_{\partial B_d} |p|^2 d\mu \ \ \forall p \in \mathbb{C}[z]$$

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For $d = 1$ one can often recover $\|p(T)x_0\|$ by use of this μ.
For $d > 1$ this is not the case.

$d = 1$ Model theorem, Agler-Stankus version

If $T \in \mathcal{B}(\mathcal{H})$ is a 2-isometry with $D \neq 0$,
if $x_0 \in \ker(\|D\| - D)$, $\|x_0\| = 1$,
if

$$\|Dp(T)x_0\|^2 = \int |p|^2 d\mu$$

then

$$\|p(T)x_0\|^2 = \|p\|_{AS}^2 = \int_{\partial\mathbb{D}} |p|^2 \frac{d\mu}{\|\mu\|} + \int_{\partial\mathbb{D}} D_\alpha(p) d\mu(\alpha)$$

$\|\mu\| = \|D\|^2$

$$D_\alpha(f) = \int_{\partial\mathbb{D}} \frac{|f(z) - f(\alpha)|^2}{|z - \alpha|^2} \frac{|dz|}{2\pi} \quad \text{(local Dirichlet integral)}$$

$d \geqslant 1$

Defn

Let $\alpha \in \partial\mathbb{B}_d$, $\sigma \in M_+(\partial\mathbb{B}_d)$, *then*

$$\sigma \in \mathcal{M}_\alpha$$

$$\Leftrightarrow$$

- $\sigma(\partial\mathbb{B}_d) = 1$,
- $\sigma(\{\alpha\}) = 0$, *and*
- $$\int_{\partial\mathbb{B}_d} p(z)\frac{(1-\langle z,\alpha\rangle)^2}{\|z-\alpha\|^2_{\mathbb{C}^d}}d\sigma(z) = 0 \ \ \forall p \in \mathbb{C}[z].$$

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Defn

Let $\alpha \in \partial\mathbb{B}_d$, $\sigma \in \mathcal{M}_\alpha$, then define the local Dirichlet integral

$$D_\alpha(p) = D_\alpha(p, \sigma) = \int_{\partial\mathbb{B}_d} \frac{|p(z) - p(\alpha)|^2}{\|z - \alpha\|^2_{\mathbb{C}^d}} d\sigma(z)$$

For $\sigma \in \mathcal{M}_\alpha$ we have

$$\sum_{i=1}^{d} D_\alpha(z_i p, \sigma) = D_\alpha(p, \sigma) + |p(\alpha)|^2 \quad \forall p \in \mathbb{C}[z].$$

Thus:

Let $\mu \in M_+(\partial\mathbb{B}_d)$, $\mu \neq 0$, let $\sigma_\alpha \in \mathcal{M}_\alpha$, continuously depending on $\alpha \in \partial\mathbb{B}_d$, then define

$$\|p\|_{AS}^2 = \int_{\partial\mathbb{B}_d} |p|^2 \frac{d\mu}{\|\mu\|} + \int_{\partial\mathbb{B}_d} D_\alpha(p, \sigma_\alpha) d\mu(\alpha).$$

Let $\mathcal{H}_{AS} = \mathcal{H}_{AS}(\mu, \sigma_\alpha)$ be the completion of the polys w.r.t. $\|\cdot\|_{AS}$.

Then $(M_z, \mathcal{H}_{AS})$ is a 2-isometric d-tuple with

- $\|Dp\|_{AS}^2 = \int_{\partial\mathbb{B}_d} |p|^2 d\mu \ \ \forall p \in \mathbb{C}[z]$
- $1 \in \ker(\|D\| - D)$, $\|1\|_{AS} = 1$.

Theorem

Let T be a two-isometric d-tuple with

- rank $D < \infty$,
- T has no nontrivial finite dimensional reducing subspace.

Let $x_0 \in \ker(\|D\| - D)$, $\|x_0\| = 1$. Then

$$\|p(T)x_0\|^2 = \|p\|_{AS}^2 + \langle Ep, p\rangle_{AS} \quad \forall p \in \mathbb{C}[z].$$

Here

- $\|Dp(T)x_0\|^2 = \int |p|^2 d\mu = \sum_{j=1}^{N} \mu_j |p(\alpha_j)|^2$
- $\exists \ \sigma_j \in \mathcal{M}_{\alpha_j} \quad j = 1, ..., N$
- $\mathcal{H}_{AS} = \mathcal{H}_{AS}(\mu, \{\sigma_j\})$
- $E \in \mathcal{B}(\mathcal{H}_{AS})$ with
  - E is s.a. with $I + E \geqslant 0$
  - $E1 = 0$
  - $\sum_{i=1}^{d} \langle Ez_i p, z_i p\rangle = \langle Ep, p\rangle \ \ \forall p \in \mathbb{C}[z]$