Not Playing With a Full Deck?

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A standard deck of 52 playing cards has 4 suits: clubs (♣), spades (♠), hearts (♥) and diamonds (♦). Each suit has 13 denominations, or ranks, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (Jack), Q (Queen), K (King) and A (Ace). But sometimes, a card goes missing – lost between the couch cushions, clothes-pinned in the bike spokes, filched by the cat, eaten by the dog (see Figure 1), or maybe misplaced in a matching deck. You may find yourself not playing with a full deck. Building a house of cards? No problem. Playing poker? Problem. Or is it? Ask yourself the following question:

> What is the probability of being dealt a Two-Pair hand of five cards from a well-shuffled deck when the Ace of Spades is missing from the deck? Specifically, how does it compare to the probability of being dealt a Two-Pair hand from a standard 52-card deck?

What does your 'probability intuition' tell you? On the one hand, with a missing Ace of Spades there are fewer Two-Pair hands. On the other hand, there are also fewer total five-card hands. The ratio of these two numbers defines the probability; which way it goes seems unclear. Let's do the calculation – first for the standard 52-card deck and then for the 51-card deck missing the Ace of Spades.

Figure 1: A 51-card deck – missing the Ace of Spades.

1 Two-Pair Hands in a Standard Deck

First note that since the order of the cards in the hand does not matter, the total number of different five-card hands in a deck of 52 cards is

$$N^{52}_{TOT} = \binom{52}{5} \quad = \quad 2,598,960. \tag{1}$$

A Two-Pair hand has two sets of matching denominations and a third denomination that does not match the other two denominations. [1] The total number of Two-Pair hands can be computed as

$$N^{52}_{2P} = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1} = 123,552. \tag{2}$$

That is, from the thirteen denominations choose two to be the denominations of the pairs (e.g. J and 5). Then, for each chosen denomination choose two of four suits (e.g. J♣, J♦,

[1] To be clear, we exclude from the count hands that would qualify as a Full House (three of one denomination and two of another) or as Four of a Kind (four of one denomination and one of another).

5♣, 5♥). For the fifth card any of 44 cards work – that is, choose one of the remaining 11 denominations (i.e. not a J or 5) and for that denomination choose 1 of the 4 suits (e.g. 9♠ for the Two-Pair hand [J♣, J♦, 5♣, 5♥, 9♠]). It follows that the probability of being dealt a Two-Pair hand is

$$P_{2P}^{52} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{123,552}{2,598,960} \approx 0.04754. \tag{3}$$

So, on average, fewer than 5 hands in 100 are expected to be Two-Pair hands.

2 Two-Pair Hands in a Deck Missing the Ace of Spades

Here the total number of five-card hands from a 51-card deck is

$$N_{TOT}^{51} = \binom{51}{5} = \frac{51!}{46!5!} = \frac{47}{52}\frac{52!}{47!5!} = \frac{47}{52}\binom{52}{5} = \frac{47}{52}N_{TOT}^{52} = 2,349,060. \tag{4}$$

That is, when one card is missing, the reduction factor in the total number of hands relative to the standard deck is 47/52.

The total number of Two-Pair hands in the 51-card deck with the missing Ace of Spades can be counted by adding up cases where (1) no Ace appears, (2) a single Ace appears, and (3) a pair of Aces appears. We find

$$\begin{aligned} N_{2P}^{51} &= \underbrace{\binom{12}{2}\binom{4}{2}\binom{4}{2}\binom{10}{1}\binom{4}{1}}_{\text{no Ace appears}} + \underbrace{\binom{12}{2}\binom{4}{2}\binom{4}{2}\binom{1}{1}\binom{3}{1}}_{\text{a single Ace appears}} + \underbrace{\binom{1}{1}\binom{3}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}}_{\text{a pair of Aces appear}}, \\ &= 111,672. \end{aligned} \tag{5}$$

If we compare this to the number of Two-Pair hands from a standard deck, N_{2P}^{52}, a quick calculation reveals that

$$N_{2P}^{51} = \frac{47}{52}N_{2P}^{52}. \tag{6}$$

That is, the reduction factor in the total number of Two-Pair hands when the Ace of Spades is missing from the deck is also 47/52.

The corresponding probability of getting a Two-Pair hand in the 51-card deck is

$$P_{2P}^{51} = \frac{N_{2P}^{51}}{\binom{51}{5}} = \frac{47}{52}\frac{N_{2P}^{52}}{\binom{51}{5}} = \frac{N_{2P}^{52}}{\binom{52}{5}} = P_{2P}^{52}. \tag{7}$$

So, curiously, the probability of getting a Two-Pair hand in the 51-card deck with the missing Ace of Spades is exactly the same as for a 52-card standard deck. That is, the reduction factor in the number of Two-Pair hands (47/52) exactly matches that in the total number of hands, leaving the probability unchanged. Further, nothing about these Two-Pair arguments is specific to the Ace of Spades; the result holds for any missing card!

3 What about Other Poker Hands?

It turns out that this seemingly magical result for a deck with a missing Ace of Spades is not limited to Two-Pair hands. Indeed, examining the traditional hierarchy of 5-card poker hands (see Tables 1 and 2), we find an analogous result holds for One-Pair hands, Three-of-a-Kind, Full Houses and Four-of-a-Kind. Further, as with the Two-Pair hands, this remains true irrespective of which card goes missing from the deck. These types of hands share a symmetry property making them what we shall refer to as *magic* sets of hands.

Next, we define what we mean by a magic set of hands and present a theorem that can help identify such a magic set. In the section that follows we present a second theorem that helps explain the curious probability results observed for missing cards.

Definition 3.1 (Magic Set of Hands)**.** Consider a finite set $\mathcal{D}$ of m elements and let $c \in \mathcal{D}$ (i.e. card c in deck $\mathcal{D}$). Let $\mathcal{H}$ denote a set of n-card hands with $n < m$. Let $\mathcal{H}(c)$ denote the subset of $\mathcal{H}$ which contains all of the hands that involve a particular card c. We call the set of hands $\mathcal{H}$ *magic* if all subsets $\mathcal{H}(c)$ have the same number of elements (cardinality).

For example, if the type of hand in question is a Two-Pair hand from a standard, 52-card deck – using the notation $\mathcal{H}_{2P}^{52}$ for the set of all such hands – then, for example,

$$\begin{aligned} \mathcal{H}_{2P}^{52}(A\spadesuit) &= \Big\{\ldots,[A\spadesuit, A\clubsuit, 4\heartsuit, 4\spadesuit, 8\heartsuit],\ldots,[A\spadesuit, A\clubsuit, J\heartsuit, J\clubsuit, K\diamondsuit], \\ &\quad \ldots,[Q\spadesuit, Q\heartsuit, 10\heartsuit, 10\spadesuit, A\spadesuit],\ldots\Big\}, \qquad (8) \\ \mathcal{H}_{2P}^{52}(10\heartsuit) &= \Big\{\ldots,[10\heartsuit, 10\diamondsuit, 4\heartsuit, 4\spadesuit, 8\heartsuit],\ldots,[K\diamondsuit, K\spadesuit, J\heartsuit, J\clubsuit, 10\heartsuit], \\ &\quad \ldots,[Q\spadesuit, Q\heartsuit, 10\heartsuit, 10\spadesuit, A\spadesuit],\ldots\Big\}. \qquad (9) \end{aligned}$$

Observe that these subsets are not disjoint. Also note that any particular element of $\mathcal{H}_{2P}^{52}$ (i.e. any Two-Pair hand) will appear in five different subsets. Arguments similar to those in the previous sections can be used to show that all such subsets $\mathcal{H}_{2P}^{52}(c)$ have the same cardinality. Therefore, the set of Two-Pair hands $\mathcal{H}_{2P}^{52}$ is *magic*. In contrast, the set of all possible Straights dealt from a standard deck is not a magic set (e.g. there are more Straights that involve an 8 than there are that involve a 2). A non-poker example of a set of hands that is not magic is the set of all two-card hands from a standard 52-card deck in which both cards are red. Here, for example, $|\mathcal{H}_{2\,Red}^{52}(Q\heartsuit)| = 25$ but $|\mathcal{H}_{2\,Red}^{52}(Q\clubsuit)| = 0$, where we use $|\cdot|$ to denote cardinality.

For decks composed of cards for which both a rank and a suit is assigned, the magic property is more readily detected using a symmetry property described in the following theorem. The sets of poker hands just identified all fall into this category.

Theorem 3.1. Suppose that $\mathcal{D}$ is a set for which each element c has assigned to it a rank (e.g. 2 through A) and a suit (e.g. $\clubsuit$, $\diamondsuit$, $\heartsuit$, $\spadesuit$). Sets of hands, $\mathcal{H}$, are magic if they are invariant under the permutation of both the rank and suit.

Proof. One can show this using any cyclical permutation of the rank and suits, for example, for the case of a standard deck:

$$P_r = (A \to 2, 2 \to 3, \ldots, Q \to K, K \to A),$$

	Description	Example
Royal Flush	10–A all of the same suit	10♥, J♥, Q♥, K♥, A♥
Straight Flush	5 sequential denominations all of the same suit	7♠, 8♠, 9♠, 10♠, J♠
Four of a Kind	4 cards with the same denomination + any other card	3♣, 3♠, 3♥, 3♦, 10♣
Full House	3 of a kind + one pair	A♣, A♠, A♥, 8♥, 8♣
Flush	Five cards of the same suit	Q♦, J♦, 7♦, 6♦, 3♦
Straight	5 sequential denominations	3♠, 4♥, 5♥, 6♦, 7♣
Three of a Kind	3 cards with the same denomination	J♣, J♠, J♥, 4♥, K♠
Two Pair	Two sets of paired denominations	A♥, A♠, 5♣, 5♠, 9♣
One Pair	One paired denomination	Q♠, Q♣, 9♠, 8♣, 2♠
High Card	None of the above combinations	K♣, Q♠, 10♥, 9♥, 4♠

Table 1: Standard five-card Poker hands. Magic hands are shown in bold type.

	Number of Hands	Standard Probability	Number of Hands (missing Ace of Spades)	Probability Ratio (P^{51}/P^{52})
Royal Flush	4	0.000001539	3	0.8298
Straight Flush	36	0.00001385	35	1.0757
Four of a Kind	624	0.0002401	564	1
Full House	3,744	0.001441	3,384	1
Flush	5,108	0.001965	4,615	0.9996
Straight	10,200	0.003925	9,690	1.0511
Three of a Kind	54,912	0.02113	49,632	1
Two Pair	123,552	0.04754	111,672	1
One Pair	1,098,240	0.4226	992,640	1
High Card	1,302,540	0.5012	1,176,825	0.9996

Table 2: Poker hand hierarchy. The number of hands and probabilities for five-card poker hands dealt from a standard deck can be found in many introductory books on probability (e.g. see Siegrist [1]) as well as poker websites [2, 3]. These have been explored alongside fascinating variations by Cheung [4], Gadbois [5], Iiams [6], and Lanphier & Taalman[7]. Here we make a specific comparison of these well-known results to the corresponding results when the deck is missing the Ace of Spades. The High Card category (called 'Junk' in [5] and 'Trash' in [6]) counts all hands that do not qualify as any of the other types. For an additional bit of curiosity, it turns out the ratio of probabilities P^{51}/P^{52} for Flush is exactly that of High Card. Magic hands are shown in bold type.

$$P_s = (\clubsuit \to \diamondsuit, \diamondsuit \to \heartsuit, \heartsuit \to \spadesuit, \spadesuit \to \clubsuit).$$

Under P_r the subsets $\mathcal{H}(c)$ generated by the presence of each individual card transform as

$\mathcal{H}(2\clubsuit) \to \mathcal{H}(3\clubsuit), \ldots, \mathcal{H}(K\clubsuit) \to \mathcal{H}(A\clubsuit)$ and similarly for each of the other three suits. Also under P_s we have $\mathcal{H}(2\clubsuit) \to \mathcal{H}(2\diamondsuit)$, $\mathcal{H}(2\diamondsuit) \to \mathcal{H}(2\heartsuit)$, $\mathcal{H}(2\heartsuit) \to \mathcal{H}(2\spadesuit)$, and $\mathcal{H}(2\spadesuit) \to \mathcal{H}(2\clubsuit)$, and similarly for other ranks. Note that each of these transformations is a bijection: each hand in a set that is being transformed gets mapped to a unique target hand that is also in the set. Thus the set that is being transformed and the target set have the same number of elements. It then follows that $|\mathcal{H}(2\clubsuit)| = |\mathcal{H}(3\clubsuit)| = \ldots = |\mathcal{H}(K\clubsuit)| = |\mathcal{H}(A\clubsuit)|$ and similarly for each of the other three suits. Further, we also have $|\mathcal{H}(2\clubsuit)| = |\mathcal{H}(2\diamondsuit)| = |\mathcal{H}(2\heartsuit)| = |\mathcal{H}(2\spadesuit)|$, showing that the set magic. $\square$

It is easy to see that these transformations leave the sets of One Pair, Two Pair, Three of a Kind, Full House, and Four of a Kind hands unchanged. From this we can also see that if Straights were defined to extend the "wrap around" behavior of an Ace to cards ranking 2, 3, and 4, so that, for example, $[K\spadesuit, A\clubsuit, 2\heartsuit, 3\spadesuit, 4\heartsuit]$ was considered a four-high Straight, the set of Straights, Straight Flushes (inclusive of the Royal Flush), Flushes, and the High-Card hands would all become magic sets as well.

The basic idea of Theorem 3.1 also applies to a decks of cards, $\mathcal{D}$, for which each card of rank r and suit s appears duplicated d times. Here $m = r \times s \times d$. One example of such a deck is a 'Double-Deck' ($d = 2$) composed of two standard 52-card decks for a total of 104 cards. Another example is the Pinochle deck which has six different denominations (9, 10, J, Q, K, A) and four suits (♣, ♦, ♥, ♠), with each card appearing twice for a total of 48 cards. Assuming that the duplicate cards are indistinguishable the theorem can be applied as written. If there is something about the hand under consideration that distinguishes the duplicate cards (e.g. the color/design on the back of the card), then invariance under permutation with respect to decks should be added as a requirement along with the other two permutations. Note that in the definition of a Magic Set of Hands the removal of one card in this context means that only one of the duplicates, not all of them, is removed.

Note that the condition in Theorem 3.1 is sufficient, but not necessary, for a set to be magic. For example, if we define a set of two-card hands $\mathcal{H}$ by a specific partition a 52-card deck into 26 disjoint, two-card hands, we have a magic set, as the subset of hands involving any of the 52 cards contains exactly one hand. That is $|\mathcal{H}| = 26$, and $|\mathcal{H}(c)| = 1$ for all cards c in the deck. However, the resulting set of hands will not be invariant under the permutation of rank and suit. For example, if one of the 26 elements in the set is $[A\spadesuit, 10\clubsuit]$ there is no guarantee that the rank-permuted and suit-permuted results $[2\spadesuit, J\clubsuit]$ and $[A\clubsuit, 10\diamondsuit]$ are elements in the set.

4 How the magic works

The following theorem allows us to use the magic property to easily identify sets of hands where the probability is unchanged by the removal of any *single* card.

Theorem 4.1. The probability that a hand from a given set of hands $\mathcal{H}$ is dealt from a deck $\mathcal{D}_{-c}$ with any *single* missing card c is the same as the probability of that type of hand being dealt from a full deck $\mathcal{D}$ if and only if the set of hands $\mathcal{H}$ is magic.

Proof. First, assume the set $\mathcal{H}$ is magic, so that the number of hands containing a specific card $|\mathcal{H}(c)| \equiv h$ is independent of c. Let $H = |\mathcal{H}|$ be the number of hands in $\mathcal{H}$. The total number of n-card hands that can be dealt from an m-card deck is

$$B = \binom{m}{n}. \tag{10}$$

When one card is removed from the deck we lose b from this total, where

$$b = \binom{m}{n} - \binom{m-1}{n} = \binom{m-1}{n-1} = \frac{n}{m}B. \tag{11}$$

The probability of getting a hand from the set $\mathcal{H}$ with the full deck of m cards is then

$$P(\mathcal{H}) = \frac{H}{B}, \tag{12}$$

while the probability of getting the same type of hand from a deck with any card c removed from it is

$$P_{-c}(\mathcal{H}) = \frac{H-h}{B-b}, \tag{13}$$

where we have used the fact that $\mathcal{H}$ is magic. That is, the total number of hands available from the orignal set $\mathcal{H}$ is reduced by h (independent of which card is removed), and the total number of hands of any kind is reduced by b.

Now $\mathcal{H}$ is the union over all m subsets $\mathcal{H}(c)$ and for an n-card hand the sum $\Sigma_c|\mathcal{H}(c)|$ counts each hand n times. Therefore, for a magic set of hands, it follows that

$$H = \frac{\sum_c |\mathcal{H}(c)|}{n} = \frac{mh}{n}. \tag{14}$$

From this we have

$$P_{-c}(\mathcal{H}) = \frac{H-h}{B-b} = \frac{H - \frac{n}{m}H}{B - \frac{n}{m}B} = \frac{H}{B} = \frac{h}{b} = P(\mathcal{H}). \tag{15}$$

For the converse, assume a set of hands $\mathcal{H}$ where

$$P_{-c}(\mathcal{H}) = P(\mathcal{H}), \tag{16}$$

irrespective of which card c is removed from the deck. From this we have

$$P_{-c}(\mathcal{H}) = \frac{H-h(c)}{B-b} = P(\mathcal{H}) = \frac{H}{B}, \tag{17}$$

independent of c, which means that $|H(c)| \equiv h(c)$ must also be independent of c, so that $\mathcal{H}$ is a magic set.

$\square$

Notice that the "magic" comes from the fact that both H and B are reduced by the same factor $\frac{m-n}{m}$ (see equation (15)). Now we can use the absence/presence of the more readily identified magic property to identify sets of hands whose probablility will/will not change upon the removal of a card.

As we have seen, magic sets are not limited to five-card hands. In today's most popular poker variant, Texas Hold'em, many players are aware that the probability of being dealt a two-card "pocket pair" is $\frac{1}{17}$.[2] Pocket pairs are a two-card magic set, so the probability of recieving one is not changed by the removal of any single card from the deck. Similar statements can be made about the sets of seven-card hands that are ultimately formed in Texas-Hold'em.

Theorem 4.1 holds for generalized decks with $m > n$ cards: As long as the definition of a magic set of hands is met, the probability of being dealt a particular type of n-card hand from an m-card deck will match the corresponding probability when one card is missing from the deck. Examples of generalized decks include that used in the game 'Heartless Poker' [7] in which the entire suit of hearts is removed and the game is played with 39 cards, and the 'Fat Pack' deck, with 13 denominations and 8 different suits. In the more general deck considered by Lanphier & Taalman [7] with r ranks and s suits for a total $m = rs$ cards, a magic set of hands – such as a Two-Pair Hand – occurs with the same probability when a single card is removed and the game is played with $rs - 1$ cards. Similar statements apply to a deck of $m = rsd$ cards with r ranks, s suits, and d duplicates of each card (e.g. the Double-Deck or Pinochle Deck mentioned earlier) when one card goes missing and the modified deck has $rsd - 1$ cards. Indeed, there are many other magic sets of hands that could be identified, in these and other card games.

Note that Theorem 4.1 also allows us to see that after a card has been removed, what is left of the sets that were previously magic are no longer magic, so that a second card cannot be removed without changing the probability of the various categories of poker hands. The modified probabilities for such hands can, of course, be computed directly by similar counting argument to those used earlier. Also note that while the probability of receiving a pair is not affected by the removal of the Ace of Spades, the probability of being dealt of pair of aces certainly is.

Finally, it is possible that there may be a subset of cards that can be removed from the deck without changing the probability of getting a hand in a particular set of hands. To see this, we revisit (14), relaxing the magic assumption:

$$H = \frac{\sum_c |\mathcal{H}(c)|}{n} = \frac{m\bar{h}}{n}, \tag{18}$$

where $\bar{h} = \frac{1}{m}\sum_c |\mathcal{H}(c)|$. Equation (15) will still hold with h replaced by $\bar{h}$, provided $\bar{h} = \frac{nH}{m}$. This does not happen with any of the sets of conventional poker hands discussed above. An example where this does occur is the following set of two-card hands formed from a standard

[2] In Texas Hold'em each player is initially dealt two cards whose identity they keep to themselves. Players will eventually make their best five-card hand out of their two hidden cards and five "community" cards revealed sequentially during the course of play. A pocket pair is a pair that occurs in a player's hidden cards (e.g. 10♥, 10♠).

52-card deck: $\mathcal{H}$={red pairs, non-heart two-card straight-flushes}. That is,

$$\begin{aligned}\mathcal{H} \; = \; & \Big\{[A\heartsuit, A\diamondsuit], [K\heartsuit, K\diamondsuit], \ldots, [2\heartsuit, 2\diamondsuit], \\ & [A\diamondsuit, 2\diamondsuit], [2\diamondsuit, 3\diamondsuit], \ldots, [K\diamondsuit, A\diamondsuit], \\ & [A\clubsuit, 2\clubsuit], [2\clubsuit, 3\clubsuit], \ldots, [K\clubsuit, A\clubsuit], \\ & [A\spadesuit, 2\spadesuit], [2\spadesuit, 3\spadesuit], \ldots, [K\spadesuit, A\spadesuit]\Big\}, \end{aligned} \tag{19}$$

for which $H = |\mathcal{H}| = 4 \times 13 = 52$. Now, removing any of the 13 hearts removes one hand from $\mathcal{H}$, removing one of the 13 diamonds removes three hands from $\mathcal{H}$, and removing any of the 26 black cards removes two hands from $\mathcal{H}$. Thus, since $|\mathcal{H}(c)|$ is not independent of card c, this set is not magic. However, $\sum_c |\mathcal{H}(c)| = 13 \times 1 + 13 \times 3 + 26 \times 2 = 52 \times 2$. Therefore, the average number of hands removed is $\bar{h} = 2$ so that $\bar{h} = \frac{2H}{52}$. Thus, removing any of the black cards, c, will yield an unchanged probability

$$P_{-c}(\mathcal{H}) = P(\mathcal{H}). \tag{20}$$

5 Poker Hand Hierarchy and Extra Cards

It is worth noting that in Table 2 when the Ace of Spades goes missing, the ordering of the poker hands with respect to their probabilities does not change and so the standard hierarchy of hands is still consistent with the 51-card deck.[3]

Now, a number of previous authors have shown that it is possible to 'shuffle' this hierarchy. One poker variation where this occurs is Lanphier & Taalman's 'Heartless Poker' [7]. With only three suits in play, the Flush is a more likely hand than the Straight and so in the proper hierarchy for Heartless Poker – based on the frequency of appearance of hands in the modified deck as opposed to the traditional ordering – these hands would be reversed (i.e. a Straight beats a Flush in Heartless Poker). Lanphier & Taalman [7] also point out that in the 'Fat Pack' deck, with 13 denominations and 8 different suits, the probability of getting a Flush is smaller than that for a Full House, reversing these hands in the hierarchy. In fact, in their generalization to s suits and r ranks any ordering of Straight, Flush, and Full House hands is possible. However, in the absence of a way to play poker with non-integer numbers of suits and ranks Lanphier & Talmaan showed that it is not possible for the probabilities of the Straight, Flush, and Full House to all be equal.[4] Interestingly, as we outline below, this exchange of positions in the hierarchy can also be pushed to the level of unresolvable paradox [5] with respect to the standard hand types when Wild Cards are shuffled into the deck.

[3] By 'hierarchy' we mean the ordering of types of hands from least likely (the best hand) to most likely (the worst hand), for example, as shown in Table 1 or Table 2. Inherent in this definition is the players ability to choose their hand as the best one possible in the established hierarchy. For example, the hand – A♣, A♦, A♥, 8♥, 8♣ – could be seen as qualifying as a One-Pair hand (e.g. 8s), Two-Pair (Aces and 8s), or Three of a Kind (Aces); in the standard poker-hand hierarchy this hand would be considered a Full House.

[4] At the risk of living up to the title of this contribution, we might argue that it is completely rational to play poker with $s = 3\,\frac{12}{13}$ suits and $r = 12\,\frac{3}{4}$ ranks (i.e. one missing card)!

5.1 Wild Cards: Not Magical but Paradoxical

Suppose a Joker gets mixed into a standard deck giving a 53-card deck. Under one set of house rules, the Joker is considered a 14th denomination that has a suit of its own – so the Joker is *not* Wild.[5] Here the probability of a Two-Pair hand is

$$P_{2P}^{52+JNW} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\left[\binom{11}{1}\binom{4}{1}+1\right]}{\binom{53}{5}}. \tag{21}$$

This follows the standard count for Two-Pair but now includes 44 + 1 cards that can be the non-paired card. Relative to the standard case this has the ratio

$$\frac{P_{2P}^{52+JNW}}{P_{2P}^{52}} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\times 45}{\binom{53}{5}} \times \frac{\binom{52}{5}}{\binom{13}{2}\binom{4}{2}\binom{4}{2}\times 44} = \frac{45}{44}\times\frac{48}{53} \approx 0.9262. \tag{22}$$

So, these house rules for the Joker make the Two-Pair hand less likely.

If you play in a house where Jokers are Wild, there are a lot more ways to make a pair. As we have already seen, there are 123,552 Two-Pair hands where the Joker does not appear. However, when a Joker does appear, hands with one other pair are promoted to Three of a Kind (e.g. [Q♠, Q♦, 4♣, 3♥, Joker]) and hands with two other pair sets are promoted to Full House (e.g. [10♣, 10♥, 8♠, 8♥, Joker]). In fact, *there are no Two-Pair hands that involve a Wild Joker.* Then

$$\frac{P_{2P}^{52+JW}}{P_{2P}^{52}} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\times 44}{\binom{53}{5}} \times \frac{\binom{52}{5}}{\binom{13}{2}\binom{4}{2}\binom{4}{2}\times 44} = \frac{48}{53} \approx 0.9057. \tag{23}$$

So, relatively speaking, it is *less likely* to get a Two-Pair hand when the Joker is Wild.

The change in probabilities with/without a Joker shows that, according to Theorem 4.1, the Two-Pair hand in a 53-card deck is not a magic hand; if it were magic and the Joker was removed to make the standard 52-card deck, the probabilities would necessarily stay the same.

The real problem when one attempts to play poker with Wild Jokers is the Gadbois paradox [5]. The inclusion of a Joker (or two) allows for flexibility by the player to choose the Joker to optimize their hand in the established hierarchy. This is exactly the situation that Gadbois [5] analyzed for five-card poker hands formed from a 54-card deck with two Wild Jokers. There, Gadbois found, paradoxically, that if the Two-Pair hand is placed lower in the hierarchy than the Three of a Kind hand (i.e. the usual poker hand hierarchy) a player with a hand such as [Q♠, Q♣, 9♠, 8♣, Joker] would always choose the Joker to make Three-of-a-Kind instead of Two-Pair to optimize their hand in the established hierarchy. This actually makes the Two-Pair hand occur in play less frequently which would suggest the need to change the hierarchy accordingly. However, if one attempts to correct this situation by placing the Two-Pair hand *higher* in the hierarchy than Three of a Kind, the player with the hand noted above would instead choose the Joker to make a Two-Pair hand; this has

[5] A Wild Joker means to choose it as any denomination and suit to make your best hand.

the effect of making the Three-of-a-Kind hand occur in play less frequently. The paradoxical situation is also present when playing with only one Wild Joker.[6]

5.2 Don't Blame the Dog: The Ace of Spades Found!

Let us consider a final scenario regarding the whereabouts of the Ace of Spades. Suppose that the missing Ace of Spades got shuffled into a matching deck of cards and now you are dealing five-card poker hands from a 53-card deck that has an extra Ace of Spades. Care to venture a guess at the probability of dealing a Two-Pair hand from this deck? It turns out that this probability *does not* match that of the standard deck – no magic. In particular, using an approach analogous to that outlined in equation (5)[7] we find that the probability of a Two-Pair hand dealt from a 53-card deck with an extra Ace of Spades relative to the standard case is

$$\frac{P_{2P}^{52+A}}{P_{2P}^{52}} = \left(1 + \frac{11}{13 \times 53}\right) > 1. \tag{24}$$

Therefore, the chances of getting a Two-Pair hand are slightly better in this 53-card deck!

6 Closing Thoughts

There are a number of poker variations one can find in the casinos (e.g. Five Card Draw, Texas Hold'em). Add to these other variations played at home games using hands of different sizes [4], Wild Jokers [5], 'Heartless' decks [7], and variations thereof [7, 6]. All of these come with their adjusted set of probabilities for the standard hands (e.g. One Pair, Two Pair, Three of a Kind), the introduction of new types of hands (e.g. Five of a Kind with a Wild Joker), and the potential for a rearranged hierarchy of hands [7] including paradoxical situations where a proper hierarchy of these hands does not exist [5].

In the very special kind of variation introduced at the beginning of this article – a single missing card from a standard deck – many of the poker hand probabilities remain, remarkably, unchanged. We have connected these results to the notion of magic sets and more generalized decks including a theorem on missing cards.

At home and at the casino, amateur and professional poker players may actually encounter a 51-card deck in the event that the dealer accidentally exposes a card during the deal.[8] The results we outlined here show that in this case the mathematically-savvy player need not contemplate a whole new set of probabilities – the probabilities for One-Pair, Two-Pair, Three of a Kind, Full-House, and Four of a Kind do not change at all (hands related to Straights are the minor exception and depend on the missing card's denomination). Texas

[6] In the standard hierarchy there are 137,280 Three of a Kind hands (the 54,912 'normal' ones that do not involve the Joker and 82,368 promoted upward from One Pair by the Joker) compared to 123,552 Two-Pair hands. In an alternate hierarchy (Three of a Kind ranked below Two-Pair in the hierarchy) then there are 205,920 Two-Pair hands (123,552 'normal' ones and 82,368 promoted upward from One Pair) compared to 54,912 Three of a Kind Hands.

[7] In equation (5) replace $\binom{3}{1}$ in the second term with $\binom{5}{1}$ and replace $\binom{3}{2}$ in the third term with $\binom{5}{2}$.

[8] Assuming this happens only once, this card is revealed to all players and removed from play (i.e. burned).

Hold'em players, for example, can continue play with the bonus knowledge of the whereabouts and identity of the mis-dealt card and the knowledge that the probability of a 'pocket pair' has remained exactly the same.

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