

**AN ELEMENTARY INTRODUCTION
TO WHAT I DO FOR MY RESEARCH.**

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My research falls into the category of foundations research. While there are general applications looming in the background, in my day to day work a typical "application" of a theorem would be to answer other mathematical questions. Specifically, I work in an area called *function theoretic operator theory*. The basic goal is to use methods and concepts from Complex Analysis to answer questions from Functional Analysis. Functional Analysis is a fancy (and shorter) name for Linear Algebra on infinite dimensional vector spaces with a little topology thrown in.

I will now try to motivate and present some of the highlights of an early success story of this theory and then tell you that I work on related questions .

One of the main theorems in linear algebra on finite dimensional vector spaces is the Jordan decomposition theorem for linear transformations

$$T : \mathbb{C}^n \rightarrow \mathbb{C}^n .$$

Here \mathbb{C} denotes the complex numbers. Rough spoken it is a decomposition of the space \mathbb{C}^n into subspaces \mathcal{M}_i , $i = 1, \dots, p$ such that each \mathcal{M}_i corresponds to an eigenvalue λ_i of T . The subspaces \mathcal{M}_i correspond to the Jordan blocks of the Jordan canonical form of T . Recall that $\lambda \in \mathbb{C}$ is an eigenvalue for T if there is a nonzero vector $x \in \mathbb{C}^n$ such that $Tx = \lambda x$, and that the Jordan canonical form for T looks something like

$$\left[\begin{array}{cccccccc} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \lambda_1 & 1 & & & & \\ & & & \lambda_1 & 1 & & & \\ & & & & \lambda_1 & 1 & & \\ & & & & & \lambda_2 & 1 & \\ & & & & & & \lambda_2 & \\ & & & & & & & \lambda_3 \\ & & & & & & & \cdot \\ & & & & & & & \cdot \\ & & & & & & & \cdot \end{array} \right]$$

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In this example \mathcal{M}_1 and \mathcal{M}_2 correspond to the eigenvalue λ_1 and are 3- and 2-dimensional subspaces respectively. \mathcal{M}_3 corresponds to λ_2 and is 2-dimensional, etc.

The subspaces \mathcal{M}_i in this decomposition are **invariant subspaces** for T , i.e. $Tx \in \mathcal{M}_i$ for every $x \in \mathcal{M}_i$.

Now let $\mathcal{H} = l^2 = \{(a_0, a_1, a_2, \dots) : a_i \in \mathbb{C}, \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$, the space of square summable sequences, and let

$$T : \mathcal{H} \rightarrow \mathcal{H}$$

be linear and continuous. T is called a linear operator.

Open Question. (*The invariant subspace problem*)

If $T : \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator, does T have nontrivial invariant subspaces? That is, is there a closed subspace $\mathcal{M} \subseteq \mathcal{H}$, $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$, and such that $Tx \in \mathcal{M}$ for every $x \in \mathcal{M}$.

Of course, if T has an eigenvalue, then the corresponding eigenspace is an invariant subspace. Since every linear transformation $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ has eigenvalues it follows that such T has nontrivial invariant subspaces whenever $n \geq 2$. However, on \mathcal{H} there are many linear operators that do not have any eigenvalues. An example is given by the **unilateral shift operator** $S : \mathcal{H} \rightarrow \mathcal{H}$. It is defined by $S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots)$.

Exercise. *Show that the unilateral shift S is a linear operator that has no eigenvalues.*

It is easy to see that the unilateral shift S has nontrivial invariant subspaces. For example, we can let \mathcal{M}_n be the subspace of square summable sequences such that the first n components are 0. Then it is clear that for each $x \in \mathcal{M}_n$ we have $Sx \in \mathcal{M}_{n+1} \subseteq \mathcal{M}_n$. Does S have any other invariant subspaces? Plenty! Many more will become apparent once we reformulate the question by use of the Fourier transform. We define the Hardy space H^2 to be a space of complex-valued analytic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C}$. More precisely, we set

$$H^2 = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for } z \in \mathbb{D}, \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\},$$

and define the Fourier transform from \mathcal{H} to H^2 by

$$(a_0, a_1, a_2, \dots) \rightarrow f, \text{ where } f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Note that H^2 contains all the polynomials. On the space H^2 we can define the operator of multiplication by z , i.e.

$$(M_z f)(z) = z f(z) \text{ for } f \in H^2.$$

Then $M_z(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=1}^{\infty} a_{n-1} z^n$, and it is easy to see that the Fourier transform provides a unitary equivalence between S and M_z .

Now let $\{c_1, c_2, \dots, c_n\}$ be a finite subset of \mathbb{D} . Then we can set

$$\mathcal{M} = \{f \in H^2 : f(c_i) = 0, i = 1, \dots, n\}.$$

It is easy to see that \mathcal{M} is a closed subspace of H^2 and it is clear that \mathcal{M} is invariant for M_z . It is also easy to see that \mathcal{M} is a nontrivial subspace of H^2 . In fact, since $1 \notin \mathcal{M}$ we have $\mathcal{M} \neq H^2$, and we also note that $p(z) = \prod_{i=1}^n (c_i - z)$ is a polynomial in \mathcal{M} , $p \neq 0$, hence $\mathcal{M} \neq \{0\}$. Thus, we have already found new invariant subspaces of S , and one can go one step further. Let $\{c_1, c_2, \dots\}$ be an infinite subset of \mathbb{D} , and set

$$\mathcal{M} = \{f \in H^2 : f(c_i) = 0, i = 1, \dots\}.$$

As before, it is easily seen that $\mathcal{M} \subseteq H^2$ is an invariant subspace of M_z , and that $\mathcal{M} \neq H^2$. But it is not clear that $\mathcal{M} \neq \{0\}$! The first temptation would be to try to set $f(z) = \prod_{i=1}^{\infty} (c_i - z)$. But such an infinite product never defines a function in H^2 , there is a problem with the convergence here. On the other hand, if the sequence $\{c_1, c_2, \dots\}$ is dense in \mathbb{D} , then every continuous function f that is zero at each point $z = c_i$ must be zero at every $z \in \mathbb{D}$. It is a good advanced calculus exercise to verify this. The following precise description of when $\mathcal{M} \neq \{0\}$ is due to F. Riesz and goes back to the 1920s.

Theorem. $\mathcal{M} \neq \{0\}$ if and only if $\sum_{n=1}^{\infty} 1 - |c_n| < \infty$. Furthermore, if $\sum_{n=1}^{\infty} 1 - |c_n| < \infty$, then $B(z) = \prod_{i=1}^{\infty} d_i \frac{c_i - z}{1 - \bar{c}_i z}$ converges and defines a function in H^2 . Here $d_i = 1$, if $c_i = 0$ and $d_i = \frac{c_i}{|c_i|}$ otherwise. The function B is called a Blaschke product.

It turns out that there are even more invariant subspaces for M_z . One can show that functions $f \in H^2$ have "boundary values" $f(w)$ that are defined for a.e. $w \in \partial\mathbb{D}$ (with respect to Lebesgue measure on $\partial\mathbb{D}$). One then needs an interpretation what one means by a zero of a function f in $\partial\mathbb{D}$. The function theoretic description of all invariant subspaces of S was given by A. Beurling in 1949.

Theorem. There is a 1-1 correspondence between the nonzero invariant subspaces \mathcal{M} of the unilateral shift and the "inner" functions φ . An inner function is a function that can be written as $\varphi = BS$, where B is a Blaschke product as in the previous theorem and S is a singular inner function of the form $S(z) = e^{-\int_{\partial\mathbb{D}} \frac{w+z}{w-z} d\sigma(w)}$ for some measure σ on $\partial\mathbb{D}$ that is singular with respect to Lebesgue measure on $\partial\mathbb{D}$.

The singular inner functions S can be seen to exponentially decay near the (small) set in $\partial\mathbb{D}$ where the measure σ lives.

Much of my research here at UT I do with my colleague Carl Sundberg. Motivated by a set-up similar to what was outlined above, we investigate similar questions for other linear operators. Often linear operators can be seen to be unitarily equivalent to M_z on spaces of analytic functions other than H^2 , or they may be unitarily equivalent to M_φ ($M_\varphi f = \varphi f$ for some function φ). For example, in the paper A. Aleman, S. Richter, C. Sundberg, Beurling's Theorem for the Bergman space, *Acta Math.* 177 (1996), 275-310, MR 98a:46034 we prove an analogue of the above theorem of Beurling for the Bergman space L_a^2 . L_a^2 consists of all analytic functions f in \mathbb{D} that are square area integrable, i.e. $\int_{\mathbb{D}} |f(z)|^2 dx dy < \infty$, $z = x+iy$. An overview about some results about the Bergman space (through about 2001) can be obtained by reading a book review I wrote: S. Richter (reviewer), *Theory of Bergman spaces*, by Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu, *Bull. Amer. Math. Soc.* 39 (2002), 121-127.

Recently, we also got interested in trying to understand certain properties of tuples of linear operators (T_1, T_2, \dots, T_n) . In that case one can transform questions about the operator tuple into questions about analytic functions of several complex variables, see e.g. D. Greene, S. Richter, C. Sundberg, The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels, *J. Funct. Anal.* 194 (2002), 311-331, and J. Gleason, S. Richter, C. Sundberg, On the index of invariant subspaces in spaces of analytic functions of several complex variables, preprint.