The index of invariant subspaces in spaces of analytic functions.

1. The unit disc: A quick overview.

2. The unit ball in $\mathbb{C}^d$

(joint work by:

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$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$\mathcal{H} \subseteq$ Hol$(\mathbb{D})$

$M(\mathcal{H}) = \{\varphi \in$ Hol$(\mathbb{D}) : \varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}\}$ (the multiplier algebra)

Assumptions:

1. $z \in M(\mathcal{H})$, so $M_z : \mathcal{H} \to \mathcal{H}$.

2. $\sigma(M_z) = \overline{\mathbb{D}}$, $\sigma_e(M_z) = \partial\mathbb{D}$

3. $\dim \mathcal{H}/z\mathcal{H} = 1$

Then by **2.** ran$(M_z - \lambda)$ is closed $\forall \lambda \in \mathbb{D}$, hence $||(M_z - \lambda)f|| \geq c_\lambda ||f||$ for all $f \in \mathcal{H}, \lambda \in \mathbb{D}$.

If $1 \in \mathcal{B}$, then **2.** and **3.** hold, if and only if

$$f \in \mathcal{H},\ \lambda \in \mathbb{D},\ f(\lambda) = 0 \implies \frac{f}{z - \lambda} \in \mathcal{H}.$$

$\mathcal{M} \in$ Lat$(M_z, \mathcal{H})$

$$\text{ind } \mathcal{M} = \dim \mathcal{M}/z\mathcal{M}$$

Why consider ind $\mathcal{M}$?

it is a unitary invariant for $M_z|\mathcal{M}$

$\sigma_e(M_z|\mathcal{M}) = \partial\mathbb{D}$ if and only if ind$\mathcal{M} < \infty$

$\sigma_e(M_z|\mathcal{M}) = \overline{\mathbb{D}}$ if and only if ind$\mathcal{M} = \infty$.

It is related to the Fredholm index of $M_z|\mathcal{M}$:

$$\text{ind}\mathcal{M} = -\text{ind}M_z|\mathcal{M} = -\text{ind}(M_z - \lambda)|\mathcal{M}$$

$$= \dim \mathcal{M}/(z-\lambda)\mathcal{M} \quad \text{for all } \lambda \in \mathbb{D}$$

Easy fact: If $\mathcal{M} \neq (0)$ and if

$$\mathcal{M} = [f]$$

or

$$\mathcal{M} = \text{ zero set based,}$$

then ind$\mathcal{M} = 1$.

Thm 1. *(Beurling's Thm)*

If $\mathcal{H} = H^2(\mathbb{D})$, then ind$\mathcal{M} = 1$ *for all $\mathcal{M} \neq (0)$ and if $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $||\varphi|| = 1$, then*

$$\mathcal{M} = \varphi H^2(\mathbb{D}).$$

Thm 2. *(a) (ABFP) If $\mathcal{H}$ is a Hilbert space as above and such that*

$$||zf|| \leq ||f|| \text{ and } ||z^n f|| \to 0 \text{ for all } f \in \mathcal{H},$$

then for each $n = 1, 2, ..\infty$ there is an $\mathcal{M}$ with ind$\mathcal{M} = n$.

(b) (Abakumov-Borichev) $p > 2$
$\mathcal{B} = l^p = \{f \in$ Hol$(\mathbb{D}) : \sum |\widehat{f}(n)|^p < \infty\}$, then for each $n = 1, 2, ..\infty$ there is an $\mathcal{M}$ with ind$\mathcal{M} = n$.

Bergman space

$$L_a^2 = \{f \in \text{ Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty\}$$

Thm 3. *(ARS) If $\mathcal{M} \in$ Lat (M_z, L_a^2), if*

$$\mathcal{D} = \mathcal{M} \ominus z\mathcal{M},$$

then there is a space $\mathcal{K}$ of $\mathcal{D}$-valued analytic functions such that

$$H^2 \otimes \mathcal{D} = \; H^2_{\mathcal{D}} \; \subseteq \; \mathcal{K} \; \subseteq \; L^2_{a\mathcal{D}} \; = L_a^2 \otimes \mathcal{D}$$

and the $\mathcal{D}$-valued polynomials are dense in $\mathcal{K}$ and $M_z|\mathcal{M}$ u.e. $M_z|\mathcal{K}$.

(HJS) (McCullough-R) In fact, $\mathcal{K}$ has an operator-valued reproducing kernel of the type

$$k_\lambda(z) = \frac{I_{\mathcal{D}} - z\overline{\lambda}V(z)V(\lambda)^*}{(1 - z\overline{\lambda})^2}$$

for some contractive analytic $V(z) : \mathcal{D} \to \mathcal{D}$.

Thm 4. *(a) If* $M(\mathcal{H}) \subseteq \mathcal{H}$ *dense, then*

whenever $\mathcal{M} \cap M(\mathcal{H}) \neq (0)$, *then* $\text{ind}\mathcal{M} = 1$.

This applies to many spaces $\mathcal{H} \subseteq H^2(\mathbb{D})$ *and all* $\mathcal{M} \neq (0)$ *(Dirichlet-type spaces).*

(b) (ARS) Let $\mu > 0$, *supp* $\mu \subseteq \overline{\mathbb{D}}$, $1 \leq t < \infty$, $\mathcal{H} = P^t(\mu) =$ *closure of polys in* $L^t(\mu)$.

If $P^t(\mu)$ *is irreducible and abpe* $P^t(\mu) = \mathbb{D}$, *then*

TFAE:

(1) every $\mathcal{M} \neq (0)$ *has* $\text{ind}\mathcal{M} = 1$,

(2) $\mu|\partial\mathbb{D} \neq 0$, *i.e.* $\exists f \; ||z^n f|| \not\to 0$,

(3) $\exists E \subseteq \partial\mathbb{D}$, $|E| > 0$: *all* $f \in \mathcal{H}$ *have nontangential limits a.e. on* E.

The <u>majorization function</u>

$$k_{\mathcal{M}}(\lambda) = \frac{||P_{\mathcal{M}} k_\lambda||}{||k_\lambda||},$$

where k_λ is the reproducing kernel for $\mathcal{H}$, $< f, k_\lambda >= f(\lambda)$, and $P_{\mathcal{M}} =$ projection onto $\mathcal{M}$.

For $\mathcal{M} = \varphi H^2 \in$ Lat(M_z, H^2) one checks that $k_{\mathcal{M}}(\lambda) = |\varphi(\lambda)|$.

Thm 5. *(ARS, 2002) If $\mathcal{M} \in$ Lat(M_z, L^2_a) with* ind$\mathcal{M} = 1$, *then*

<u>TFAE:</u>

(1) every $\mathcal{N}$ with $\mathcal{M} \subseteq \mathcal{N}$ has ind$\mathcal{M} = 1$,

(2) $\exists E \subseteq \partial\mathbb{D}$, $|E| > 0$*:*

$$nt\text{-}\underline{\lim}_{\lambda \to z} k_{\mathcal{M}}(\lambda) > 0 \quad \forall z \in E,$$

(3) $\exists E \subseteq \partial\mathbb{D}$, $|E| > 0$*:*

$$nt\text{-}\lim_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 1 \ \forall z \in E.$$

(1) $\Leftrightarrow$ (2) holds for many other $\mathcal{H}$.

$d > 1, \;\; \mathcal{B}_d = \{z \in \mathbb{C}^d : |z| < 1\}$

$\mathcal{H} \subseteq \text{ Hol}(\mathcal{B}_d)$

Examples: $< z, w >= \sum_{i=1}^{d} z_i \overline{w}_i$.

(1) $\mathcal{H} = H^2(\partial\mathcal{B}_d), \;\; M(H^2(\partial\mathcal{B}_d)) = H^\infty(\mathcal{B}_d)$

reproducing kernel $k_w(z) = \frac{1}{(1-<z,w>)^d}$.

(2) $\mathcal{H} = L_a^2(\mathcal{B}_d), \;\; M(L_a^2(\mathcal{B}_d)) = H^\infty(\mathcal{B}_d)$

reproducing kernel $k_w(z) = \frac{1}{(1-<z,w>)^{d+1}}$.

(3) $\mathcal{H} = H_d^2, \;\; M(H_d^2) \subsetneq H^\infty(\mathcal{B}_d)$

reproducing kernel $k_w(z) = \frac{1}{1-<z,w>}$.

$M_z = (M_{z_1}, ..., M_{z_d})$

If $\lambda = (\lambda_1, ..., \lambda_d) \in \mathcal{B}_d$, then

$$M_z - \lambda = (M_{z_1} - \lambda_1 I, ..., M_{z_d} - \lambda_d I)$$

$\mathcal{M} \in$ LatM_z i.e. $z_i\mathcal{M} \subseteq \mathcal{M}$ for $i = 1, ..., d$

Facts:

(1) If $\mathcal{M} \in$ Lat$(M_z, H^2(\partial\mathcal{B}_d))$, $\mathcal{M} \neq (0)$, then

$$\text{nt-}\lim_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 1 \quad \text{a.e. } z \in \partial\mathcal{B}_d.$$

(follows easily from the existence of nontangential limits of functions in $H^2(\partial\mathcal{B}_d)$ and the form of the norm $||f||^2 = \int_{\partial B_d} |f|^2 d\sigma$.)

(2) If $\mathcal{M} \in$ Lat$(M_z, L^2_a(\mathcal{B}_d))$, then ?

(Expect all hell to break loose in general.)

(3) If $\mathcal{M} \in$ Lat(M_z, H^2_d), $\mathcal{M} \neq (0)$, then

(a) (Arveson, McCullough-Trent)
$\exists \varphi_n \in \mathcal{M} \cap M(H^2_d)$ such that

$$P_{\mathcal{M}} = \sum_n M_{\varphi_n} M^*_{\varphi_n},$$

and in fact

$$\mathcal{M} = \Phi(H^2_d \otimes \mathcal{E}),$$

$\Phi = (M_{\varphi_1}, M_{\varphi_2}, ...) : H^2_d \otimes \mathcal{E} \to H^2_d.$

(b) (Greene-R-S) If φ_n are as in (a) , then

$$\sum_n |\varphi_n(z)|^2 = 1 \quad \text{a.e. } z \in \partial \mathcal{B}_d.$$

Note:

$$k^2_{\mathcal{M}}(\lambda) = \frac{<P_{\mathcal{M}} k_\lambda, k_\lambda>}{||k_\lambda||^2} = \frac{\sum_n ||\overline{\varphi_n(\lambda)} k_\lambda||^2}{||k_\lambda||^2} = \sum_n |\varphi_n(\lambda)|^2$$

Want to consider

$$\mathcal{M}/(z_1\mathcal{M} + .. + z_d\mathcal{M})$$

or

$$\mathcal{M}/((z_1 - \lambda_1)\mathcal{M} + .. + (z_d - \lambda_d)\mathcal{M})$$

<u>Problem 1:</u>

If $z_1\mathcal{H} + .. + z_d\mathcal{H}$ is closed in $\mathcal{H}$, is $z_1\mathcal{M} + .. + z_d\mathcal{M}$ closed in $\mathcal{M}$ for all $\mathcal{M}$?

Open even for all spaces as above.

Problem 2:

Is

$$\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+..+(z_d-\lambda_d)\mathcal{M})$$

independent of λ?

NO!

$\mathcal{H}$ as above, $\mathcal{M}=\{f\in\mathcal{H}: f(0)=0\}$

Then $z_1,...,z_d \in \mathcal{M}\ominus(z_1\mathcal{M}+..+z_d\mathcal{M})$,

In fact,

$$\dim \mathcal{M}/(z_1\mathcal{M}+..+z_d\mathcal{M})=d,$$

but

$$\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+..+(z_d-\lambda_d)\mathcal{M})=1$$

for all $\lambda\neq 0$ (easy, but also follows from main theorem)

Note:

$$0 \in Z(\mathcal{M}) = \{\lambda \in \mathcal{B}_d : f(\lambda) = 0 \text{ for all } f \in \mathcal{M}\}.$$

In $d = 1$ the index on $Z(\mathcal{M})$ is finessed with the Fredholm index.

In $d > 1$ the definition of fredholmness and Fredholm index involves the Koszul complex.

The Koszul complex for $(M_z - \lambda)|\mathcal{M}$:

$$0 \to \Lambda^0(\mathcal{M}) \xrightarrow{\partial_{0,\lambda}} \Lambda^1(\mathcal{M}) \xrightarrow{\partial_{1,\lambda}} \cdots \xrightarrow{\partial_{d-1,\lambda}} \Lambda^d(\mathcal{M}) \to 0,$$

where $\Lambda^p(\mathcal{M})$ is a Hilbert space and

$\partial_{p,\lambda}$ is a bounded linear operator dependent on $(M_z - \lambda)|\mathcal{M}$.

$\lambda \notin \sigma_e(M_z|\mathcal{M})$ or $(M_z - \lambda)|\mathcal{M}$ is Fredholm, if ran $\partial_{p,\lambda}$ is closed for all p and

$$\dim \frac{\ker \partial_{p,\lambda}}{\text{ran}\partial_{p-1,\lambda}} < \infty \text{ for all } p$$

$$\text{ind}(M_z - \lambda)|\mathcal{M} = \sum_{p=1}^{d} (-1)^p \dim \frac{\ker \partial_{p,\lambda}}{\text{ran}\partial_{p-1,\lambda}}$$

If $p = d$ then $\frac{\ker \partial_{d,\lambda}}{\text{ran}\partial_{d-1,\lambda}} = \mathcal{M}/((z_1 - \lambda_1)\mathcal{M} + .. + (z_d - \lambda_d)\mathcal{M})$.

Continuity property of Fredholm index $\Longrightarrow$

$$\text{ind}(M_z - \lambda)|\mathcal{M} = constant$$

for $\lambda \in$ connected component of $\mathbb{C}^d \setminus \sigma_e(M_z|\mathcal{M})$

Thm 6. *(Gleason-R-S) Let $\mathcal{H}$ be the Hardy or Bergman space of the ball or polydisc of $\mathbb{C}^d$, or let $\mathcal{H} = H^2_d$.*

$\mathcal{M} \in$ LatM_z and if $\exists\ \varphi \in \mathcal{M} \cap M(\mathcal{H})$, $\varphi \neq 0$. Then

$$\partial \mathcal{B}_d \subseteq \sigma_e(M_z|\mathcal{M}) \subseteq \partial \mathcal{B}_d \cup Z(\varphi)$$

and for all $\lambda \in \mathcal{B}_d \setminus \sigma_e(M_z|\mathcal{M})$

$$\text{ind}(M_z - \lambda)|\mathcal{M} = (-1)^d.$$

In fact, for all $\lambda \in \mathcal{B}_d \setminus Z(\varphi)$ we have

$$\dim \mathcal{M}/((z_1 - \lambda_1)\mathcal{M} + .. + (z_d - \lambda_d)\mathcal{M}) = 1$$

Cor 7. *In $\mathcal{H} = H^2_d$ the Theorem applies to all $\mathcal{M} \neq (0)$ and with $Z(\mathcal{M})$ instead of $Z(\varphi)$.*

The Corollary follows from the theorem and the Arveson/McCullough-Trent theorem.

The Exclusion of $Z(\mathcal{M})$ is necessary!

$\exists \mathcal{M} \subseteq H^2_d$ such that $Z(\mathcal{M}) \neq \emptyset$ and $\sigma_e(M_z|\mathcal{M}) = \partial\mathcal{B}_d \cup Z(\mathcal{M})$.

Reason: $d = 2$

$$\frac{1}{1-(x+y)} = \frac{1}{(1-x)(1-\frac{y}{1-x})} = \frac{1}{1-x} + \frac{y}{(1-x)^2} + \frac{y^2}{(1-x)^3} +$$

$$k_w(z) = \frac{1}{1-(\overline{w_1}z_1+\overline{w_2}z_2)} = \frac{1}{1-\overline{w_1}z_1} + \frac{\overline{w_2}z_2}{(1-\overline{w_1}z_1)^2} + \ldots$$

Hence

$$H^2_2 = H^2(\mathbb{D}) \oplus z_2 L^2_a(\mathbb{D}) \oplus z_2^2 \mathcal{K} \quad \text{isometrically}$$

Take $\mathcal{M} = (0) \oplus z_2\mathcal{N} \oplus z_2^2\mathcal{K}$

for $\mathcal{N} \in$ Lat(M_z, L^2_a), $ind\mathcal{N} = \infty$, then
$\dim \mathcal{M}/((z_1-\lambda)\mathcal{M} + z_2\mathcal{M}) = \infty$ for all $|\lambda| < 1$.

Similarly for $H^2(\partial\mathcal{B}_d)$. Actually, even worse:
$\exists\mathcal{M}$ such that $\sigma_e(M_z|\mathcal{M}) \cap (\mathcal{B}_d \setminus Z(\mathcal{M})) \neq \emptyset$.

Tool needed for the proof of Theorem 6:

One can solve Gleason's problem for $M(\mathcal{H})$, i.e. $\varphi \in M(\mathcal{H}) \implies \exists \varphi_1, ..., \varphi_d \in M(\mathcal{H})$ such that

$$\varphi(z) - \varphi(\lambda) = \sum_{i=1}^{d} (z_i - \lambda_i)\varphi_i(z).$$

Suppose $\varphi \in \mathcal{M} \cap M(\mathcal{H})$, let $\lambda \in \mathcal{B}_d$ with $\varphi(\lambda) = 1$.

Let $f \in \mathcal{M}$, then

$$f = f(\lambda)\varphi + \varphi(f - f(\lambda)) - (\varphi - 1)f.$$

If one can solve Gleason's problem for both the space $\mathcal{H}$ and $M(\mathcal{H})$, then $\exists\ f_1, \ldots, f_d \in \mathcal{H}$ and $\varphi_1, \ldots, \varphi_d \in M(\mathcal{H})$ such that

$$f = f(\lambda)\varphi + \sum_{i=1}^{d} (z_i - \lambda_i)(\varphi f_i - \varphi_i f).$$

Thus, $\dim \mathcal{M}/((z_1 - \lambda_1)\mathcal{M} + ... + (z_d - \lambda_d)\mathcal{M}) = 1$.

Gleason's problem for $\mathcal{H}$ = Hardy or Bergman space, or $\mathcal{H} = H^2_d$, easy.

Gleason's problem for $M(\mathcal{H}) = H^\infty(\mathcal{B}_d)$ known.

For $M(\mathcal{H}^2_d)$ we use a theorem of Eschmeier-Putinar which involves the representation of an $M(\mathcal{H}^2_d)$-function as the transfer function corresponding to a certain unitary colligation.

The vector-valued version of Theorem 6 holds.

$\mathcal{D}$ separable Hilbert space

$$\mathcal{H}_{\mathcal{D}} = \{f : \mathcal{B}_d \to \mathcal{D} : f(\lambda) = \sum_n f_n(\lambda) e_n\}$$

$f_n \in \mathcal{H}, \{e_n\}$ o.n. basis for $\mathcal{D}$, $||f||^2 = \sum_n ||f_n||^2$.

$\mathcal{M} \in$ Lat$(M_z, \mathcal{H}_{\mathcal{D}}), \lambda \in \mathcal{B}_d$

$$\mathcal{M}_\lambda = \overline{\{f(\lambda) : f \in \mathcal{M}\}} \subseteq \mathcal{D}$$

Defn 8. *The* <u>*fiber dimension*</u> *of* $\mathcal{M}$ *is*

$$m = \sup\{\dim \mathcal{M}_\lambda : \lambda \in \mathcal{B}_d\},$$

$Z(\mathcal{M}) = \{\lambda \in \mathcal{B}_d : \dim \mathcal{M}_\lambda < m\}$.

<u>Fact:</u> If $m < \infty$, then $Z(\mathcal{M}) \subseteq Z(h)$ for some $h \neq 0$ analytic on $\mathcal{B}_d$. In fact, then the collection

$$\{\mathcal{M}_\lambda\}_{\lambda \in \mathcal{B}_d \setminus Z(\mathcal{M})}$$

forms a vectorbundle.

Thm 9. *(GRS)* $\mathcal{M} \in$ *Lat*$(M_z, H^2_d(\mathcal{D}))$, $\mathcal{M} \neq (0)$

fiber dimension of $\mathcal{M} = m < \infty$.

Then

$$\partial\mathcal{B}_d \subseteq \sigma_e(M_z|\mathcal{M}) \subseteq \partial\mathcal{B}_d \cup Z(\mathcal{M})$$

and for all $\lambda \in \mathcal{B}_d \setminus \sigma_e(M_z|\mathcal{M})$

$$\text{ind}(M_z - \lambda)|\mathcal{M} = (-1)^d m.$$

In fact, for all $\lambda \in \mathcal{B}_d \setminus Z(\mathcal{M})$ *we have*

$$\dim \mathcal{M}/((z_1 - \lambda_1)\mathcal{M} + .. + (z_d - \lambda_d)\mathcal{M}) = m.$$

Note:

If $\mathcal{N} \subseteq \mathcal{M}$, then fiber dim $\mathcal{N} \leq$ fiber dim $\mathcal{M}$.

Application - we can answer a question of Arveson's

Let $T = (T_1, T_2, ..., T_d)$ be a tuple of commuting operators on a Hilbert space $\mathcal{K}$.

Defn 10. *$T = (T_1, T_2, ..., T_d)$ is called a* <u>*d-contraction*</u>, *if*

$$||\sum_{i=1}^{d} T_i x_i||^2 \leq \sum_{i=1}^{d} ||x_i||^2 \quad \textit{for all } x_1, .., x_d \in \mathcal{K}$$

$$\left(\Leftrightarrow \sum_{i=1}^{d} T_i T_i^* \leq I \right)$$

The defect operator is $D = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2}$.

Let $\Phi(A) = \sum_{i=1}^{d} T_i A T_i^*$, then T is called <u>*pure*</u>, *if $\Phi^n(I) \to 0$ (SOT).*

Thm 11. *(?, maybe from 70's)*
If T is a pure d-contraction, if $\mathcal{D} = \overline{\text{ran} D}$, then $\exists \mathcal{M} \in$ Lat$(M_z, H_d^2(\mathcal{D}))$ such that

$$T_i \text{ u.e. } P_{\mathcal{M}^\perp} M_{z_i} | \mathcal{M}^\perp, \quad \textit{same unitary } \forall i = 1, .., d.$$

Cor 12. *(Drury, 78) If T is a d-contraction, then*

$$||p(T)|| \leq ||p||_{M(H_d^2)} \text{ for every poly } p(z_1, .., z_d).$$

Cor 13. *(GRS) If T is a pure d-contraction of finite rank (i.e. $\text{rank}D < \infty$), then*

$\exists E = Z(h), h \in H^\infty(\mathcal{B}_d), h \neq 0$ *such that*

$$\sigma_e(T) \subseteq \partial\mathcal{B}_d \cup E$$

and

$$\kappa(T) = \text{ind}(T - \lambda) \ \forall \lambda \in B_d \setminus \sigma_e(T).$$

But $\exists$ such d-contractions T that are <u>not</u> Fredholm and <u>not</u> essentially normal.

Curvature invariant (Arveson) :
$\kappa(T) = \lim_{r \to 1-} \int_{\partial B_d} \text{trace } F(rz) d\sigma(z)$,

$F(\lambda) = (1 - |\lambda|^2) D(I - T(\lambda)^*)^{-1}(I - T(\lambda))^{-1}D$,
$T(\lambda) = \sum_{i=1}^d \bar{\lambda}_i T_i$.

$F(\lambda)$ is related to the unitary from Theorem 11.