Analytic contractions, nontangential limits, and the index of invariant subspaces

joint work by

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$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \partial\mathbb{D}$,

$(0) \neq \mathcal{H} \subseteq \text{Hol}(\mathbb{D})$, $(M_z f)(z) = zf(z)$

$\mathcal{M} \in \text{Lat}(M_z, \mathcal{H})$, if $\mathcal{M}$ is an invariant subspace for M_z.

Examples: Hardy space

$$H^2(\mathbb{D}) = \{f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$$

$$||f||^2_{H^2} = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \int_{\mathbb{T}} |f(z)|^2 \frac{|dz|}{2\pi}$$

Then:

- $||z^n f|| = ||f||$ for all $f \in H^2(\mathbb{D})$,

- every $f \in H^2(\mathbb{D})$ has nontangential limits a.e. on $\mathbb{T}$,

- $\forall \mathcal{M} \in \text{Lat}(M_z, H^2), \mathcal{M} \neq (0)$ we have

$\mathcal{M} = \varphi H^2(\mathbb{D})$ for φ inner, and so

$$\dim \mathcal{M}/z\mathcal{M} = 1.$$

Bergman space

$$L_a^2 = \{f \in \text{ Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$||f||^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi}$,

$dA = dxdy =$ area measure.

Then:

- $||z^n f|| \to 0$ for all $f \in L_a^2$,

- $\exists f \in L_a^2$ such that f has no nontangential limits on any set $E \subseteq \mathbb{T}$ of positive Lebesgue measure,

- $\forall n \in \{1, 2, ...\} \cup \{\infty\}$ $\exists \mathcal{M} \in$ Lat(M_z, L_a^2) such that

$$\dim \mathcal{M}/z\mathcal{M} = n.$$

Assume: $(0) \neq \mathcal{H} \subseteq$ Hol$(\mathbb{D})$, reprod. kernel k_λ

(1) $\|zf\| \leq \|f\|$ for all $f \in \mathcal{H}$,

(2) $\sigma_e(M_z) = \partial\mathbb{D}$ &

(3) $\dim \mathcal{H}/z\mathcal{H} = 1$

Def: If $\mathcal{M} \in$ Lat$(M_z, \mathcal{H})$, then

$$\text{ind } \mathcal{M} = \dim \mathcal{M}/z\mathcal{M}$$

$$\text{ind}\mathcal{M} = -\text{ind} M_z|\mathcal{M} = -\text{ind}(M_z - \lambda)|\mathcal{M}$$

$$= \dim \mathcal{M}/(z-\lambda)\mathcal{M} \quad \text{for all } \lambda \in \mathbb{D}$$

Nontangential limits:

$z \in \mathbb{T}$, $0 < \alpha < 1$

$\Gamma_\alpha(z) =$

If f is meromorphic in $\mathbb{D}$ and $A \in \mathbb{C}$, then

$$\text{nt-}\lim_{\lambda \to z} f(\lambda) = A,$$

if, whenever $\{\lambda_n\} \to z, \{\lambda_n\} \subseteq \Gamma_\alpha(z)$ for some α, then $\{f(\lambda_n)\} \to A$.

Def: Let $E \subseteq \mathbb{T}$ be measurable, we say that $\mathcal{H}$ admits nontangential limits on E, if the meromorphic function

g/f has nontangential limits a.e. on E

for all $g \in \mathcal{H}$ and some $f \in \mathcal{H}, f \neq 0$.

Remark The definition is independent of f.
$g/h = \frac{g/f}{h/f}$, $h/f \neq 0$ for $h \neq 0$ follows from

Luzin-Privalov uniqueness theorem:

If k is a meromorphic function on $\mathbb{D}$ that has 0 nontangential limit on $E \subseteq \mathbb{T}$ with $|E| > 0$, then $k = 0$.

Hence:

If $1 \in \mathcal{H}$, then $\mathcal{H}$ admits nontangential limits on E, iff every function in $\mathcal{H}$ has nontangential limits a.e. on E.

Example: If $\mathcal{H}_1 = H^2(\mathbb{D})$ and $\mathcal{H}_2 = fH^2(\mathbb{D})$, $||fg||_2 = ||g||_{H^2}$ for some analytic function f, then

$(M_z, \mathcal{H}_1)$ and $(M_z, \mathcal{H}_2)$ are unitarily equivalent,

so they both satisfy (1), (2), and (3), and both admit nontangential limits on $\mathbb{T}$, no matter what f is.

Thm 1. *Assume $\mathcal{H}$ satisfies only (1), (2), (3)*

A. $\forall f \in \mathcal{H} \quad \|z^n f\| \to 0$ *as* $n \to \infty$.

(trivial) $\Downarrow$ *not* $\Uparrow$ *(ARS)*

B. $\exists f \in \mathcal{H}, f \neq 0 \quad \|z^n f\| \to 0$ *as* $n \to \infty$.

(Khin.-Kol.) $\Downarrow$ *not* $\Uparrow$ *(ARS)*

C. $\mathcal{H}$ *does not admit nt-limits on any* $\Delta \subseteq \mathbb{T}$, $|\Delta| > 0$.

(ARS) $\Downarrow$ *?* $\Uparrow$ *?*

D. $\exists \mathcal{M}$ *such that ind* $\mathcal{M} > 1$.

B $\Rightarrow$ D follows from ABFP: $\mathbb{A}_1 \cap C_{00} = \mathbb{A}_{\aleph_0} \cap C_{00}$.

Thm 2. *We have* $A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$, *whenever* $\mathcal{H}$ *satisfies (1), (2), (3) and either*

- M_z *is cyclic subnormal, i.e.* $\mathcal{H} = P^2(\mu)$ *for some* μ, *supp*$\mu \subseteq \overline{\mathbb{D}}$, *or*
- $\exists c > 0 \ \|\frac{z-\lambda}{1-\overline{\lambda} z} f\| \geq c\|f\|$ *for all* $f \in \mathcal{H}, \lambda \in \mathbb{D}$.

Theorem (Khinchin-Kolmogorov)
If $\sum_{n\geq 0} |a_n|^2 = \infty$, then for some choice of $\varepsilon_n \in \{-1, 1\}$ the function

$$g(z) = \sum_{n\geq 0} \varepsilon_n a_n z^n$$

has no nontangential limits on any $\Delta \subseteq \mathbb{T}$, $|\Delta| > 0$.

B $\Rightarrow$ C:

If $||z^n f|| \to 0$ for $f \in \mathcal{H}, f \neq 0$, then choose $\{n_k\}$ such that $\sum_{k\geq 0} ||z^{n_k} f|| < \infty$. Then

$$g(z) = \sum_{k\geq 0} \varepsilon_{n_k} z^{n_k} f(z) \in \mathcal{H}$$

and g/f has no nontangential limits.

C $\Rightarrow$ D:

Case 1: The polynomials are dense in $\mathcal{H}$.

Use the hypothesis to show that there is a dominating sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{D}$ that is interpolating for $\mathcal{H}$.

Then $\mathcal{M} = \{f \in \mathcal{H} : f(\lambda_n) = 0 \text{ for all } n\}$ satisfies that $M_z^*|\mathcal{M}^\perp$ is similar to a diagonal normal operator with $\overline{\lambda_n}$ on the diagonal. Now use a result of Wermer from 1950 to finish it off.

Case 2: General $\mathcal{H}$.

Use case 1 and then apply the Bercovici-Chevreau Theorem ($\mathbb{A} = \mathbb{A}_1$) to $M_z^*|\mathcal{M}^\perp$ for appropriate $\mathcal{M}$.

For the construction of the interpolating sequence $\{\lambda_n\}$ and the rest of Theorem 1 one needs to know more about the relationship of when $\mathcal{H}$ will admit nontangential limits on a set Δ and the asymptotic size of the reproducing kernel $||k_\lambda||$.

$$|f(\lambda)| = (1-|\lambda|^2)|\frac{f}{1-\overline{\lambda}z}(\lambda)| \leq (1-|\lambda|^2)||\frac{f}{1-\overline{\lambda}z}|| \, ||k_\lambda||$$

$$\left|\frac{f}{g}(\lambda)\right|^2 \leq \underbrace{\left((1-|\lambda|^2)\|\frac{f}{1-\overline{\lambda}z}\|^2\right)}_{I_\lambda} \underbrace{\left((1-|\lambda|^2)\frac{||k_\lambda||^2}{|g(\lambda)|^2}\right)}_{II_\lambda}$$

(a) $M_z = PU|\mathcal{H}$, U unitary dilation with spectral measure E:

$$\begin{aligned} I_\lambda &= (1-|\lambda|^2)||P(1-\overline{\lambda}U)^{-1}f||^2 \\ &\leq \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} d < E(z)f, f > \\ &= v_f(\lambda) \; = \; \text{harmonic function} \end{aligned}$$

(b) $E \subseteq \mathbb{T}$ closed, $\Omega_E = \bigcup_{z \in E} \Gamma_\alpha(z)$

If $\quad II_\lambda \leq M \quad$ in Ω_E, then

$$\left|\frac{f}{g}(\lambda)\right|^2 \leq M v_f(\lambda) \;\; \text{in } \Omega_E, \text{ i.e. } f/g \in H^2(\Omega_E),$$

hence $\mathcal{H}$ admits nontangential limits on E.

Def:

$$\Delta(\mathcal{H}) = \left\{ z \in \mathbb{T} : \text{nt-}\overline{\lim_{\lambda \to z}} (1 - |\lambda|^2) \frac{||k_\lambda||^2}{|g(\lambda)|^2} < \infty \right\}$$

Fact: Up to a.e. this definition is independent of $g \in \mathcal{H}$, $g \neq 0$.

Thm 3. *(a) $\mathcal{H}$ admits nt-limits on $\Delta(\mathcal{H})$.*

(b) If $\mathcal{H}$ admits nt-limits on E, then $E \subseteq \Delta(\mathcal{H})$ a.e.

If $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$ satisfies (1),(2),(3) and if $\mathcal{H} = P^2(\mu)$, supp$\mu \subseteq \overline{\mathbb{D}}$, then $d\mu|\mathbb{T} = h\frac{|dz|}{2\pi}, h \in L^1$ and

$$I_\lambda = \int_{\mathbb{D}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2}|f|^2 d\mu + \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2}|f|^2 h\frac{|dz|}{2\pi}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$0 \text{ a.e.} \qquad\qquad\qquad w_f = |f|^2 h \text{ a.e.}$$

$\Longrightarrow I_\lambda \to w_f(z)$ a.e. as $\lambda \to z$ nontangentially.

Claim: If $f \neq 0$, then a.e.

$\Delta(\mathcal{H}) \subseteq \{z : w_f(z) > 0\} \subseteq \{z : h(z) > 0\} =: \Sigma(\mathcal{H})$.

Proof: For a.e z, if $I_\lambda \to w_f(z) = 0$ and $z \in \Delta(\mathcal{H})$, then

$$\text{nt-lim}_{\lambda\to z}(\tfrac{f}{g})(\lambda) = 0.$$

Hence if $f \neq 0$ by Luzin-Privalov this happens on a set of measure 0.

Note: $\mathcal{H} = P^2(\mu)$

$$||z^n f|| \to 0 \ \forall f \iff h = 0 \iff |\Sigma(\mathcal{H})| = 0$$

Thm 4. *If $\mathcal{H} \subseteq Hol(\mathbb{D})$ satisfies (1),(2),(3) and if $\mathcal{H} = P^2(\mu)$, supp$\mu \subseteq \overline{\mathbb{D}}$, then*

$$\Delta(\mathcal{H}) \ = \ \Sigma(\mathcal{H}) \ \ \textit{a.e.}$$

and $\forall f \in \mathcal{H}$ and a.e. $z \in \mathbb{T}$

$$\begin{aligned} nt\text{-}\lim_{\lambda \to z} & \frac{|f(\lambda)|^2}{(1-|\lambda|^2)||k_\lambda||^2} \\ & = w_f(z) = (|f|^2 h)(z) \\ & = \ nt\text{-}\lim_{\lambda \to z} (1-|\lambda|^2)||\frac{f}{1-\overline{\lambda} z}||^2 \end{aligned}$$

Works for $P^t(\mu)$, $1 \leq t < \infty$.

Proof: Rework the proof of Thomson's Theorem and use Tolsa's results on analytic capacity.

If $\mathcal{H} \neq P^2(\mu)$ use <u>minimal co-isometric extension</u> of M_z:

$$M_z = (S^* \oplus R)\,|\mathcal{H} \quad \text{on } \mathcal{K}_1 \oplus \mathcal{K}_2 \supseteq \mathcal{H}$$

S^* = backward shift, R unitary, E = spectral measure for R

<u>Facts:</u> (a) $I_\lambda \to w_f(z) = 2\pi \, \frac{d<E(z)P_2 f, P_2 f>}{|dz|}$ a.e. as $\lambda \to z$ nontangentially.

(b) $\exists w \in L^1(\mathbb{T})$ such that

$$||E(F)|| = 0 \;\Leftrightarrow\; \int_F w(z)|dz| = 0.$$

(c) If $\mathcal{H} = P^2(\mu)$ as above, then $\{z \in \mathbb{T} : w(z) > 0\} = \{z \in \mathbb{T} : h(z) > 0\}$ a.e.

<u>Def.:</u> $\Sigma(\mathcal{H}) = \{z \in \mathbb{T} : w(z) > 0\}$

<u>Facts:</u>(a) $||z^n f|| \to 0$ for all $f \in \mathcal{H}$, iff $|\Sigma(\mathcal{H})| = 0$.

(b) $\Delta(\mathcal{H}) \subseteq \Sigma(\mathcal{H})$ a.e.

(c) In general, $\Delta(\mathcal{H}) \neq \Sigma(\mathcal{H})$.

Thm 5. *If $\mathcal{H}$ satisfies (1),(2),(3) and if* $\exists c > 0$ *such that*

$$\|\frac{z-\lambda}{1-\bar{\lambda}z}f\| \geq c\|f\|$$

for all $f \in \mathcal{H}, \lambda \in \mathbb{D}$, *then*

$$\Delta(\mathcal{H}) \ = \ \Sigma(\mathcal{H}) \ \ \textit{a.e.}$$

and $\forall f \in \mathcal{H}$ *and a.e.* $z \in \mathbb{T}$

$$\begin{aligned} \textit{nt-}\lim_{\lambda\to z} & \frac{|f(\lambda)|^2}{(1-|\lambda|^2)||k_\lambda||^2} \\ & = w_f(z) \\ & = \ \textit{nt-}\lim_{\lambda\to z}(1-|\lambda|^2)||\frac{f}{1-\bar{\lambda}z}||^2 \end{aligned}$$

Thm 6. *Suppose* $k_\lambda(z) = \frac{l_\lambda(z)}{1-\bar{\lambda}z}$, *where*

$$l_\lambda(z) = \sum_{k\geq 0} \overline{\varphi_k(\lambda)}\varphi_k(z),$$

$\varphi_k \in N$, N = *Nevanlinna class*

If for a.e. $z \in \mathbb{T}$ *we have*

$$\textit{nt-}\overline{\lim_{\lambda\to z}}\ l_\lambda(\lambda) = \infty \Rightarrow \sum_{k\geq 0}|\varphi_k(z)|^2 = \infty,$$

then **same conclusion**.