Invariant subspaces for the Backward shift on Hilbert spaces of analytic functions with regular norm.

(joint work by:

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for a preprint see:

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$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \zeta(z) = z$ for all $z \in \mathbb{D}$

$\mathcal{H} \subseteq$ Hol($\mathbb{D}$) - Hilbert space

Assumptions:

1. $\forall \lambda \in \mathbb{D}$ the map $f \to f(\lambda)$ is continuous on $\mathcal{H}$.

2. The analytic polynomials are dense in $\mathcal{H}$.

3. If $f \in \mathcal{H}$, then $Sf = M_\zeta f = \zeta f \in \mathcal{H}$.

4. If $f \in \mathcal{H}$, then $Lf = \frac{f - f(0)}{\zeta} \in \mathcal{H}$.

5. $\sigma(S) = \sigma(L) = \overline{\mathbb{D}}$.

Examples: $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$

H^2 Hardy space,

$f \in H^2 \iff ||f||_{H^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$

L_a^2 Bergman space,

$f \in L_a^2 \iff ||f||_{L_a^2}^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} |\hat{f}(n)|^2 < \infty$

D Dirichlet space,

$f \in D \iff ||f||_D^2 = \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2 < \infty$

H_w weighted shift space,

$w = \{w_n\}_{n \geq 0}$, $w_n > 0, \frac{w_{n+1}}{w_n} \to 1$ as $n \to \infty$

$f \in H_w \iff ||f||_w^2 = \sum_{n=0}^{\infty} w_n |\hat{f}(n)|^2 < \infty$

$\mathcal{M} \in$ Lat T

$\iff$ $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ with $T\mathcal{M} \subseteq \mathcal{M}$.

Example 1: If $\mathcal{H} = H^2$, then $L = S^*$.

Thus Beurling's theorem implies:

$\mathcal{M} \in$ Lat $L, \mathcal{M} \neq H^2 \iff \mathcal{M}^\perp = \varphi H^2$, φ inner.

Thm (Moeller, 62) If $\mathcal{M} \in$ Lat $L, \mathcal{M} \neq H^2$, then

$$\sigma(L|\mathcal{M}) \cap \mathbb{D} = Z(\varphi)^* = Z(\mathcal{M}^\perp)^*.$$

In fact,
$\sigma(L|\mathcal{M}) = \underline{Z}(\varphi)^* = \underline{Z}(\mathcal{M}^\perp)^* =$ (supportφ)*.

Here
$A^* = \{\overline{\lambda} : \lambda \in A\}$
$Z(f) = \{\lambda \in \mathbb{D} : f(\lambda) = 0\}$
$\underline{Z}(f) = Z(f) \cup \{z \in \partial\mathbb{D} : \underline{\lim}_{\lambda \to z} |f(\lambda)| = 0\}$.

$$Z(\mathcal{M}^\perp) = \bigcap_{f \in \mathcal{M}^\perp} Z(f)$$

$$\underline{Z}(\mathcal{M}^\perp) = \bigcap_{f \in \mathcal{M}^\perp} \underline{Z}(f)$$

Thm (Douglas, Shapiro, Shields, 70) $f \in H^2$. Then
$\exists \mathcal{M} \in$ Lat $L, \mathcal{M} \neq H^2$ with $f \in \mathcal{M}$
$\iff$ f has a meromorphic pseudocontinuation $\tilde{f} \in N(\mathbb{D}_e)$ in the exterior disc $\mathbb{D}_e$.

$N(\mathbb{D}_e) =$ Nevanlinna class $=$ quotients of bounded analytic functions

Defn If $f \in$ Hol$(\mathbb{D})$ and if $\tilde{f}$ is a meromorphic function in $\mathbb{D}_e$, then they are pseudocontinuations of one another, if

$$\text{ntl-}\lim_{\lambda \to z, \lambda \in \mathbb{D}} f(\lambda) = \text{ntl-}\lim_{\lambda \to z, \lambda \in \mathbb{D}_e} \tilde{f}(\lambda)$$

for a.e. $z \in \partial\mathbb{D}$.

<u>Connection:</u>

If $\frac{1}{\lambda} \in \mathbb{C} \setminus \sigma(L|\mathcal{M})$, then

$$(I - \lambda L|\mathcal{M})^{-1} f(z) = \begin{cases} \frac{zf(z) - \lambda f(\lambda)}{z - \lambda} & |\lambda| < 1 \\ \frac{zf(z) - \lambda \tilde{f}(\lambda)}{z - \lambda} & |\lambda| > 1 \end{cases}$$

$\mathcal{H} \neq H^2 \Longrightarrow L \neq S^*$

Suppose $\mathcal{H}$ and $\mathcal{H}'$ are **Cauchy duals** of one another, e.g. L^2_a and D or more generally $\mathcal{H}_w$ and $\mathcal{H}_{1/w}$, then

$$\langle f, G\rangle_{\mathcal{H},\mathcal{H}'} = \sum_{n\geq 0} \widehat{f}(n)\overline{\widehat{G}(n)}, f \in \mathcal{H}, G \in \mathcal{H}'$$

and

$$L = S'^* \text{ where } L = (L, H), S' = (M_\zeta, \mathcal{H}')$$

Thus

$$\mathcal{M} \in \text{ Lat}(L, \mathcal{H}) \Longleftrightarrow \mathcal{M}^\perp \in \text{ Lat}(S', \mathcal{H}')$$

Thm (R) If $\mathcal{M} \in$ Lat$(L, \mathcal{H})$, $\mathcal{M} \neq (0), \mathcal{H}$, then

$$\sigma(L|\mathcal{M}) \cap \mathbb{D} = Z(\mathcal{M}^\perp) \Leftrightarrow \text{ ind } \mathcal{M}^\perp = 1$$

$$\sigma(L|\mathcal{M}) = \overline{\mathbb{D}} \Leftrightarrow \text{ ind } \mathcal{M}^\perp > 1$$

Thus, $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is either discrete or all of $\mathbb{D}$.

If $\mathcal{N} \in$ Lat$(S', \mathcal{H}')$, then

$$\text{ind } \mathcal{N} = \dim \mathcal{N} \ominus S'\mathcal{N} = \dim \mathcal{N}/S'\mathcal{N}$$

Fact: (easy, but mild extra hypothesis on $\mathcal{H}$)
If $\mathcal{M} \in \text{Lat}(L, \mathcal{H})$ such that $\sigma(L|\mathcal{M})$ omits an arc $I \subseteq \partial\mathbb{D}$, then every $f \in \mathcal{M}$ has an analytic continuation $\tilde{f}$ across I that extends to be meromorphic in $\mathbb{D}_e$ and whenever $\frac{1}{\lambda} \in \mathbb{C} \setminus \sigma(L|\mathcal{M})$, then

$$(*)\ (I - \lambda L|\mathcal{M})^{-1} f(z) = \begin{cases} \frac{zf(z)-\lambda f(\lambda)}{z-\lambda} & |\lambda| < 1 \\ \frac{zf(z)-\lambda \tilde{f}(\lambda)}{z-\lambda} & |\lambda| > 1 \end{cases}$$

Thm (RS, 94) $\mathcal{H} = L^2_a, \mathcal{H}' = D$
If $\mathcal{M} \in \text{Lat}\ (L, L^2_a), \mathcal{M} \neq L^2_a$, then

(a) $\sigma(L|\mathcal{M}) \cap \mathbb{D} = Z(\mathcal{M}^\perp)^*$ is discrete.

(b) $\sigma(L|\mathcal{M}) = \underline{Z}(\mathcal{M}^\perp)^*$.

(c) Furthermore, if $f \in \mathcal{M}$, then $f \in N(\mathbb{D})$ and f has a meromorphic pseudocontinuation $\tilde{f} \in N(\mathbb{D}_e)$ which satisfies (*).

(ARR, 98) extend (a) and (c) to all radially weighted Bergman spaces.

Thm/Cor $\mathcal{H} = D, \mathcal{H}' = L_a^2$ Everything fails.
Then
(a) (ABFP) there exists $\mathcal{M} \in \text{Lat}(L, D)$ such that $\sigma(L|\mathcal{M}) = \overline{\mathbb{D}}$, (i.e. $\mathcal{M}^\perp \in \text{Lat}(M_\zeta, L_a^2)$ with ind $\mathcal{M}^\perp > 1$).
(ARR/ARS) In fact, one can even get such $\mathcal{M}$ where every $f \in \mathcal{M}$ has a meromorphic pseudocontinuation in $\mathbb{D}_e$.

(b) there exists $\mathcal{M} \in \text{Lat}(L, D)$, $f \in \mathcal{M}$, but f has no pseudocontinuation in $\mathbb{D}_e$.

(c) there exists $\mathcal{M} \in \text{Lat}(L, D)$, $f \in \mathcal{M}$ such that $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is discrete, f has a pseudocontinuation $\tilde{f}$ in $\mathbb{D}_e$, but for $\frac{1}{\lambda} \in \mathbb{C} \setminus \sigma(L|\mathcal{M})$ we have

$$(I - \lambda L|\mathcal{M})^{-1} f(z) = \begin{cases} \frac{zf(z)-\lambda f(\lambda)}{z-\lambda} & |\lambda| < 1 \\ \frac{zf(z)-\lambda F(\lambda)}{z-\lambda} & |\lambda| > 1 \end{cases}$$

where $F \neq \tilde{f}$.

Thm (ARR) Suppose $\mathcal{H}$ satisfies 1.-5. and $\mathcal{M} \in$ Lat L, $\mathcal{M} \neq (0), \mathcal{H}$. Then

$\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is discrete

$$\iff \tilde{f}_g(\lambda) = \frac{\langle \frac{\zeta f}{\zeta-\lambda}, g\rangle}{\langle \frac{\lambda}{\zeta-\lambda}, g\rangle}, |\lambda| > 1 \text{ is indep. of } g \in \mathcal{M}^\perp$$

$(g \neq 0)$

If $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is discrete $g \in \mathcal{M}^\perp, g \neq 0$, and if $\frac{1}{\lambda} \in \mathbb{C} \setminus \sigma(L|\mathcal{M})$, then

$$(I - \lambda L|\mathcal{M})^{-1} f(z) = \begin{cases} \frac{zf(z)-\lambda f(\lambda)}{z-\lambda} & |\lambda| < 1 \\ \frac{zf(z)-\lambda \tilde{f}_g(\lambda)}{z-\lambda} & |\lambda| > 1. \end{cases}$$

Note: If $||M_\zeta|| \leq 1$, then M_ζ has a unitary dilation U with spectral measure E, hence for $|\lambda| > 1$ we have

$$\langle \frac{\zeta f}{\zeta - \lambda}, g\rangle = \langle (U-\lambda)^{-1} U f, g\rangle = \int_{\partial\mathbb{D}} \frac{z}{z-\lambda} d\langle E(z) f, g\rangle$$

is a Cauchy transform in $N(\mathbb{D}_e)$.

Hence

If $||M_\zeta|| \leq 1$, then for all $f \in \mathcal{M}$, $g \in \mathcal{M}^\perp, g \neq 0$ we have

$$\tilde{f}_g(\lambda) = \frac{\langle \frac{\zeta f}{\zeta - \lambda}, g\rangle}{\langle \frac{\lambda}{\zeta - \lambda}, g\rangle} \in N(\mathbb{D}_e).$$

Question: Under what extra hypothesis is $\tilde{f}_g$ a pseudocontinuation of f ?

If $\exists \mathcal{N} \in$ Lat$(M_\zeta, \mathcal{H}), \mathcal{M} \in$ Lat $(L, \mathcal{H})$ such that $\mathcal{N} \subseteq \mathcal{M} \neq \mathcal{H}$, then $\forall g \in \mathcal{M}^\perp$, $f \in \mathcal{N}$, $|\lambda| > 1$

$$\langle \frac{\zeta f}{\zeta - \lambda}, g\rangle = 0, \text{ hence } \tilde{f}_g = 0.$$

Thm (Esterle) If $w = \{w_n\}$, $w_n \geq w_{n+1} \to 0$ is given, then $\exists v = \{v_n\}$, $v_n \geq v_{n+1} \to 0$ and $w_n \geq v_n$ so that $||(M_\zeta, \mathcal{H}_v)|| \leq 1$ and $H^2 \subseteq \mathcal{H}_v \subseteq \mathcal{H}_w$ and the above happens in $\mathcal{H}_v$ and in fact

$$\sigma(L|\mathcal{M}) = \overline{\mathbb{D}}.$$

Thm 1. *If $\mathcal{H}$ satisfies 1.-5., $||M_\zeta|| \leq 1$, and $\exists c > 0$ such that*

$$||\frac{\zeta - \lambda}{1 - \overline{\lambda}\zeta} f|| \geq c||f|| \quad \forall f \in \mathcal{H}, \lambda \in \mathbb{D},$$

then whenever $\mathcal{M} \in Lat L, \mathcal{M} \neq \mathcal{H}$, then every $f \in \mathcal{M}$ has a pseudocontinuation $\tilde{f} \in N(\mathbb{D}_e)$ and

$$\tilde{f} = \tilde{f}_g \quad \forall g \in \mathcal{M}^\perp.$$

Thus, $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is discrete.

Thm 2. *If $\mathcal{H}$ satisfies 1.-5., $||M_\zeta|| \leq 1$, and $\exists A = \{r_n\} \nearrow 1$ such that for $r \in A$*

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_r + \langle \cdot, \cdot \rangle_{1/r}$$

with for all $f \in \mathcal{H}$

$$||zf||_r \leq r||f||_r \text{ and } ||Lf||_{1/r} \leq \frac{1}{r}||f||_{1/r},$$

then whenever $\mathcal{M} \in Lat L, \mathcal{M} \neq \mathcal{H}$, then

$$\mathcal{M} \subseteq N(\mathbb{D})$$

and every $f \in \mathcal{M}$ has a pseudocontinuation $\tilde{f} \in N(\mathbb{D}_e)$ and

$$\tilde{f} = \tilde{f}_g \quad \forall g \in \mathcal{M}^\perp.$$

Thus, $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is discrete.

Example: If $d\mu(z) = \omega(r) dr dt$, then the decomposition

$$\int_{\mathbb{D}} |f|^2 d\mu = \int_{r\mathbb{D}} |f|^2 d\mu + \int_{\mathbb{D}\setminus r\mathbb{D}} |f|^2 d\mu$$

will satisfy the hypothesis of Thm 2.

If $w = \{w_n\}_{n\geq 0}$, $w_n > 0$, $\frac{w_{n+1}}{w_n} \to 1$ as $n \to \infty$, then

$$||(M_\zeta, H_w)|| \leq 1 \Longleftrightarrow \frac{w_{n+1}}{w_n} \leq 1 \quad \forall n$$

Define

$$\alpha_+ = \overline{\lim}_{n\to\infty}\ n(1 - \frac{w_{n+1}}{w_n})$$

$$\alpha_- = \underline{\lim}_{n\to\infty}\ n(1 - \frac{w_{n+1}}{w_n})$$

Ex.: If $w_n = (n+1)^{-\beta}$, then $\alpha_+ = \alpha_- = \beta$

If $0 < \alpha_+ = \alpha_- = \alpha < \infty$, then $1 - \frac{w_{n+1}}{w_n} = \frac{\alpha}{n} + o(\frac{1}{n})$

Thm 3. *If*

$$\alpha_+ < \infty \text{ and } \alpha_+ - \alpha_- < 1,$$

then H_w satisfies the hypothesis and conclusion of Theorem 1.

Thm 4. *(a) If $\frac{w_{n+1}}{w_n} \nearrow 1$, then H_w satisfies the hypothesis and conclusion of Theorem 2.*

(b) If $0 < \alpha_- \le \alpha_+ < \infty$ and

$$\alpha_+ \left(\log \frac{\alpha_+}{\alpha_-} - (1 - \frac{\alpha_-}{\alpha_+}) \right) e^{\alpha_+ - \alpha_-} < 1,$$

then H_w satisfies the conclusion of Theorem 2 (equivalent norm).

(c) If $1 - \frac{w_{n+1}}{w_n} = \frac{\alpha}{n^\gamma} + o(\frac{1}{n})$, $0 < \gamma < 1$, then then H_w satisfies the conclusion of Theorem 2 (equivalent norm).

In this case $w_n \approx b_n e^{-\frac{\alpha}{1-\gamma} n^{1-\gamma}}$.