Weak products of functions in the Dirichlet space.

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joint work with Carl Sundberg

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Introduction

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ disc
$\mathbb{T} = \partial\mathbb{D}$
$\mathbb{B}_d = \{z \in \mathbb{C}^d : |z| < 1\}$ ball
$\mathbb{T}^d$ d-torus

$\Omega \subseteq \mathbb{C}^d$
$\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ a reproducing kernel Hilbert space

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$\mathcal{H} \subseteq$ Hol(Ω) a reproducing kernel Hilbert space

$$f(z) = \langle f, k_z \rangle \quad \text{for all } f \in \mathcal{H}$$

$$\mathcal{H} \odot \mathcal{H} = \{h = \sum_{i=1}^{\infty} f_i g_i : f_i, g_i \in \mathcal{H}, \sum_{i=1}^{\infty} \|f_i\| \|g_i\| < \infty\}$$

$$\|h\|_{\mathcal{H}\odot\mathcal{H}} = \inf\{\sum_{i=1}^{\infty} \|f_i\| \|g_i\| : h(z) = \sum_{i=1}^{\infty} f_i(z) g_i(z) \text{ for all } z \in \Omega\}.$$

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Let $z \in \Omega$

$$\begin{aligned} |h(z)| &\leqslant \sum_i |f_i(z) g_i(z)| = \sum_i |\langle f, k_z \rangle \langle g, k_z \rangle| \\ &\leqslant \|k_z\|^2 \sum_i \|f_i\| \|g_i\| \end{aligned}$$

Hence

$$\mathcal{H} \odot \mathcal{H} \subseteq \text{ Hol}(\Omega) \text{ and } |h(z)| \leqslant \|k_z\|^2 \|h\|_{\mathcal{H} \odot \mathcal{H}}$$

Why consider $\mathcal{H} \odot \mathcal{H}$?

- $f, g \in \mathcal{H} \Rightarrow fg \in \mathcal{H} \odot \mathcal{H}$
  $\mathcal{H} \odot \mathcal{H} =$ a Banach space $\subseteq \mathrm{Hol}(\Omega)$

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$$H^\infty \subseteq BMOA \subseteq H^2 \subseteq H^1$$

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- If $\mathcal{H} = L_a^2$, then

$$H^\infty \subseteq \mathcal{B} \subseteq L_a^2 \subseteq L_a^1$$

  $\mathcal{B} = \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty\}$ Bloch space

  $L_a^p = \{f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f|^p dA < \infty\}$ Bergman space

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- If $\mathcal{H} = D$, then

$$M(D) \subseteq D$$

$D = \{f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 dA < \infty\}$ Dirichlet space

$M(D) = \{\varphi : \varphi f \in D \; \forall f \in D\}$ multipliers

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- If $\mathcal{H} = H^2_d$, then
$$M(H^2_d) \subseteq H^2_d$$
H^2_d Drury- Arveson space
$k_\lambda(z) = \frac{1}{1-\langle z,\lambda\rangle_{\mathbb{C}^d}}$

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- papers by Coifman-Rochberg-Weiss (1976) and by Arcozzi-Rochberg-Sawyer-Wick (2010)

If the polynomials are dense in $\mathcal{H}$ define the space of symbols for bounded Hankel forms

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \\ \exists C > 0 \ |\langle pq, b\rangle| \leqslant C\|p\|\|q\| \text{ for all polynomials } p, q\}$$

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Hankel operator $H_{\overline{b}} : \mathcal{H} \to \overline{\mathcal{H}}, \quad \|\bar{f}\|_{\overline{\mathcal{H}}} = \|f\|_{\mathcal{H}}$

$$\langle H_{\overline{b}} p, \overline{q} \rangle_{\overline{\mathcal{H}}} = H_{\overline{b}}(p, q) = \langle pq, b \rangle_{\mathcal{H}}$$

CRW:

Metatheorem: There is a connection between

- boundedness of $H_{\bar{b}} : \mathcal{H} \to \overline{\mathcal{H}}$
- $(\mathcal{H} \odot \mathcal{H})^* = \mathcal{X}(\mathcal{H})$
- Carleson measures for $\mathcal{H}$

e.g. the Hardy space

Theorem ($H^2 \odot H^2 = H^1$)

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TFAE

(a) $H_{\overline{b}} : H^2 \to \overline{H^2}$ *is bounded*

(b) $b \in \mathcal{X}(H^2)$ *i.e.* $\exists\, C > 0 \; |\langle fg, b\rangle_{H^2}| \leqslant C\|f\|_{H^2}\|g\|_{H^2}$

(c)$\exists\, C > 0 \; |\langle h, b\rangle_{H^2}| \leqslant C\|h\|_{H^1}$*, i.e.* $b \in {H^1}^*$

(d) $b \in BMOA$

(e) $d\mu = |b'|^2(1-|z|^2)dA$ *is a Carleson measure for* H^2*, i.e.*

$$\int_{\mathbb{D}} |f|^2 d\mu \leqslant C\|f\|^2_{H^2} \text{ for all } f \in H^2$$

e.g. the Bergman space

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Theorem ($L_a^2 \odot L_a^2 = L_a^1$ CRW, Horowitz)

TFAE

(a) $H_{\overline{b}} : L_a^2 \to \overline{L_a^2}$ *is bounded* *(the little Hankel operator)*

(b) $b \in \mathcal{X}(L_a^2)$ *i.e.* $\exists\, C > 0\ |\langle fg, b\rangle_{L_a^2}| \leqslant C\|f\|_{L_a^2}\|g\|_{L_a^2}$

(c)$\exists\, C > 0\ |\langle h, b\rangle_{L_a^2}| \leqslant C\|h\|_{L_a^1}$, *i.e.* $b \in {L_a^1}^*$

(d) $b \in \mathcal{B}$ *Bloch*

(e) $d\mu = |b'|^2(1-|z|^2)^2 dA$ *is a Carleson measure for* L_a^2, *i.e.*

. $\int_{\mathbb{D}} |f|^2 d\mu \leqslant C\|f\|_{L_a^2}^2$ *for all* $f \in L_a^2$

ARSW, 2010

Theorem

TFAE

(a) $H_{\overline{b}} : D \to \overline{D}$ *is bounded*

(b) $b \in \mathcal{X}(D)$ *i.e.* $\exists\, C > 0\ |\langle fg, b\rangle_{L^2_a}| \leqslant C\|f\|_D\|g\|_D$

(c)$\exists\, C > 0\ |\langle h, b\rangle_D| \leqslant C\|h\|_{D\odot D}$, *i.e.* $b \in (D \odot D)^*$

(d) (Stegenga - capacities, ARS - tree condition)

(e) $d\mu = |b'|^2 dA$ *is a Carleson measure for D, i.e.*

.

$$\int_{\mathbb{D}} |f|^2 d\mu \leqslant C\|f\|_D^2 \text{ for all } f \in D$$

In short:

If $\mathcal{H}_0 = D$,
$\mathcal{H}_1 = H^2$,
$\mathcal{H}_2 = L^2_a$, then for $j = 0, 1, 2$ we have

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Interesting: Rudin, Miles, Rosay showed that for $d \geqslant 2$
$\exists h \in H^1(\mathbb{T}^d)$ with $h \neq fg \; \forall f, g \in H^2(\mathbb{T}^d)$

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ARSW have some results, but no general intrinsic characterization:

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- $CM(D \odot D) = CM(D)$
$$\int |h| d\mu \leqslant c\|h\|_{D \odot D} \;\Leftrightarrow\; \int |f|^2 d\mu \leqslant c\|f\|_D^2$$

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- $CM(D \odot D) = CM(D)$
$$\int |h| d\mu \leqslant c \|h\|_{D\odot D} \;\Leftrightarrow\; \int |f|^2 d\mu \leqslant c \|f\|_D^2$$
- the interpolating sequences for $D \odot D$ equal the interpolating sequences for D

Theorem (Ri-Su)

If $\mathcal{H}$ has reproducing kernel k, then

$$\mathcal{H} \odot \mathcal{H} \subseteq \mathcal{H}(k^2)$$

$$\|h\|_{\mathcal{H}(k^2)} \leqslant \|h\|_{\mathcal{H} \odot \mathcal{H}} \quad \forall h \in \mathcal{H} \odot \mathcal{H}.$$

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If $\mathcal{H} = H^2$, then $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$, $k_\lambda^2(z) = \frac{1}{(1-\bar{\lambda}z)^2}$, so

$$H^1 \subseteq L_a^2, \;\; \|f\|_{L_a^2} \leqslant \|f\|_{H^1}$$

- known and due to Harold Shapiro.

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If $\mathcal{H} = D$, then $\exists\, c > 0$

$$c\sum_{n=0}^{\infty} \frac{n+1}{\log(n+2)}|\hat{h}(n)|^2 \leqslant \|h\|_{D\odot D}^2 \leqslant \|h\|_D^2 = \sum_{n=0}^{\infty}(n+1)|\hat{h}(n)|^2.$$

Open Question:

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Definition (Dirichlet space of harmonic functions)

$$D_h = D \oplus \overline{D}_0$$

If $f \in D_h \subseteq L^2(\mathbb{T})$ then

$$\|f\|_{D_h}^2 = \|f\|_{L^2}^2 + \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(z) - f(w)}{z - w} \right|^2 \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}$$

$$= \sum_{n=-\infty}^{\infty} (|n| + 1) |\hat{f}(n)|^2$$

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Theorem (Ri-Su)

(a) If $h \in D_h \odot D_h$ *is real-valued, then there are real-valued functions* $u, v \in D_h$, $u \geqslant 0$ *such that*

$$h(e^{it}) = u(e^{it})v(e^{it}) \quad \textit{and}$$

$$\|h\|_{D_h \odot D_h} = \|u\|_{D_h} \|v\|_{D_h}$$

(b) If $h = h^+ - h^- \in L^1(T)$ *is real-valued, where* $h^+ = max(h, 0), h^- = -\min(h, 0)$, *then*

$$h \in D_h \odot D_h \quad \Leftrightarrow h^+, h^- \in D_h \odot D_h$$

Definition

If $E \subseteq \mathbb{T}$, then

$$\mathrm{cap}_1(E) = \inf\{\|h\|_{D_h \odot D_h} : h \geqslant 1 \text{ on } E\}$$

$$\mathrm{cap}_2(E) = \inf\{\|f\|^2_{D_h} : f \geqslant 1 \text{ on } E\}$$

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Remark: The Theorem and Corollary hold for weighted Dirichlet spaces $D(\mu)$

Theorem (Ri-Su)

If P denotes the Cauchy projection

$$Ph(z) = \int_0^{2\pi} h(e^{it}) \frac{1}{1 - ze^{-it}} \frac{dt}{2\pi},$$

then

$P : D_h \odot D_h \to D \odot D$ is bounded.

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Hence for $h \in H^1$ we have

- $\|h\|_{D_h \odot D_h} \approx \|h\|_{D \odot D}$
- $$h \in D \odot D \Leftrightarrow h = u_1 v_1 + i u_2 v_2,\ u_j, v_j \in D_h$$

Final fact & bad news:

There is an outer function f in D such that

$$\|f^2\|_{D\odot D} \leqslant \|1\|_D \|f^2\|_D < \|f\|_D \|f\|_D$$