Boundary behavior and invariant subspaces in spaces of analytic functions.

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Overview

Nontangential limits and some spaces of analytic functions

The index of invariant subspaces, single variable
(joint work with Alexandru Aleman and Carl Sundberg)

The multivariable situation

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ disc

$\mathbb{T} = \partial\mathbb{D}$

If $E \subseteq \mathbb{T}$, then

$$|E| = \text{ Lebesgue measure of } E.$$

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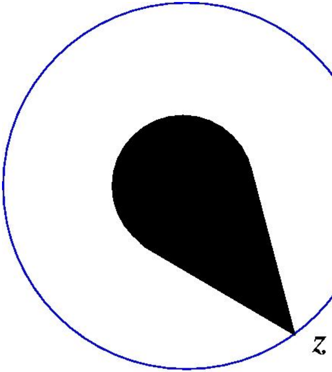
$\mathbb{B}_d = \{z \in \mathbb{C}^d : |z| < 1\}$ ball

$\mathbb{D}^d = \{z \in \mathbb{C}^d : |z_i| < 1 \ \forall i\}$ polydisc

$\Omega = \mathbb{B}_d$ or $\Omega = \mathbb{D}^d$

A nontangential approach region $\Gamma(z) \subseteq \mathbb{D}$, $z \in \mathbb{T}$.



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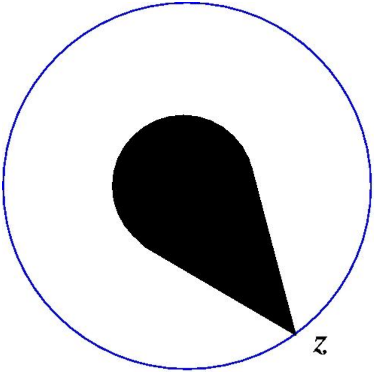

$$f \in \text{Hol}(\mathbb{D}), \qquad \text{ntl-}\lim_{\lambda \to z} f(\lambda) = A$$

Plessner's Theorem

Let $f \in \text{Hol}(\mathbb{D})$.
Then there is a measurable set $F \subseteq \mathbb{T}$ such that

1. for a.e. $z \in F$

$$\text{ntl-}\lim_{\lambda \to z} f(\lambda) = f(z) \text{ exists.}$$

2. for a.e. $z \in \mathbb{T} \setminus F$ we have

the nontangential cluster set of $f(\lambda)$ at z equals $\mathbb{C}$.

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It may happen that $|F| = 0$.

A reason for considering nontangential limits

Theorem (Luzin-Privalov)

Let $f \in$ $Hol(\mathbb{D})$ and let $E \subseteq \mathbb{T}$, $|E| > 0$, such that

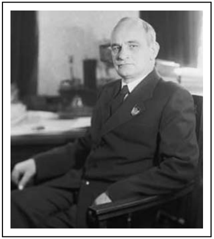
$$\textit{ntl-}\lim_{\lambda \to z} f(z) = 0 \quad \textit{for all } z \in E,$$

then $f = 0$.

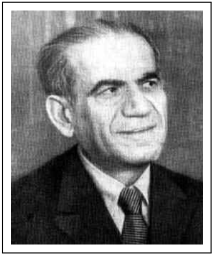
This is false for radial limits:

$$\exists \;\; f \neq 0 \quad \text{but} \quad \lim_{r \to 1^-} f(rz) = 0 \;\; a.e. \; z \in E, \; |E| > 0.$$

Nikolai Luzin
1883-1950



Ivan Privalov
1891-1941



Abraham Plessner
1900-1961

The Hardy space

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Facts:

- for every $f \in H^2$ the nontangential limit $f(e^{it})$ exists at almost every $e^{it} \in \mathbb{T}$.
- $f(e^{it}) \in L^2(\mathbb{T})$ and

$$\|f\|_{H^2}^2 = \int_0^{2\pi} |f(e^{it})|^2 \frac{dt}{2\pi}$$

The Bergman space

- $L_a^2 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|_{L_a^2}^2 = \sum_n \frac{|a_n|^2}{n+1} < \infty\}$

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Theorem (Khinchin-Kolmogorov)

If $\sum_{n\geqslant 0} |a_n|^2 = \infty$*, then for almost every choice of* $\varepsilon_n \in \{-1, 1\}$ *the function*

$$g(z) = \sum_{n\geqslant 0} \varepsilon_n a_n z^n$$

does not have nontangential limits a.e. on any $\Delta \subseteq \mathbb{T}$, $|\Delta| > 0$.

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<u>Fact:</u>

$$\|f\|_{L_a^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA}{\pi}$$

Multiplication operators

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- (M_z, H^2) unilateral shift
- (M_z, L^2_a) Bergman shift

Invariant subspaces

Let $\mathcal{H} = H^2$ or $\mathcal{H} = L^2_a$.

$\mathcal{M} \in$ Lat$(M_z, \mathcal{H})$ iff $\mathcal{M} \subseteq \mathcal{H}$ is a closed subspace and if

$$M_z \mathcal{M} \subseteq \mathcal{M}$$

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- **Cyclic invariant subspaces:** Let $f \in \mathcal{H}$, $f \neq 0$
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- **Zero-set based invariant subspaces:** Let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{D}$,
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Then

- $\dim [f] \ominus z[f] = 1$.
- $[f] = [g]$ iff they have the same wandering subspace.
- If $I(\{\lambda_n\}) \neq (0)$ is zero-set based, then $\dim \mathcal{M} \ominus z\mathcal{M} = 1$.

Beurling's Theorem, 1948

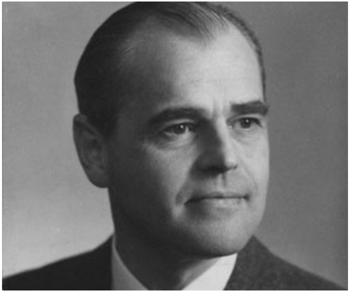
Theorem

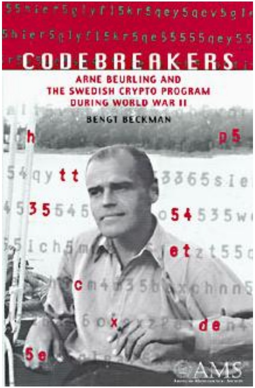
Let $(0) \neq \mathcal{M} \in Lat(M_z, H^2)$*, then*

- $\dim \mathcal{M} \ominus z\mathcal{M} = 1$,
- *if* $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, *then*

$$\mathcal{M} = [\varphi] = \varphi H^2,$$

- $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$ *is an inner function, i.e.* $|\varphi(z)| = 1$ *for a.e.* $|z| = 1$ *and* $\|\varphi f\|_{H^2} = \|f\|_{H^2}$ *for all* $f \in H^2$.





Arne Beurling (1905-1986)

Bergman space invariant subspaces

Theorem (Apostol, Bercovici, Foias, Pearcy, 1987)

If $n \in \mathbb{N} \cup \{\infty\}$, then there is $\mathcal{M} \in Lat(M_z, L^2_a)$ such that

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Corollary (Sandwich Theorem, ABFP)

If for all $\mathcal{M}, \mathcal{N} \in Lat(M_z, L^2_a)$, $\mathcal{M} \subseteq \mathcal{N}$, $dim\, \mathcal{N} \ominus \mathcal{M} > 1$*, there is* $\mathcal{K} \in Lat(M_z, L^2_a)$,

$$\mathcal{M} \subsetneqq \mathcal{K} \subsetneqq \mathcal{N},$$

then every operator on a Hilbert space of dim > 1 *has a nontrivial invariant subspace.*

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$$\mathcal{M} \subsetneq \mathcal{K} \subsetneq \mathcal{N},$$

then every operator on a Hilbert space of dim > 1 has a nontrivial invariant subspace.

Such a theorem was known to hold for the infinite multiplicity unilateral shift

$$S^{\infty} = (M_z, H^2) \oplus (M_z, H^2) \oplus ...$$

The index of invariant subspaces

- ind $\mathcal{M}$ $=$ dim $\mathcal{M} \ominus z\mathcal{M} =$ dim $\mathcal{M} \cap (z\mathcal{M})^{\perp} =$ dim$\mathcal{M}/z\mathcal{M}$

- ind$\mathcal{M} = 0 \Leftrightarrow \mathcal{M} = (0)$

- $f \neq 0 \Rightarrow$ ind$[f] = 1$

- If $\mathcal{M}, \mathcal{N} \in$ Lat$(M_z, \mathcal{H})$, then

$$\text{ind}(\mathcal{M} \cap \mathcal{N}) + \text{ind}(\mathcal{M} \vee \mathcal{N}) \leqslant \text{ind}\mathcal{M} + \text{ind}\mathcal{N}.$$

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- If dist$(\mathcal{M}_1, \mathcal{N}_1) > 0$, then $\mathcal{M} + \mathcal{N} = \mathcal{M} \vee \mathcal{N}$ and ind$\mathcal{M} \vee \mathcal{N} = 2$.

- **Question 1:** For which f, g do we have $\text{ind}[f] \vee [g] = 2$?
- **Question 2:** If $\text{ind}\mathcal{M} = \text{ind}\mathcal{N} = 1$, what is $\text{ind}\mathcal{M} \vee \mathcal{N}$?

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- Note that if $[f] = \mathcal{H}$, then for all $g \in \mathcal{H}$ one has ind$[f] \vee [g] =$ ind$\mathcal{H} = 1$.

- **Question 3:** Under what conditions on $\mathcal{M}$, ind$\mathcal{M} = 1$, is there an $\mathcal{N} \supseteq \mathcal{M}$ with ind$\mathcal{N} > 1$?

Reproducing kernels

- $\exists \quad k_\lambda \in \mathcal{H}$ - the reproducing kernel for $\mathcal{H}$
- If $f \in \mathcal{H}$, $\lambda \in \mathbb{D}$, then

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$$

- If $\mathcal{H} = H^2$, then $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$.
- If $\mathcal{H} = L_a^2$, then $k_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^2}$

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- k_λ satisfies $\|k_\lambda\|^2 = \langle k_\lambda, k_\lambda \rangle = k_\lambda(\lambda)$ and

  $\|k_\lambda\| = \sup\{|f(\lambda)| : f \in \mathcal{H}, \|f\| = 1\}$, equality for $f = \frac{k_\lambda}{\|k_\lambda\|}$.

The majorization function

$$
\begin{aligned}
k_{\mathcal{M}}(\lambda) &= \frac{\|P_{\mathcal{M}} k_\lambda\|}{\|k_\lambda\|} \\
&= \frac{\sup\{|g(\lambda)| : g \in \mathcal{M}, \|g\| = 1\}}{\sup\{|f(\lambda)| : f \in \mathcal{H}, \|f\| = 1\}}
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- $0 \leqslant k_{\mathcal{M}}(\lambda) \leqslant 1$
- If $\lambda_0 \in Z(\mathcal{M})$, i.e. $f(\lambda_0) = 0 \ \forall f \in \mathcal{M}$, then $k_{\mathcal{M}}(\lambda_0) = 0$.
- If $\lambda_n \to z \in \mathbb{T}$, $k_{\mathcal{M}}(\lambda_n) \to 0$, that should mean that

"$\mathcal{M}$ has a zero at $z \in \mathbb{T}$."

The majorization function in H^2

- If $\mathcal{M} = \varphi H^2$, φ inner, then $k_{\mathcal{M}}(\lambda) = |\varphi(\lambda)|$.

$$
\begin{aligned}
k_{\mathcal{M}}(\lambda) &= \frac{\sup\{|g(\lambda)| : g \in \varphi H^2, \|g\| = 1\}}{\sup\{|f(\lambda)| : f \in H^2, \|f\| = 1\}} \\
&= \frac{\sup\{|\varphi(\lambda) f(\lambda)| : f \in H^2, \|f\| = 1\}}{\sup\{|f(\lambda)| : f \in H^2, \|f\| = 1\}} \\
&= |\varphi(\lambda)|
\end{aligned}
$$

- $\text{ntl-}\lim_{\lambda \to z} k_{\mathcal{M}}(\lambda) = \text{ntl-}\lim_{\lambda \to z} |\varphi(\lambda)| = 1$ for a.e. $z \in \mathbb{T}$.

The majorization function in L^2_a.

If $z \in \mathbb{T}$ and if $f \in L^2_a$ extends to be continuous at z with $f(z) \neq 0$, then

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- $f \in \mathcal{M} \Rightarrow g = \frac{k_\lambda f}{\|k_\lambda f\|} \in \mathcal{M}, \quad k_\lambda = \frac{1}{(1-\bar{\lambda}z)^2}$ = Bergman kernel
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- $$\begin{aligned} 1 \geqslant k_{\mathcal{M}}(\lambda)^2 &= \frac{\|P_{\mathcal{M}} k_\lambda\|^2}{\|k_\lambda\|^2} \geqslant \frac{|g(\lambda)|^2}{\|k_\lambda\|^2} \\ &= \frac{1}{\|k_\lambda\|^2} \frac{\|k_\lambda\|^4 |f(\lambda)|^2}{\|k_\lambda f\|^2} \\ &= \frac{|f(\lambda)|^2}{\frac{\|k_\lambda f\|^2}{\|k_\lambda\|^2}} \end{aligned}$$

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- $\lim_{\lambda \to z} |f(\lambda)|^2 = |f(z)|^2 \neq 0$

To show: $\lim_{\lambda\to z} \frac{\|k_\lambda f\|^2}{\|k_\lambda\|^2} = |f(z)|^2$

Approximate identity:

$$\frac{\|k_\lambda f\|^2}{\|k_\lambda\|^2} = \int_{\mathbb{D}} |f(z)|^2 d\mu_\lambda(z) \quad \text{where}$$

$$d\mu_\lambda(z) = \frac{1}{\pi}\frac{|k_\lambda(z)|^2 dA(z)}{\|k_\lambda\|^2} = \frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}z|^4}\frac{dA(z)}{\pi}$$

and $\mu_\lambda \to \delta_z$ (wk*) as $\lambda \to z$ and the result follows since f is continuous at z.

The majorization function in L^2_a.

Theorem (ARS, 2002)

$\mathcal{M} \in$ *Lat*(M_z, L^2_a), *ind*$\mathcal{M} = 1$, $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$.

Then $\exists\, F \subseteq \mathbb{T}$ *such that*

1. *for a.e.* $z \in F$
   - *ntl*-$\lim_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 1$ *and*
   - *ntl*-$\lim_{\lambda \to z} \varphi(\lambda) = \varphi(z)$ *exists with* $|\varphi(z)| \geqslant 1$.
2. *for a.e.* $z \in \mathbb{T} \setminus F$ *we have*
   - *ntl*-$\liminf_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 0$,
   - *the nontangential cluster set of* $\varphi(\lambda)$ *at* z *equals* $\mathbb{C}$.

Def.: The Fatou set for $\mathcal{M} = F =$ the Fatou set for φ (as in Plessner's Theorem).

Theorem (ARS)

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TFAE

1. $\exists\, \mathcal{N} \supseteq \mathcal{M}$ *ind*$\mathcal{N} > 1$,
2. *for a.e.* $z \in \mathbb{T}$ *ntl*-$\liminf_{\lambda \to z} k_{\mathcal{M}}(\lambda) < 1$,
   i.e. the Fatou set for $\mathcal{M}$ *has measure 0*
3. $M_z^*|\mathcal{M}^\perp \in \mathbb{A}$.

$$M_z^*|\mathcal{M}^\perp \in \mathbb{A} \Leftrightarrow \|M_g^*|\mathcal{M}^\perp\| = \|g\|_\infty \;\forall\, g \in H^\infty.$$

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2. *for a.e.* $z \in \mathbb{T} \quad ntl\text{-}\liminf_{\lambda \to z} k_{\mathcal{M}}(\lambda) < 1$,
   i.e. the Fatou set for $\mathcal{M}$ *has measure 0*
3. $M_z^* | \mathcal{M}^\perp \in \mathbb{A}$.

$$M_z^* | \mathcal{M}^\perp \in \mathbb{A} \Leftrightarrow \| M_g^* | \mathcal{M}^\perp \| = \|g\|_\infty \ \forall \ g \in H^\infty .$$

The Theorem holds for many other spaces $\mathcal{H} \subseteq$ Hol$(\mathbb{D})$ including H^2:

$\mathcal{H} = \mathcal{H}(k)$,
$\|M_z\| \leqslant 1, \ \text{ind}\mathcal{H} = 1, \quad \| \frac{z-\lambda}{1-\bar{\lambda} z} f \| \geqslant c \|f\| \quad \forall f \in \mathcal{H}, \lambda \in \mathbb{D}$

Theorem (opposite equivalences)

$\mathcal{M} \in$ *Lat*(M_z, L_a^2), *ind*$\mathcal{M} = 1$.
TFAE

1. $\forall\, \mathcal{N} \supseteq \mathcal{M}$ *ind*$\mathcal{N} = 1$,
2. $\exists\, F \subseteq \mathbb{T}, |F| > 0$ *ntl-* $\lim_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 1$ *for a.e.* $z \in F$.
   i.e. the Fatou set for $\mathcal{M}$ *has positive measure.*
   Fatou set of $\mathcal{M}$ = *Fatou set of* $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\varphi \neq 0$.

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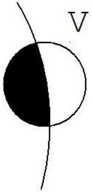
- If the Fatou set for every nonzero function in $\mathcal{M}$ has measure zero, then the Fatou set for $\mathcal{M}$ has measure zero.
  Example: If $\{\lambda_n\}$ is dominating for $\mathbb{T}$, and if $\mathcal{M} = I(\{\lambda_n\}) \neq (0)$, then $|F| = 0$.

Local Example in L^2_a



Let $f \in L^2_a \cap H^\infty(V \cap \mathbb{D})$,
then the Fatou set of $[f]$ contains $\mathbb{T} \cap V$.

Hence any $\mathcal{M}$ containing f has index 1.

In fact: If $f \in \mathcal{M} \Rightarrow k_{\mathcal{M}}(\lambda) \to 1$ a.e. nontangentially on $V \cap \mathbb{T}$.

Proof:
A refinement of the argument given earlier.

$d > 1$

$\Omega = \mathbb{B}_d$ or $\mathbb{D}^d$
$\mathcal{H} = H^2(\Omega)$ Hardy space
or $\mathcal{H} = L^2_a(\Omega)$ Bergman space
or $\mathcal{H} = H^2_d$ Drury-Arveson space

reproducing kernels:

H^2_d $\quad k_\lambda(z) = \frac{1}{1-\langle z,\lambda\rangle}$

$H^2(\mathbb{B}_d)$ $\quad k_\lambda(z) = \frac{1}{(1-\langle z,\lambda\rangle)^d}$

$L^2_a(\mathbb{B}_d)$ $\quad k_\lambda(z) = \frac{1}{(1-\langle z,\lambda\rangle)^{d+1}}$

$(M_z, \mathcal{H}) = ((M_{z_1}, \dots, M_{z_d}), \mathcal{H})$

Koszul complex

$\lambda = (\lambda_1, ..., \lambda_d) \in \Omega$

$$T_\lambda : \mathcal{H}^d \to \mathcal{H},$$
$$T_\lambda(f_1, ..., f_d) = \sum_{i=1}^{d} (z_i - \lambda_i) f_i.$$

$$K(M_{z-\lambda}, \mathcal{H}) : 0 \to \Lambda^1(\mathcal{H}) \to ... \to \Lambda^{d-1}(\mathcal{H}) \to \Lambda^d(\mathcal{H}) \to 0$$
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For all $\mathcal{H}$ as above $T_\lambda(\mathcal{H})$ is closed and

$$\dim \mathcal{H}/T_\lambda(\mathcal{H}) = \dim \mathcal{H} \ominus T_\lambda(\mathcal{H}^d) = 1.$$

and the $K(M_{z-\lambda}, \mathcal{H})$ is exact at all other stages.

Invariant subspaces and index

$\mathcal{M} \in$ Lat$(M_z, \mathcal{H}) \Leftrightarrow M_{z_i}\mathcal{M} \subseteq \mathcal{M}$ for all $i = 1, \ldots, d$

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Question: Is $T_\lambda(\mathcal{M}^d)$ always closed?

More generally: If $(z_1 - \lambda_1)\mathcal{H} + ... + (z_d - \lambda_d)\mathcal{H}$ is closed in $\mathcal{H}$, under what hypothesis on the invariant subspace $\mathcal{M}$ is

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Definition: $\text{ind}_\lambda \mathcal{M} = \dim \mathcal{M}/\overline{T_\lambda(\mathcal{M}^d)} = \mathcal{M} \ominus T_\lambda(\mathcal{M}^d)$

Conjecture: If $(0) \neq \mathcal{M} \in \text{Lat}(M_z, H^2(\Omega))$, then

$$\text{ind}_\lambda \mathcal{M} = 1 \text{ for all } \lambda \text{ in some large subset of } \Omega.$$

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- True for $d = 1$ for all $\lambda \in \mathbb{D}$.
- $H^2(\mathbb{B}_d)$: If $k_{\mathcal{M}}(\lambda) = \frac{\|P_{\mathcal{M}} k_\lambda\|}{\|k_\lambda\|}$, then

$$\text{ntl-}\lim_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 1 \text{ for a.e. } z \in \partial\mathbb{B}_d$$

— distinguished boundary for $H^2(\mathbb{D}^d)$ —

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- True for $\mathcal{M}_0 = \{f \in H^2(\Omega) : f(0) = 0\}$:

$$\text{ind}_\lambda \mathcal{M}_0 = 1 \quad \text{for all } \lambda \in \Omega \setminus \{0\},$$

  i.e. outside the common zero set of $\mathcal{M}_0$.

Fact:

$$T : H^2(\mathbb{D}^2) \to L^2_a,$$
$$(Tf)(\lambda) = f(\lambda, \lambda)$$

is onto and a partial isometry.

By use of this one can see that there are $\mathcal{M} \in \text{Lat}(M_z, H^2(\mathbb{D}^2))$ with $Z(\mathcal{M}) = \emptyset$ and

$$\text{ind}_{(\lambda,\lambda)}\mathcal{M} = \infty \text{ for all } \lambda \in \mathbb{D}.$$

These examples satisfy $\text{ind}_{(\lambda,\mu)}\mathcal{M} = 1$ for all $\lambda \neq \mu$.

Similar examples exist in $H^2(\mathbb{B}_d)$.

If $f, g, h \in \mathcal{H}$, if

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- If $f, g \in H^4(\Omega)$, then $fg \in [f] \cap [g] \subseteq H^2(\Omega)$, hence

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- If $\mathcal{M} \in \text{Lat}(M_z, H^2_d)$, then

$$\text{ind}_\lambda \mathcal{M} = 1 \quad \forall \lambda \in \Omega \setminus Z(\mathcal{M}).$$

(”global results”)

A local result

$f, g \in H^2(\mathbb{D}^2)$
Let V be an open disc in $\mathbb{C}$ centered at a boundary point of $\mathbb{D}$ in the z_1-variable, and suppose that

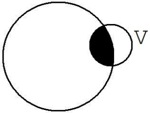
$$\exists C > 0 \quad \forall \lambda \in (\mathbb{D} \cap V) \times \mathbb{D} : \quad |f(\lambda)|, |g(\lambda)| \leqslant C.$$

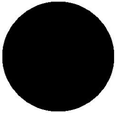
Then

$$\mathrm{ind}_\lambda [f] \vee [g] = 1$$

for all

$$\lambda \in (\mathbb{D} \cap V) \times \mathbb{D}, \lambda \notin Z(f) \cup Z(g).$$





In $L_a^2(\Omega)$, there are $\mathcal{M}$ with $\text{ind}_\lambda \mathcal{M} = \infty$ for at least a dense set of $\lambda \in \Omega$.

(Bercovici for $L_a^2(\Omega)$ (1987), or newer $\mathbb{A}_1$-type results of Eschmeier (2001), Ambrozie-Mueller (2009))