Cyclic vectors in the Drury-Arveson space.

Stefan Richter
joint work with Carl Sundberg

Department of Mathematics
The University of Tennessee, Knoxville

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Multiindex notation: If $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$, then

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$$

$$\alpha! = \alpha_1! \cdots \alpha_d! \qquad |\alpha| = \alpha_1 + ... + \alpha_d$$

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Other spaces of analytic functions in $\mathbb{B}_d$:

- $H^2(\partial \mathbb{B}_d)$ – $k_w(z) = \left(\frac{1}{1-\langle z,w \rangle} \right)^d$
- $L^2_a(\mathbb{B}_d)$ – $k_w(z) = \left(\frac{1}{1-\langle z,w \rangle} \right)^{d+1}$

Interesting fact about (M_z, H^2_d):

Each M_{z_i} is subnormal, but (M_z, H^2_d) is not jointly subnormal.

- (M_{z_1}, H^2_d) u.e. $(M_z, H^2(\mathbb{D})) \oplus \big((M_z, L^2_a(\mathbb{D})) \otimes I_d\big) \oplus ...$

- $M(H^2_d) \subsetneqq H^\infty(\mathbb{B}_d)$

H^2_d has been proposed as a possible analogue of $H^2(\mathbb{D})$, useful for the study of commuting row contractions $T = (T_1, ..., T_d)$

$$\|(T_1, \ldots, T_d) \begin{pmatrix} x_1 \\ \\ x_d \end{pmatrix} \|^2 = \| \sum_j T_j x_j \|^2 \leqslant \|x_1\|^2 + \cdots + \|x_d\|^2$$

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<u>Evidence and successes:</u>

- dilation theorem (Drury, Mueller-Vasilescu, Arveson)
  $\exists\, \mathcal{M} \in \ \mathrm{Lat}(M_z \otimes I) \oplus U$

$$T_i = P_{\mathcal{M}^\perp}\left((M_{z_i} \otimes I) \oplus U_i\right) |\mathcal{M}^\perp, \ \ i = 1, ..., d$$

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- Corona theorem for $M(H_d^2)$ (Costea-Sawyer-Wick)

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Ideally a function theory of H_d^2 would take advantage of results about Besov spaces of the ball and the special form of k_w.

Thm (Gleason-R-Sundberg)

If $\mathcal{M} \in Lat(M_z, H^2_d)$*, if* $T_i = P_{\mathcal{M}^\perp} M_{z_i} | \mathcal{M}^\perp$ *for all i, then*

$$\sigma(T) \cap \mathbb{B}_d = Z(\mathcal{M}) = \{z \in \mathbb{B}_d : f(z) = 0 \, \forall f \in \mathcal{M}\}.$$

Remark: This is false for some invariant subspaces of $H^2(\partial B_d)$, $d > 1$.

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What about $\sigma(T) \cap \partial \mathbb{B}_d$?

invariant subspaces - inner sequences

Thm (McCullough-Trent, Arveson)

If $\mathcal{M} \in Lat(M_z, H^2_d)$*, then there are* $\varphi_1, \varphi_2, \ldots \in M(H^2_d)$ *such that*

$$P_{\mathcal{M}} = \sum_n M_{\varphi_n} M^*_{\varphi_n}$$

$$\mathcal{M} = \left\{ \sum_n \varphi_n f_n : f_n \in H^2_d, \sum_n \|f_n\|^2 < \infty \right\}$$

Thm (Greene-R-Sundberg)

$$\sum_n |\varphi_n(z)|^2 = 1 \ a.e. \ z \in \partial\mathbb{B}_d$$

Defn

$$\underline{Z}(\mathcal{M}) = Z(\mathcal{M}) \cup \{w \in \partial\mathbb{B}_d : \liminf_{z\to w} \sum_n |\varphi_n(z)|^2 = 0\}$$

Conjecture:

$$\underline{Z}(\mathcal{M}) = \sigma(T)$$

Question

If $w \in \sigma(T) \cap \partial\mathbb{B}_d$, then what can one conclude about the functions in $\mathcal{M}$ near w?

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$f(z) = 1$ is cyclic since polynomials are dense in H_d^2

$$f \text{ is cyclic } \Leftrightarrow \exists \text{ polys } p_n \text{ such that } p_n f \to 1.$$

Hence, of course: If f is cyclic, then $f(z) \neq 0$ for all $z \in \mathbb{B}_d$.

$f(z_1, ..., z_d) = 1 - z_1$ is cyclic in H_d^2.

The H^2_d-norm

$$k_w(z) = \frac{1}{1-\langle z,w\rangle} = \sum_{n\geqslant 0}\langle z,w\rangle^n = \sum_{n\geqslant 0}\sum_{|\alpha|=n}\frac{|\alpha|!}{\alpha!}z^\alpha\overline{w}^\alpha$$

If $f \in \operatorname{Hol}(\mathbb{B}_d)$, then

$$f(z) = \sum_{n\geqslant 0}\left(\sum_{|\alpha|=n}\hat{f}(\alpha)z^\alpha\right)$$

and

$$\|f\|^2_{H^2_d} = \sum_{n\geqslant 0}\sum_{|\alpha|=n}\frac{\alpha!}{|\alpha|!}|\hat{f}(\alpha)|^2$$

Slice functions

If $f \in H_d^2$, set $f_1(\lambda) = f(\lambda, 0, ..., 0)$, then

$$\begin{aligned} \|f_1\|_{H^2(\mathbb{D})}^2 &= \sum_{n \geqslant 0} |\hat{f}(n, 0..., 0)|^2 \\ &\leqslant \sum_{n \geqslant 0} \sum_{|\alpha| = n} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2 = \|f\|_{H_d^2}^2 \end{aligned}$$

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$$\leqslant \sum_{n \geqslant 0} \sum_{|\alpha| = n} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2 = \|f\|^2_{H^2_d}$$

Hence $f_1 \in H^2(\mathbb{D})$ and, if f is cyclic in H^2_d, then

$$\|p_{n,1} f_1 - 1\|^2_{H^2(\mathbb{D})} \leqslant \|p_n f - 1\|_{H^2_d} \to 0$$

Thus, f_1 is outer in $H^2(\mathbb{D})$.

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Thm

If $f \in H^2_d$ is cyclic, then for all $z \in \partial\mathbb{B}_d$

$$f_z \text{ is outer in } H^2(\mathbb{D}).$$

$$f_z(\lambda) = f(\lambda z), \quad \lambda z = (\lambda z_1, ..., \lambda z_d)$$

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Thm

If $f \in H_d^2$ is cyclic, then f must be cyclic in $P^2(\mu)$ for every representing measure for the ball algebra.

$$\int_{\partial \mathbb{B}_d} p d\mu = p(0), \qquad \mu \geqslant 0,$$

The converse is false.

Take functions with Taylor coefficients supported on the diagonal:

$$f(z_1, ..., z_d) = \sum_{\alpha} \hat{f}(\alpha) z_1^{\alpha_1} \cdots z_d^{\alpha_d}$$

$$\hat{f}(\alpha) = 0 \text{ unless } \alpha_1 = \alpha_2 = ... = \alpha_d.$$

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Let $R_d = d^{d/2}$, then

$$|z|^2 = \sum_{j=1}^{d} |z_j|^2 < 1 \quad \Rightarrow \quad R_d |z_1 z_2 \cdots z_d| < 1$$

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If $g \in \mathrm{Hol}(\mathbb{D})$, then set

$$f(z) = T(g)(z_1, ..., z_d) = g(R_d z_1 z_2 \cdots z_d) \in \mathrm{Hol}(\mathbb{B}_d)$$

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$$\|f\|^2_{H^2_d} = \|T(g)\|^2_{H^2_d} \approx \sum_{n \geqslant 0} (n+1)^{\frac{d-1}{2}} |\hat{g}(n)|^2 = \|g\|^2_{D_{\frac{d-1}{2}}}$$

If p is any polynomial in d variables with diagonal terms $p_{diag} = p_{diag}(z_1 \cdots z_d)$, then

$$\|pT(g) - 1\|^2_{H^2_d} \geqslant \|p_{diag}T(g) - 1\|^2_{H^2_d} \approx \|p_{diag}g - 1\|^2_{D_{\frac{d-1}{2}}}$$

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For $d \geqslant 2$ there is a noncyclic $f \in H^2_d$ such that f_z is outer in $H^2(\mathbb{D})$ for all $z \in \partial\mathbb{B}_d$.

Polynomials

If $\|g\|^2_{D_{\frac{d-1}{2}}} = \sum_{n\geqslant 0}(n+1)^{\frac{d-1}{2}}|\hat{g}(n)|^2$, then $D_{\frac{d-1}{2}} \subseteq A(\overline{\mathbb{D}})$ if $d \geqslant 4$

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Question

For $d = 3$, is a polynomial p cyclic in H^2_3 if and only if $p \neq 0$ on $\mathbb{B}_d$?

Thm (Cascante-Ortega, 1995)

Let $1 < \tau \leqslant d$

$$D_\tau(\zeta) = \{z \in \mathbb{B}_d : |1 - \langle z, \zeta \rangle|^\tau < 1 - |z|\}.$$

Then $M_\tau f \in L^2(\mu)$ *for all* $f \in H^2_d$ *and all* μ *such that*

$$\mu(B(\zeta, \delta)) = O(\delta^\tau)$$

where

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Cor

If

$$Z(f) = \{\zeta \in \partial\mathbb{B}_d : D_\tau - \lim_{z\to\zeta} f(z) = 0\}$$

and if $\mu(Z(f)) > 0$ *for any* μ *as above, then* f *is not cyclic in* H^2_d*.*

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then must f be cyclic?

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Known to be true, if

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- $d \leqslant 3$ (R-Sundberg)

H^2_d-norm as a Besov space norm

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Fact:

- If d is even, then

$$\|f\|^2_{H^2_d} \approx \int_{|z|<0.5} |f|^2 dV \; + \; \|R^{d/2} f\|^2_{L^2_a(\mathbb{B}_d)}$$

- If d is odd, then

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Thus, if $d = 2, 3$, then we only need to consider Rf.

Thm ($d \leqslant 3$)

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Strategy:

- $\varphi = g/f \in H^\infty$
- $\varphi_r f \in [f]$ for all $r < 1$
- $\varphi_r(z) f(z) \to g(z)$ in $\mathbb{B}_d$
- Need $\|\varphi_r f\|_{H^2_d} \leqslant C$

The proof of $\|\varphi_r f\|_{H^2_d} \leqslant c$ for $d = 2, 3$

$\|\ \|$ norms are $H^2(\partial\mathbb{B}_d)$- or $L^2_a(\mathbb{B}_d)$-norms.

Facts:

- $\|f - f_r\| \leqslant (1-r)\|Rf\|$
- $\|R\varphi_r\|_\infty \leqslant \frac{\|\varphi\|_\infty}{1-r}$

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$$\begin{aligned}
\|R(\varphi_r f)\| &\leqslant \|R(\varphi_r(f - f_r))\| + \|R(\varphi_r f_r)\|, \quad \varphi f = g \\
&\leqslant \|(R\varphi_r)(f - f_r)\| + \|\varphi_r R(f - f_r)\| + \|Rg\| \\
&\leqslant \frac{\|\varphi\|_\infty}{1-r}\ \|f - f_r\| \ + \ \|\varphi\|_\infty \|R(f - f_r)\| \ + \ \|g\| \\
&\leqslant \frac{\|\varphi\|_\infty}{1-r}(1-r)\|Rf\| + M \leqslant M'
\end{aligned}$$

J. Xia's argument:

Suppose $f \in M(H^2_d)$, $1/f \in H^\infty(\mathbb{B}_d)$

to show: $1/f \in M(H^2_d)$

$d = 2, 3$

- $R\left(\frac{g}{f}\right) = \frac{fRg - gRf}{f^2}$
- $R(fg) = gRf + fRg$
- $R\left(\frac{g}{f}\right) = 2\frac{Rg}{f} - \frac{R(fg)}{f^2}$

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$d = 4, 5$

$$R^2\left(\frac{g}{f}\right) = 4\frac{R^2 g}{f} - 2\frac{R(fRg)}{f^2} - 3\frac{R^2(fg)}{f^2} + 2\frac{R(fR(fg))}{f^3}$$