Graph transformations of the Bergman shift.

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$\mathbb{D}$ unit disc, $\quad \mathbb{T} = \partial\mathbb{D}$

$H^2 = H^2(\mathbb{D})$ the Hardy space

$L^2_a = \{f \in \text{ Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f|^2 dA < \infty\}$ the Bergman space

$D = \{f \in \text{ Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 dA < \infty\}$ the Dirichlet space

If $\mathcal{H} \subseteq \text{Hol}(\Omega)$, then

$$M(\mathcal{H}) = \{M_\varphi : \varphi\mathcal{H} \subseteq \mathcal{H}\} \subseteq \mathcal{B}(\mathcal{H}),$$

the algebra of multiplication operators on $\mathcal{H}$.

$\mathcal{M} \in \operatorname{Lat}(M_z, \mathcal{H})$ iff $z\mathcal{M} \subseteq \mathcal{M}$

$$\underbrace{D \quad \subseteq \quad H^2}_{} \qquad \underbrace{\subseteq L_a^2}_{}$$

$\forall n \in \mathbb{N}$

$\forall \mathcal{M} \in \operatorname{Lat}(M_z, \mathcal{H}), \mathcal{M} \neq (0)$ $\quad$ $\exists \mathcal{M}_n \in \operatorname{Lat}(M_z, \mathcal{H})$

$\dim \mathcal{M} \ominus z\mathcal{M} = 1$ $\quad$ $\dim \mathcal{M} \ominus z\mathcal{M} = n$

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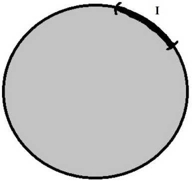
$\dim \mathcal{M} \ominus z\mathcal{M} = n$

$$d\mu = dA|\mathbb{D} + \chi_I |dz|, \quad I \subseteq \partial\mathbb{D}$$



$$\underbrace{D \quad \subseteq \quad H^2}_{\substack{\forall \mathcal{M},\mathcal{N} \in \mathrm{Lat}(M_z,\mathcal{H}) \\ \mathcal{M},\mathcal{N} \neq (0) \Rightarrow \mathcal{M}\cap\mathcal{N} \neq (0)}} \quad \subseteq \quad \underbrace{P^2(\mu) \quad \subseteq L^2_a}_{\substack{\exists\ \mathcal{M},\mathcal{N} \in \mathrm{Lat}(M_z,\mathcal{H}) \\ \mathcal{M}\cap\mathcal{N} = (0) \text{ and} \\ \mathcal{M}+\mathcal{N} \text{ is dense in } \mathcal{H}}}$$

Well-known for L^2_a. Horowitz uses zero sets, applies to $P^2(\mu)$.

We'll see that it is interesting to have a refinement of this fact, not based on zero sets.

$\mathcal{H}$ separable Hilbert space, $T \in \mathcal{B}(\mathcal{H})$

$\text{Lat}T = \{\mathcal{M} : T\mathcal{M} \subseteq \mathcal{M}\}$

(ISP) - invariant subspace problem

(HISP) hyperinvariant subspace problem
If $T \neq \lambda I$, then $\exists\ \mathcal{M}$, $\mathcal{M} \neq (0), \mathcal{H}$ such that $S\mathcal{M} \subseteq \mathcal{M}$ for all S with $ST = TS$?

TAP - The transitive algebra problem, (Kadison, 1957)

Definition: An algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called transitive, if

- $1 \in \mathcal{A}$
- $\mathcal{A}$ is SOT-closed
- $\operatorname{Lat} \mathcal{A} = \{(0), \mathcal{H}\}$

Example: $\mathcal{A} = \mathcal{B}(\mathcal{H})$

Problem

(TAP) If $\mathcal{A}$ is transitive, then $\mathcal{A} = \mathcal{B}(\mathcal{H})$?

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(TAP) $\Rightarrow$ (HISP)

$\mathcal{A} = \{T\}'$ and suppose $\mathrm{Lat}\,\mathcal{A} = \{(0), \mathcal{H}\}$, then assuming (TAP) $\mathcal{A} = \mathcal{B}(\mathcal{H})$, hence $T = \lambda I$.

Known results on TAP

Theorem

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- $\mathcal{A}$ contains a unilateral shift of finite multiplicity
  multiplicity 1 -Arveson, 1967,
  higher finite multiplicities - Nordgren, 1970

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- $\mathcal{A}$ contains the Dirichlet shift (M_z, D) (R, 1988)

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  higher finite multiplicities - Nordgren, 1970
- $\mathcal{A}$ contains the Dirichlet shift (M_z, D) (R, 1988)
- $M(\mathcal{H}) \subseteq \mathcal{A}$ -
  the multiplier algebra for a space $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ with complete NP kernel.
  (Cheng, Guo, Wang, 2010)
  Actually, this is more general, it includes finite multiplicities and restrictions to invariant subspaces.
  It includes unilateral and Dirichlet shifts.

Open Problem: If $\mathcal{A}$ is transitive and if $B = (M_z, L_a^2) \in \mathcal{A}$, then $\mathcal{A} = \mathcal{B}(\mathcal{H})$?

More generally: If $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ and if $M(\mathcal{H}) \subseteq \mathcal{A}$, then what extra hypothesis is needed to imply $\mathcal{A} = \mathcal{B}(\mathcal{H})$?

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The basic tool is Arveson's Lemma, which uses invariant graph subspaces.

Invariant graph subspaces

$n \geqslant 1$
$\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$
$A^{(n)} = A \oplus \ldots \oplus A$

$\mathcal{M}$ is an invariant graph subspace of $\mathcal{A}$, IGS of $\mathcal{A}$, if

- $\mathcal{M}$ is a closed subspace of $\mathcal{H}^{(n)}$
- $A^{(n)}\mathcal{M} \subseteq \mathcal{M}$ for all $A \in \mathcal{A}$
- $\mathcal{M} = \{f, T_1 f, \ldots, T_{n-1} f) : f \in \mathcal{D}\}, \quad T_i : \mathcal{D} \to \mathcal{H}$

i.e. $\mathcal{M} \in \operatorname{Lat} \mathcal{A}^{(n)}$ is determined by the 1st component

Note:

$$x = (f, T_1 f, \ldots, T_{n-1} f) \in \mathcal{M}$$

$$A^{(n)} x = (Af, AT_1 f, \ldots, AT_{n-1} f) \in \mathcal{M}$$

$$\Leftrightarrow \qquad \forall\, i : \; AT_i = T_i A, \;\; A\mathcal{D} \subseteq \mathcal{D}$$

T_i are the linear graph transformations of $\mathcal{A}$

Examples

Example (Multiplication by a meromorphic function)

$f, g \in \mathcal{H} \subseteq \text{Hol}(\Omega) \qquad \mathcal{A} = M(\mathcal{H})$

$[f] = \overline{\{\varphi f : \varphi \in M(\mathcal{H})\}}$

$\mathcal{D} = \{h \in [f] : \frac{g}{f} h \in [g]\}$

Then $\{\varphi f : \varphi \in M(\mathcal{H})\} \subseteq \mathcal{D} \subseteq [f]$

$T = M_{\frac{g}{f}}$ is a multiplication

$\mathcal{M} = \{(h, \frac{g}{f} h) : h \in \mathcal{D}\}$ is an IGS of $M(\mathcal{H})$.

The main example

Let $\mathcal{L}, \mathcal{N} \in \text{Lat}(M_z, L^2_a)$ with $\mathcal{L}, \mathcal{N} \neq (0)$ and $\mathcal{L} \cap \mathcal{N} = (0)$
(such $\mathcal{L}$ and $\mathcal{N}$ exist in L^2_a, but not in D, H^2)

The main example

Let $\mathcal{L}, \mathcal{N} \in \operatorname{Lat}(M_z, L_a^2)$ with $\mathcal{L}, \mathcal{N} \neq (0)$ and $\mathcal{L} \cap \mathcal{N} = (0)$

$\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$

$\mathcal{D} = \mathcal{L} + \mathcal{N}$ $\quad T(f+g) = \varphi f + \psi g$

$\mathcal{M} = \{(f+g, \varphi f + \psi g) : f \in \mathcal{L}, g \in \mathcal{N}\}$

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$\mathcal{M} = \{(f+g, \varphi f + \psi g) : f \in \mathcal{L}, g \in \mathcal{N}\}$

$\mathcal{M}$ is closed:

$$\left\{ \begin{array}{c} f_n + g_n \to u \\ \varphi f_n + \psi g_n \to v \\ \varphi f_n + \varphi g_n \to \varphi u \end{array} \right\} \quad \Rightarrow (\psi - \varphi) g_n \to v - \varphi u \in \mathcal{N}$$

$$\Rightarrow g_n \to \frac{v - \varphi u}{\psi - \varphi} \in \mathcal{N}$$

$$\Rightarrow f_n \to f \in \mathcal{L}$$

Example ($\mathcal{A} = \mathcal{B}(\mathcal{H})$)

If $\mathcal{M}$ is an IGS of $\mathcal{B}(\mathcal{H})$, then

- $A\mathcal{D} \subseteq \mathcal{D} \ \forall A \in \mathcal{B}(\mathcal{H}) \Rightarrow \mathcal{D} = \mathcal{H}$
- $AT_i = T_iA \ \forall A \in \mathcal{B}(\mathcal{H}) \Rightarrow T_i = \lambda_i I$

Theorem (Arveson's Lemma)

If

- *$\mathcal{A}$ is transitive, and*
- *whenever $\mathcal{M}$ is an IGS for $\mathcal{A}$, then $T_i = \lambda_i I$,*

then $\mathcal{A} = \mathcal{B}(\mathcal{H})$

Suppose $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$

Let $\mathcal{M}$ be an IGS for $M(\mathcal{H})$, set

$$\begin{aligned} \mathcal{A}_{\mathcal{M}} &= \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{D} \subseteq \mathcal{D}, T_i A = A T_i \ \forall\, i\} \\ &= \text{ the largest subalgebra such that } \mathcal{M} \text{ is an IGS of } \mathcal{A} \end{aligned}$$

Then

$$M(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathcal{M}}$$

An $\mathcal{A}_{\mathcal{M}} \neq \mathcal{B}(\mathcal{H})$ with $\mathrm{Lat}\mathcal{A}_{\mathcal{M}} = \{(0), \mathcal{H}\}$ would be a counterexample to TAP.

$$\mathcal{A}_{\mathcal{M}} = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{D} \subseteq \mathcal{D}, T_i A = A T_i \ \forall\, i\}$$

Lat$\mathcal{A}_{\mathcal{M}}$ - some obvious examples

$\alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in \mathbb{C}^n \quad L_\alpha = \alpha_0 I + \sum_{i=1}^{n-1} \alpha_i T_i$

Then $L_\alpha A = A L_\alpha$ for all $A \in \mathcal{A}_{\mathcal{M}}$, hence

$$\overline{\ker L_\alpha}, \quad \overline{\operatorname{ran} L_\alpha} \in \ \mathrm{Lat}\mathcal{A}_{\mathcal{M}}$$

Easy fact: If $T_i = M_\varphi$, then

$$\overline{\operatorname{ran}\,(M_\varphi - \varphi(\lambda))} \neq \mathcal{H}$$

Consequence: If $\mathcal{A}_{\mathcal{M}}$ is transitive, then any T_i that is a multiplication is $T_i = \lambda_i I$.

Definition:

$$\begin{aligned}\mathcal{M}_\lambda &= \{(f(\lambda), (T_1 f)(\lambda), ..., (T_{n-1} f)(\lambda)) : f \in \mathcal{D}\} \subseteq \mathbb{C}^n \\ &= \text{ the fiber of } \mathcal{M} \text{ at } \lambda\end{aligned}$$

$$\begin{aligned}fd\mathcal{M} &= \sup_{\lambda \in \Omega} \dim \mathcal{M}_\lambda \\ &= \text{ the fiber dimension of } \mathcal{M}\end{aligned}$$

Proposition: If $\mathcal{M}$ is an IGS of $M(\mathcal{H})$, then

$$fd\mathcal{M} = 1 \;\Leftrightarrow\; \text{ each } T_i \text{ is a multiplication.}$$

Corollary: If each non-trivial IGS $\mathcal{M}$ of $M(\mathcal{H})$ has fiber dimension 1, then
if $\mathcal{A}$ is transitive and if $M(\mathcal{H}) \subseteq \mathcal{A}$, then $\mathcal{A} = \mathcal{B}(\mathcal{H})$.
Cheng, Guo, Wang: If $\mathcal{H}$ has an NP kernel, then $fd\mathcal{M} = 1\ \forall \mathcal{M}$.

Corollary:

Given $M(\mathcal{H})$, then TFAE

- whenever $\mathcal{A}$ is transitive with $M(\mathcal{H}) \subseteq \mathcal{A}$, then $\mathcal{A} = \mathcal{B}(\mathcal{H})$.
- whenever $\mathcal{M}$ is an IGS of $M(\mathcal{H})$ with $fd\mathcal{M} > 1$, then $\operatorname{Lat}\mathcal{A}_{\mathcal{M}}$ is non-trivial.

$$\mathcal{L} \cap \mathcal{N} = (0),\ \mathcal{M} = \{(f+g, \varphi f + \psi g) : f \in \mathcal{L}, g \in \mathcal{N}\}$$

$$\mathcal{M}_\lambda = \left\{ f(\lambda) \begin{pmatrix} 1 \\ \varphi(\lambda) \end{pmatrix} + g(\lambda) \begin{pmatrix} 1 \\ \psi(\lambda) \end{pmatrix} : f \in \mathcal{L}, g \in \mathcal{N} \right\}$$

$$\Rightarrow \quad \dim \mathcal{M}_\lambda = 2 \quad \Leftrightarrow \quad \lambda \notin Z(\mathcal{L}) \cup Z(\mathcal{N})$$

If $\dim \mathcal{M}_{\lambda_0} < 2$, say $\lambda_0 \in Z(\mathcal{L})$, then with $\mu = \psi(\lambda_0)$

$$(T - \mu)(f + g) = (\varphi - \psi(\lambda_0))f + (\psi - \psi(\lambda_0))g$$

$$\Rightarrow k_{\lambda_0} \perp \text{ran } (T - \mu),\ \text{since } f(\lambda_0) = 0 \quad \forall f \in \mathcal{L}$$

$$\Rightarrow \quad \mathcal{A}_\mathcal{M} \text{ is not transitive.}$$

$$\mathcal{M} = \{(f+g, \varphi f + \psi g) : f \in \mathcal{L}, g \in \mathcal{N}\}$$

Theorem 1: $\exists\ \mathcal{L}, \mathcal{N} \in \operatorname{Lat}(M_z, L^2_a)$ such that

- $\mathcal{L} \cap \mathcal{N} = (0)$
- $\mathcal{L} + \mathcal{N}$ is dense in L^2_a
- $Z(\mathcal{L}) = Z(\mathcal{N}) = \emptyset$.

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Theorem 2: $\exists\ \mathcal{L}, \mathcal{N} \in \text{Lat}(M_z, L^2_a)$ and
$\exists\ \varphi, \psi \in H^\infty$, $\frac{1}{\varphi - \psi} \in H^\infty$ such that $\forall\ \alpha \in \mathbb{C}^n$

$$\overline{\ker L_\alpha}, \quad \overline{\text{ran}\ L_\alpha} \in \{(0), \mathcal{H}\}.$$

Thus, $\mathcal{A}_\mathcal{M}$ has no non-trivial invariant subspaces defined by the linear graph transformations of $\mathcal{M}$.

$$\mathcal{M} = \{(f + g, \varphi f + \psi g) : f \in \mathcal{L}, g \in \mathcal{N}\}$$

Note:

- many φ, ψ will work in Theorem 2
- If $(\varphi(\mathbb{D}) \setminus \overline{\psi(\mathbb{D})}) \cup (\psi(\mathbb{D}) \setminus \overline{\varphi(\mathbb{D})}) \neq \emptyset$, then

$$\mathcal{N} \text{ or } \mathcal{L} \in \text{ Lat}\mathcal{A}_{\mathcal{M}}$$

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Theorem 3: $\exists\ \varphi, \psi \in H^\infty, \frac{1}{\varphi - \psi} \in H^\infty$
$\exists\ \mathcal{L}, \mathcal{N} \in \text{ Lat}(M_z, L^2_a)$ such that

- $\mathcal{L} \cap \mathcal{N} = (0)$
- $\mathcal{L} + \mathcal{N}$ is dense in L^2_a
- $\mathcal{L}, \mathcal{N}$ are zero based
- $\mathcal{L}, \mathcal{N} \notin \text{ Lat}\mathcal{A}_{\mathcal{M}}$

Proof idea for Theorem 1

$w_n > 0, a_n \in \partial\mathbb{D}, \quad \mu = \sum_n w_n \delta_{a_n}, \; |\mu| = \sum_n w_n$

$|\mu| < \infty, \quad S_\mu(z) = e^{-\sum_n w_n \frac{a_n+z}{a_n-z}}$ singular inner

$[S_\mu] \subsetneqq L^2_a$

$$I_\mu = \bigcap \{[S_\nu] : 0 \leqslant \nu \leqslant \mu, |\nu| < \infty\}$$

μ is admissable, if $I_\mu \neq (0)$

Thm 1: $\exists\ \mu_1, \mu_2$ admissable, $\mu_1 + \mu_2$ not admissable, and $I_{\mu_1} + I_{\mu_2}$ is dense in L_a^2

Thm (Horowitz, 1974)

$$f(z) = \prod_n (1 - \frac{5}{4} z^{3^n}) \in L_a^2$$

$$f(z) = 0 \quad \Leftrightarrow \quad z = \left(\frac{4}{5}\right)^{\frac{1}{3^n}} e^{2\pi i \frac{k}{3^n}}, \;\; k = 0, ..., 3^n - 1$$

Thm (Korenblum, 1990)

$f \in L_a^2 \;\; f(b_n) = 0, \quad w_n = \frac{1-|b_n|}{1+|b_n|}, \quad a_n = \frac{b_n}{|b_n|}$

$$\Rightarrow g(z) = \prod_n \frac{S_{w_n, a_n}(z)}{\varphi_{b_n}(z)} \text{ converges and}$$

$$\|gf\|_{L_a^2} \leqslant \|f\|_{L_a^2}$$

$\nu = \sum_n w_n \delta_{b_n}$

$\theta_1, \theta_2, ...$ linearly indep. over $\mathbb{Q}$

$\nu_j = \nu$ rotated by $2\pi i \theta_j$

Then for sufficiently large J

$$\sum_{j=1}^{J} \nu_j \text{ is not admissable.}$$

$\mu_1 = \sum_{j=1}^{J_0} \nu_j, \quad \mu_2 = \nu_{J_0+1}$

$J_0 = \sup\{J : \sum_{j=1}^{J} \nu_j \text{ is admissable } \}$