Extremals for the families of commuting spherical contractions and their adjoints.

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Topic: Dilations and extensions of d-tuples of commuting Hilbert space operators

Example for $d = 1$:

Thm 1. *(the Sz.-Nagy dilation theorem)*
$T \in \mathcal{B}(\mathcal{H}), \|T\| \leq 1$
$\Rightarrow \exists\ V \in \mathcal{B}(\mathcal{K}), \quad \mathcal{H} \subseteq \mathcal{K},$
$\quad V\mathcal{H} \subseteq \mathcal{H}, \quad \|V^*x\| = \|x\|,$
$\quad T = V|\mathcal{H}.$

$V =$ co-isometric extension of T

$V = S^* \oplus U$,

S unilateral shift of some multiplicity,
U unitary

A study of such V leads to function theory in $\mathbb{D}$.

All Hilbert spaces in the following are supposed to be separable.

$d \in \mathbb{N}$, $\mathbb{B}^d = \{z \in \mathbb{C}^d : |z| < 1\}$

Defn 2. *(Agler) A* **family** $\mathcal{F}$ *is a collection of* d*-tuples* $T = (T_1, .., T_d)$ *of Hilbert space operators,* $T_i \in \mathcal{B}(\mathcal{H})$ *such that*

(a) $\mathcal{F}$ *is bounded,*
$\exists c > 0 \ \forall T = (T_1, .., T_d) \in \mathcal{F} : \|T_i\| \leq c \quad \forall i$

(b) restrictions to invariant subspaces
$T \in \mathcal{F}, \mathcal{M} \subseteq \mathcal{H}, T_i\mathcal{M} \subseteq \mathcal{M} \ \forall i \Rightarrow T|\mathcal{M} \in \mathcal{F}$

(c) direct sums
$T_n \in \mathcal{F} \Rightarrow \oplus_n T_n \in \mathcal{F}, \quad T_n = (T_{1n}, .., T_{dn})$

*(d) unital * -representations*
$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}), \pi(I) = I,$
$T = (T_1, .., T_d) \in \mathcal{F}$
$\Rightarrow \pi(T) = (\pi(T_1), .., \pi(T_d)) \in \mathcal{F}.$

Examples:

$d = 1$:

$\mathcal{F} =$ contractions, $T^*T \leq I$
isometries, $T^*T = I$
subnormal contractions

$d \geq 1$:

$\mathcal{F} =$ contractions
commuting contractions
isometries
commuting isometries

$\mathcal{F} =$ commuting spherical contractions
commuting row contractions (d-contractions)
commuting spherical isometries

Defn 3. *If* $T = (T_1, .., T_d), T_i \in \mathcal{B}(\mathcal{H})$*,*
$S = (S_1, .., S_d), S_i \in \mathcal{B}(\mathcal{K})$*,* $T, S \in \mathcal{F}$*,*
then

$$T \leq S \Leftrightarrow \mathcal{H} \subseteq \mathcal{K}, S\mathcal{H} \subseteq \mathcal{H}, T = S|\mathcal{H}$$
$$\Leftrightarrow S = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix}$$
$$\Leftrightarrow S \text{ extends } T.$$

Defn 4. T *is* **extremal** *for* $\mathcal{F}$*,*

$$\Leftrightarrow S \geq T, S \in \mathcal{F} \Rightarrow S = \begin{pmatrix} T & 0 \\ 0 & Y \end{pmatrix} = T \oplus Y.$$

We will write $T \in \text{ext}(\mathcal{F})$

Thm 5. *(Agler)*
$\mathcal{F}$ *family,* $T \in \mathcal{F} \Rightarrow \exists\ S \in \mathit{ext}(\mathcal{F})\ S \geq T$

Examples:

$\mathcal{F}$ =contractions $\Rightarrow$ ext($\mathcal{F}$) = co-isometries

Cor 6. *(Sz. Nagy)*
Every contraction has a co-isometric extension.

$\mathcal{F}$ =isometries $\Rightarrow$ ext($\mathcal{F}$) = unitaries

Cor 7. *Every isometry is subnormal.*

$\mathcal{F}$ =subnormal contractions
$\Rightarrow$ ext($\mathcal{F}$) = normal contractions.

Cor 8. *Every subnormal contraction is subnormal.*

Commuting spherical isometries (A. Athavale)

$$\mathcal{F} = \{T = (T_1, .., T_d) : T_i \leftrightarrow T_j, \\ \textstyle\sum_{i=1}^d \|T_i x\|^2 = \|x\|^2 \quad \forall x \in \mathcal{H}\}$$

$$\text{ext}(\mathcal{F}) = \{U = (U_1, .., U_d) : U_i \leftrightarrow U_j, U_i \text{ normal} \\ \textstyle\sum_{i=1}^d \|U_i x\|^2 = \|x\|^2 \quad \forall x \in \mathcal{H}\}$$

$$= \text{commuting spherical unitaries}$$

The proof follows Attele-Lubin, JFA, 1996.

Cor 9. *(Athavale, 91) Every commuting spherical isometry is jointly subnormal.*

Commuting spherical contractions

Drury 78, Mueller-Vasilescu 93,
Arveson 98

$$\mathcal{F} = \{T = (T_1, .., T_d) : T_i \leftrightarrow T_j, \\ \sum_{i=1}^d \|T_i x\|^2 \leq \|x\|^2 \quad \forall x \in \mathcal{H}\}$$

$T \in \mathcal{F} \quad \Leftrightarrow \quad \sum_{i=1}^d T_i^* T_i \leq I,$ commuting

ext($\mathcal{F}$) $= \{S^* \oplus U\}$

$U =$ commuting spherical unitary
$S = d$-shift of some multiplicity

$S = M_z$ on $H_d^2(\mathcal{D}) = H_d^2 \otimes \mathcal{D}$

$H_d^2 \subseteq$ Hol($\mathbb{B}^d$), Drury-Arveson-Hardy space defined by reproducing kernel

$$k_w(z) = \frac{1}{1 - \langle z, w \rangle}, \; z, w \in \mathbb{B}^d$$

Thm 10. *(Richter-Sundberg) Let* $T = (T_1, ..T_d)$ *be a commuting operator tuple.*

Then the following are equivalent

(a) $T \in$ *ext*$(\mathcal{F})$

(b) $T = S^* \oplus U$

(c) (1) $\sum_{i=1}^{d} T_i^* T_i = P =$ *a projection*
(2) $\sum_{i=1}^{d} T_i T_i^* \geq I$
(3) If $x_1, .., x_d \in \mathcal{H}$ *with* $T_i x_j = T_j x_i$*, then* $\exists x \in \mathcal{H}$ *with* $x_i = T_i x$*.*

(c3) says that the Koszul complex for T *is exact at a certain stage.*

Note 1: For $d = 1$ (c) becomes

(1) $T^*T = P$, i.e. T is a partial isometry

(2) $TT^* \geq I$, so T is onto

(3) if $x_1 \in \mathcal{H}$, then $\exists x \in \mathcal{H}$ with $x_1 = Tx$, i.e. T is onto

Hence (1)&(2) or (1)&(3) are equivalent to T^* being an isometry.

For $d > 1$ let $T = M_z$ on $H^2(\partial\mathbb{B}^d)$, then (1) and (3) are satisfied, but (2) is not.

Note 2: If $T \in \mathcal{F}$ and
if $\sum_{i=1}^{d} T_i^* T_i \neq$ a projection, then $T \notin$ ext($\mathcal{F}$).

Commuting row contractions (d-contractions)

$\mathcal{F} = \{T : T^*$ is a commuting spherical contraction $\}$

$$= \{T : T_i \leftrightarrow T_j, \sum_{i=1}^d \|T_i^* x\|^2 \leq \|x\|^2 \forall x\}$$

$$= \{T : T_i \leftrightarrow T_j, \|\sum_{i=1}^d T_i x_i\|^2 \leq \sum_{i=1}^d \|x_i\|^2 \ \ \forall x_i\}$$

$T \in \mathcal{F}$

$\Leftrightarrow (T_1, .., T_d) : \mathcal{H} \oplus .. \oplus \mathcal{H} \to \mathcal{H}$ is contractive commutative

$\Rightarrow T^* = S^* \oplus U|\mathcal{H}$

(by Mueller/Vasilescu-Arveson)

$\Rightarrow T = P_{\mathcal{H}}(S \oplus U^*)|\mathcal{H}$,

$\mathcal{H} =$co-invariant for $S \oplus U^*$

ext$(\mathcal{F}) =$?

Thm 11. *(easy)*
(a) $\{T : \sum_{i=1}^{d} T_i T_i^* = I\} \subseteq$ *ext*$(\mathcal{F})$
spherical co-isometries

(b) If $\sum_{i=1}^{d} T_i T_i^* = P$ *is a projection, then*

$T \notin$ *ext*$(\mathcal{F})$
$\Leftrightarrow\ \exists x_1, .., x_d \in$ *ker* $P, \sum_{i=1}^{d} \|x_i\|^2 > 0$
with $T_i x_j = T_j x_i$

If $S = M_z = d$-shift, then $\sum_{i=1}^{d} S_i S_i^* = P$ is a projection, and ker $P =$ constants, hence $S \in$ ext$(\mathcal{F})$.

Cor 12. $\{S \oplus U^*\} \subsetneq$ *ext*$(\mathcal{F})$

defect operator

$$D_* = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2}$$

Thm 13. *(R-S)*
If $T \in \mathcal{F}$ and if D_* has rank one, i.e.

$$D_* = u \otimes u$$

for some $u \neq 0$, then

$T \in$ *ext*$(\mathcal{F}) \Leftrightarrow$ *dim span*$\{u, T_1 u, .., T_d u\} \geq 3$

If $S = (M_z, H_d^2) =$ the d-shift,
if $\mathcal{M}$ is invariant for S, $\mathcal{M} \neq H_d^2$
then

$$T = P_{\mathcal{M}^\perp} S | \mathcal{M}^\perp \in \mathcal{F},$$

and D_* has rank 1.
This can be used to produce examples $T \in$ ext$(\mathcal{F})$ such that $\sum_{i=1}^{d} T_i T_i^* = I - D_*^2$ is not a projection, so THM 2 does not not characterize all extremals.