Quantitative Homogenization of Elliptic Equations

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March 23, 2023 UTK-PDE

Elliptic Operators

- Laplace's operator

$$\mathcal{L} = -\Delta = -\text{div}\big(A\nabla\big)$$

Homogeneous and isotropic material

$$A = I_{d\times d}$$

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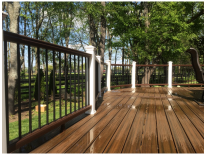
- Second-order elliptic operators in divergence form

$$\mathcal{L} = -\text{div}(A(x)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right]$$

Inhomogeneous material

$$A = A(x) = \big(a_{ij}(x)\big)_{d\times d}$$

Composite Materials (Composites)



Composite materials are widely used in industry and in our daily lives.

What is a Composite Material?

Two or more materials with different physical or chemical properties are combined in a proper fashion to create a superior new material (stronger, lighter, ...).

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Two main categories of constitutes:

- Matrix (binder)
- Reinforcement (fiber)

The constitutes are combined in some organized manner at a (relatively) small scale.

Strongly Inhomogeneous Material

- Material with rapidly oscillating and "self-similar" microstructure, such as composite materials,

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$\varepsilon > 0$ - microscopic scale

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- Direct computation of the characteristics of the material may be costly

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- Homogenization theory:
  Use asymptotic analysis to find effective (averaged, homogenized) characteristics

Elliptic Operators with Rapidly Oscillating Coefficients

Consider a family of elliptic operators in divergence form

$$\mathcal{L}_\varepsilon = -\text{div}\big(A(x/\varepsilon)\nabla\big) = -\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \qquad \varepsilon > 0$$

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Let

$$A = A(y) = \big(a_{ij}(y)\big), 1 \le i, j \le d$$

Assume that

- A is real, bounded, and elliptic
- A satisfies some structure conditions, e.g., periodic, quasi-periodic, almost-periodic, stationary random (statistically homogeneous)

Theory of Homogenization

- Goal: Describe the macroscopic properties of microscopically heterogeneous material
- Consider the boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega, \\ u_\varepsilon \text{ subject to some boundary condition,} \end{cases}$$

which describes a stationary process in a strongly inhomogeneous material with rapidly oscillating microstructure

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which describes a stationary process in a strongly inhomogeneous material with rapidly oscillating microstructure

$\varepsilon > 0$ – the inhomogeneity scale

- ε is very small relative to the linear size of the domain

Homogenization of Elliptic Equations

- As $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \quad \text{strongly in } L^2(\Omega) \text{ and weakly in } H^1(\Omega),$$

where u_0 is a solution of an elliptic equation with constant coefficients (the homogenized or effective equation),

$$\begin{cases} \mathcal{L}_0(u_0) = F & \text{in } \Omega, \\ u_0 \text{ subject to the same kind of boundary condition} \end{cases}$$

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- $\mathcal{L}_0 = -\text{div}(\widehat{A}\nabla)$ and $\widehat{A} = (\hat{a}_{ij})$ may be computed "explicitly", using $A(y)$
- *The strongly inhomogeneous material with rapidly oscillating microstructure, such as composite material, may be approximately described via an effective homogeneous material*

Basic Questions in Homogenization

- Qualitative theory: consider a general PDE

$$F(D^2 u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, x/\varepsilon) = 0$$

Does u_ε have a limit as $\varepsilon \to 0$?

If it does, what is the (effective) PDE for the limit function u_0?

Basic Questions in Homogenization

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- Quantitative theory:

  Convergence rates of u_ε to u_0;

  Regularity and geometric properties, which are uniform with respect to $\varepsilon > 0$, of solutions u_ε

Lecture Plan

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- $A = A(y)$ is real, bounded, and uniformly elliptic:

$$\mu|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \quad \text{and} \quad |a_{ij}(y)| \leq \mu^{-1}$$

for any $y \in \mathbb{R}^d$ and $\xi = (\xi_i) \in \mathbb{R}^d$, where $\mu > 0$

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- A is 1-periodic:

$$A(y+z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d$$

- Some smoothness conditions may be needed for small-scale estimates

Lecture Plan

- Lecture 1 - Qualitative Theory (introduction, correctors, effective coefficients, compactness theorem, homogenization of BVPs)
- Lecture 2 - Large-scale Regularity, Part I (method of Avellaneda - Lin by compactness)
- Lecture 3 - Large-scale Regularity, Part II (method of Armstrong - Smart by convergence rates)
- Additional Reading - Calderón-Zygmund Estimates (classical theory, dual and improved version, weak reverse Hölder inequalities, local $W^{1,p}$ estimates, global estimates)

Correctors $\chi(y) = (\chi_j(y))$

- For $1 \leq j \leq d$, $\chi_j(y)$ is a function in $H^1(\mathbb{T}^d)$ satisfying

$$\begin{cases} -\operatorname{div}\big(A(y)\nabla\chi_j\big) = \operatorname{div}\big(A(y)\nabla y_j\big) \quad \text{in } \mathbb{R}^d \\ \chi_j \text{ is 1-periodic} \\ \displaystyle\int_{\mathbb{T}^d} \chi_j \, dy = 0 \end{cases}$$

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- Existence and uniqueness: apply Lax-Milgram Theorem to

$$B[\phi, \psi] = \int_{\mathbb{T}^d} A(y)\nabla\phi \cdot \nabla\psi \, dy \quad \text{for } \phi, \psi \in H^1(\mathbb{T}^d)$$

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- $$\mathcal{L}_\varepsilon\big(x_j + \varepsilon\chi_j(x/\varepsilon)\big) = 0 \quad \text{in } \mathbb{R}^d$$

Homogenized Operator and Coefficients

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$$\widehat{a}_{ij} = \fint_{\mathbb{T}^d} \left[a_{ij} + a_{ik} \frac{\partial \chi_j}{\partial y_k} \right] = B[y_j + \chi_j, y_i + \chi_i]$$

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- The constant matrix $\widehat{A}$ is elliptic:

$$\mu|\xi|^2 \le \widehat{a}_{ij}\xi_i\xi_j \quad \text{and} \quad |\widehat{A}| \le \mu_1,$$

where $\mu_1 > 0$ depends only on μ and d.

Homogenization of Dirichlet Problems

Theorem

Suppose $A = A(y)$ is elliptic and periodic. Let Ω be a bounded Lipschitz domain. Let $u_\varepsilon \in H^1(\Omega)$ be the weak solution to

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega,$$

where $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$.

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$$\begin{aligned} u_\varepsilon &\to u_0 && \text{weakly in } H^1(\Omega), \\ A(x/\varepsilon)\nabla u_\varepsilon &\to \widehat{A}\nabla u_0 && \text{weakly in } L^2(\Omega), \end{aligned}$$

where u_0 is the solution to the homogenized problem:

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Homogenization of Neumann Problems

Theorem

Suppose $A = A(y)$ is elliptic and periodic. Let Ω be a bounded Lipschitz domain. Let $u_\varepsilon \in H^1(\Omega)$ be the weak solution to

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F + \mathrm{div}(G) \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g - n \cdot G \quad \text{on } \partial\Omega,$$

with $\int_\Omega u_\varepsilon \, dx = 0$, where $F, G \in L^2(\Omega)$, and $g \in H^{-1/2}(\partial\Omega)$.

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Quantitative Homogenization

Question: Convergence Rates

What can one say about the convergence rates for

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)}$$

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Question: Uniform (and large-scale) Regularity

Regularity estimates for u_ε that are uniform in $\varepsilon > 0$.

Uniform Regularity Estimates

- Question: Suppose that

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega,$$

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- Observation: If

$$u_\varepsilon = x_k + \varepsilon \chi_k(x/\varepsilon),$$

where $\chi_k(y)$ is the corrector, then

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \nabla u_\varepsilon = \nabla x_k + \nabla \chi_k(x/\varepsilon)$$

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- Note that ∇u_ε is bounded uniformly in $\varepsilon > 0$, but not uniformly Hölder continuous (unless $\chi_k = 0$). Thus, the optimal estimates one may prove are the Lipchitz estimates, not $C^{1,\alpha}$ estimates.

Lipschitz Estimates: Dirichlet Condition

Theorem (M. Avellaneda - F. Lin, 1987)

Assume that $A(y) = \left(a_{ij}^{\alpha\beta}(y)\right)$ is elliptic, periodic, and Hölder continuous. Let Ω be $C^{1,\alpha}$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega.$$

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$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega.$$

Then, if $p > d$ and $\sigma > 0$,

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C \left\{ \|F\|_{L^p(\Omega)} + \|f\|_{C^{1,\sigma}(\partial\Omega)} \right\},$$

where C is independent of ε.

Lipschitz Estimates: Neumann Conditions

Theorem (Kenig - Lin - S. (2013), S. Armstrong - S. (2016))

Assume that $A = A(y)$ is elliptic, periodic, and Hölder continuous. Let Ω be $C^{1,\alpha}$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega.$$

Then, if $p > d$ and $\sigma > 0$,

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C \left\{ \|F\|_{L^p(\Omega)} + \|g\|_{C^\sigma(\partial\Omega)} \right\},$$

where C is independent of ε.

$W^{1,p}$ Estimates

Theorem

Assume that $A = A(y)$ is elliptic, periodic, and belongs to VMO. Let Ω be C^1. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

Then, if $1 < p < \infty$,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \, \|f\|_{L^p(\Omega)},$$

where C is independent of ε.

Avellaneda - Lin, Caffarelli - Peral, Shen,...

Nontangential-maximal-function Estimates

Theorem

Assume that $A = A(y)$ is elliptic, periodic, and Hölder continuous. Let Ω be Lipschitz. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega$$

Then,

$$\|(u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C\,\|f\|_{L^2(\partial\Omega)}$$
$$\|(\nabla u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C\,\|f\|_{H^1(\partial\Omega)}$$

where C is independent of ε.

Avellaneda - Lin, Dahlberg, Kenig - Shen.

Stochastic Homogenization

Large-scale regularity for elliptic equations with random coefficients:

$$-\text{div}(A(x/\varepsilon, \omega)\nabla u_\varepsilon) = F$$

A. Gloria - S. Neukamm - F. Otto,
S. Armstrong - C. Smart,
S. Armstrong - T. Kuusi - J.-C. Mourrat,
...

- *Quantitative Homogenization of Elliptic Operators with Periodic Coefficients*, Harmonic Analysis and Applications, 73 - 129, IAS/Park City Math. Ser., 27, AMS (2020).
- *Periodic Homogenization of Elliptic Systems*, Operator Theory: Advances and Applications, 269. Advances in PDEs (Basel). Birkhäuser/Springer, Cham, 2018. ix+291 pp.
- The work was supported in part by the NSF

Thank You