Large-Scale Regularity in Elliptic Homogenization - Part I

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March 30, 2023 UTK-PDE

Elliptic Operators with Rapidly Oscillating Coefficients

Consider

$$\mathcal{L}_\varepsilon = -\text{div}\big(A(x/\varepsilon)\nabla\big) = -\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \qquad \varepsilon > 0.$$

Let

$$A = A(y) = \big(a_{ij}(y)\big), \; 1 \le i, j \le d$$

Assume

- A is real, bounded, and uniformly elliptic
- A is 1-periodic

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- A is 1-periodic

Basic Assumptions

- Ellipticity: there exists $\mu > 0$ such that

$$\|A\|_\infty \le \mu^{-1}$$
$$\mu|\xi|^2 \le a_{ij}(y)\xi_i\xi_j$$

for any $\xi \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$.

- Periodicity:

$$A(y+z) = A(y) \quad \text{for any } z \in \mathbb{Z}^d$$

and for a.e. $y \in \mathbb{R}^d$.

- All results hold for second-order elliptic systems in divergence form

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Uniform Regularity Estimates

- Question: Suppose that

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega,$$

$$u_\varepsilon \in \text{ what space uniformly in } \varepsilon > 0?$$

- Observation: If

$$u_\varepsilon = x_k + \varepsilon \chi_k(x/\varepsilon),$$

where $\chi_k(y)$ is the corrector, then

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \nabla u_\varepsilon = \nabla x_k + \nabla \chi_k(x/\varepsilon)$$

- Note that ∇u_ε is bounded uniformly in $\varepsilon > 0$, but not uniformly Hölder continuous (unless $\chi_k = 0$). Thus, the optimal estimates one may prove are the Lipchitz estimates, not $C^{1,\alpha}$ estimates.

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Lipschitz Estimates: Dirichlet Condition

Theorem (M. Avellaneda - F. Lin, 1987)

Assume that $A(y) = \left(a_{ij}^{\alpha\beta}(y)\right)$ is elliptic, periodic, and Hölder continuous. Let Ω be $C^{1,\alpha}$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega.$$

Then, if $p > d$ and $\sigma > 0$,

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C \left\{ \|F\|_{L^p(\Omega)} + \|f\|_{C^{1,\sigma}(\partial\Omega)} \right\},$$

where C is independent of ε.

Lipschitz Estimates: Neumann Conditions

Theorem (Kenig - Lin - S. (2013), S. Armstrong - S. (2016))

Assume that $A = A(y)$ is elliptic, periodic, and Hölder continuous. Let Ω be $C^{1,\alpha}$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega.$$

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where C is independent of ε.

Compactness Method (M. Avellaneda - F. Lin)

Theorem (large-scale interior Lipschitz estimate)

Assume $A = A(y)$ is elliptic and periodic. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_1 = B(0,1).$$

Then, for $\varepsilon \le r \le 1$,

$$⨍_{B_r} |\nabla u_\varepsilon|^2 \le C ⨍_{B_1} |\nabla u_\varepsilon|^2,$$

where C depends only on d and μ.

- No smoothness assumption is needed

Interior Lipschitz Estimate (full scale)

Assume $A = A(y)$ is elliptic, 1-periodic, and Hölder continuous,

$$|A(x) - A(y)| \le M|x - y|^\lambda \quad \text{for any } x, y \in \mathbb{R}^d,$$

where $M > 0$ and $\lambda \in (0, 1)$. Suppose that

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_1.$$

Then

$$|\nabla u_\varepsilon(0)| \le C \left(\fint_{B_1} |\nabla u_\varepsilon|^2 \right)^{1/2},$$

where C depends only d, μ, λ and M.

- The case $\varepsilon \geq 1/2$ follows from classical results, since $A(x/\varepsilon)$ is uniformly Hölder continuous in $\varepsilon \geq 1/2$
- Let $v(x) = \varepsilon^{-1} u_\varepsilon(\varepsilon x)$. Then $\mathcal{L}_1(v) = 0$. By the classical results,

$$|\nabla v(0)| \leq C \left(\fint_{B_1} |\nabla v|^2 \right)^{1/2}$$

- Since $|\nabla u_\varepsilon(\varepsilon x)| = |\nabla v(x)|$,

$$|\nabla u_\varepsilon(0)| \leq C \left(\fint_{B_\varepsilon} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2},$$

where the large-scale estimate is used for the last step.

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Compactness Theorem

Let $M(\mu)$ denote the class of all $d \times d$ 1-periodic matrices that satisfy $\|A\|_\infty \leq \mu^{-1}$ and the ellipticity condition with μ.

Theorem

Let u_k be a weak solution of

$$\mathrm{div}\big(A^k(x/\varepsilon_k)\nabla u_k\big) = 0 \quad \text{in } \Omega,$$

where $\varepsilon_k \to 0$ and $A^k \in M(\mu)$. Suppose that $\{u_k\}$ is bounded in $H^1(\Omega)$. Then there exists a subsequence, still denoted by $\{u_k\}$, such that

$$u_k \to u_0 \quad \text{weakly in } H^1(\Omega),$$
$$\mathrm{div}\big(A^0\nabla u_0\big) = 0 \quad \text{in } \Omega,$$

where A^0 is a constant and positive-definite matrix.

Proof of the Compactness Theorem

- Since $\{u_k\}$ is bounded in $H^1(\Omega)$, there exists a subsequence, still denoted by $\{u_k\}$, such that $u_k \to u_0$ weakly in $H^1(\Omega)$.
- By passing to a subsequence, we may assume that

$$\widehat{A^k} \to A^0 \quad \text{in } \mathbb{R}^{d\times d}$$

A^0 satisfies the same ellipticity condition and $\|A^0\|_\infty \le C(d,\mu)$.

- Use the Div-Curl Lemma or L. Tartar's test function method to show that

$$A^k(x/\varepsilon_k)\nabla u_k \to A^0\nabla u_0 \quad \text{weakly in } L^2(\Omega)$$

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One-Step Improvement

Lemma

Fix $\sigma \in (0,1)$. *There exist* $\varepsilon_0 \in (0,1/2)$, $\theta \in (0,1/4)$, *depending only on* d, μ *and* σ, *such that if* $0 < \varepsilon < \varepsilon_0$ *and*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \textit{in } B_1 = B(0,1),$$

then

$$\fint_{B_\theta} \left| u_\varepsilon - \fint_{B_\theta} u_\varepsilon - (x + \varepsilon\chi(x/\varepsilon)) \fint_{B_\theta} \nabla u_\varepsilon \right|^2 \le \theta^{2+2\sigma} \fint_{B_1} |u_\varepsilon|^2$$

Argue by Contradiction

- Let $\theta \in (0, 1/4)$ to be determined later. Assume that no $\varepsilon_0 > 0$ exists for this θ.
- Then there exist $\{\varepsilon_k\}$, $\{u_k\}$, $\{A^k\}$ such that

$$\varepsilon_k \to 0, \quad A^k \in M(\mu)$$

$$\mathrm{div}\left(A^k(x/\varepsilon_k)\nabla u_k\right) = 0 \quad \text{in } B_1,$$
$$\fint_{B_1} |u_k|^2 = 1$$

$$\fint_{B_\theta} \left| u_k - \fint_{B_\theta} u_k - (x + \varepsilon_k \chi^k(x/\varepsilon_k)) \cdot \fint_{B_\theta} \nabla u_k \right|^2 \geq \theta^{2+2\sigma}$$

- χ^k is the corrector for A^k

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Proof (continued)

- By Caccioppoli's inequality, $\{u_k\}$ is bounded in $H^1(B_{1/2})$. We may assume

$$u_k \to v \quad \text{weakly in } H^1(B_{1/2})$$
$$u_k \to v \quad \text{weakly in } L^2(B_1)$$

- We may assume $\widehat{A^k} \to A^0$ for some A^0
- Let $k \to \infty$ to obtain

$$\fint_{B_1} |v|^2 \leq 1,$$
$$\fint_{B_\theta} \left| v - \fint_{B_\theta} v - x \fint_{B_\theta} \nabla v \right|^2 \geq \theta^{2+2\sigma}$$

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Proof (continued)

- By the compactness theorem,

$$\operatorname{div}(A^0 \nabla v) = 0 \quad \text{in } B_{1/2}$$

- By the C^2 regularity for elliptic systems with constant coefficients,

$$\begin{aligned}
\theta^{2+2\sigma} &\leq \fint_{B_\theta} \left| v - \fint_{B_\theta} v - x \fint_{B_\theta} \nabla v \right|^2 \\
&\leq C\theta^4 \|\nabla^2 v\|^2_{L^\infty(B_{1/4})} \\
&\leq C\theta^4 \fint_{B_{1/2}} |v|^2 \\
&\leq C_0 \theta^4
\end{aligned}$$

- For contradiction, choose $\theta \in (0, 1/4)$ so that

$$C_0 \theta^4 < \theta^{2+2\sigma}$$

Iteration

Lemma

Let σ, ε_0, θ be the same as in the last lemma. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_1 \quad \text{and} \quad 0 < \varepsilon < \varepsilon_0 \theta^{k-1}$$

Then there exist $E_k \in \mathbb{R}$ and $H_k \in \mathbb{R}^d$ such that

$$\fint_{B_{\theta^k}} \left| u_\varepsilon - E_k - (x + \varepsilon\chi(x/\varepsilon)) \cdot H_k \right|^2 \leq \theta^{(2+2\sigma)k} \fint_{B_1} |u_\varepsilon|^2$$

and

$$|H_k| \leq C \sum_{\ell=1}^{k} \theta^{\sigma\ell} \left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2}$$

Proof by Induction on k

- The case $k = 1$ is given by the last lemma
- Suppose the lemma holds for some $k \geq 1$. To prove it for $k+1$, suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_1 \quad \text{and} \quad 0 < \varepsilon < \varepsilon_0 \theta^k$$

Apply the last lemma to

$$v(x) = u_\varepsilon(\theta^k x) - E_k - \left((\theta^k x + \varepsilon \chi(\theta^k x/\varepsilon)\right) \cdot H_k$$

- Note that

$$\mathcal{L}_{\frac{\varepsilon}{\theta^k}}(v) = 0 \quad \text{and} \quad 0 < \frac{\varepsilon}{\theta^k} < \varepsilon_0$$

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Proof (continued)

$$\fint_{B_\theta} \left| v - \fint_{B_\theta} v - (x + \varepsilon\theta^{-k}\chi(x\theta^k/\varepsilon)) \cdot \fint_{B_\theta} \nabla v \right|^2$$
$$\leq \theta^{2+2\sigma} \fint_{B_1} |v|^2 \quad \text{by the last lemma}$$
$$\leq \theta^{(2+2\sigma)(k+1)} \fint_{B_1} |u_\varepsilon|^2 \quad \text{by the induction assumption}$$

This leads to the inequality for $k+1$ with

$$H_{k+1} = H_k + \theta^{-k} \fint_{B_\theta} \nabla v$$

Note that

$$\left| \theta^{-k} \fint_{B_\theta} \nabla v \right| \leq C\theta^{-k} \left(\fint_{B_1} |v|^2 \right)^{1/2} \leq C\theta^{\sigma k} \left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2}$$

Proof (continued)

$$\fint_{B_\theta} \left| v - \fint_{B_\theta} v - \left(x + \varepsilon\theta^{-k}\chi(x\theta^k/\varepsilon)\right) \cdot \fint_{B_\theta} \nabla v \right|^2$$

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Proof of Large-scale Lipschitz Estimate

Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_1 and $\varepsilon \le r \le \varepsilon_0\theta$.
Choose $k \ge 1$ such that

$$\varepsilon_0\theta^{k+1} \le r < \varepsilon_0\theta^k$$

Then

$$\begin{aligned}
\fint_{B_r} |\nabla u_\varepsilon|^2 &\le \frac{C}{r^2} \fint_{B(0,\varepsilon_0\theta^{k-1})} |u_\varepsilon - E_k|^2 \qquad \text{by Caccioppoli} \\
&\le \frac{C}{r^2} \fint_{B(0,\varepsilon_0\theta^{k-1})} |u_\varepsilon - E_k - (x + \varepsilon\chi(x/\varepsilon)) \cdot H_k|^2 \\
&\quad + \frac{C}{r^2} \fint_{B(0,\varepsilon_0\theta^{k-1})} |x + \varepsilon\chi(x/\varepsilon)|^2 |H_k|^2 \\
&\le C\theta^{2k\sigma} \fint_{B_1} |u_\varepsilon|^2 + C|H_k|^2 \\
&\le C \fint_{B_1} |u_\varepsilon|^2
\end{aligned}$$

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\end{aligned}$$

Large-scale $C^{1,\alpha}$ estimates

Let $0 < \alpha < 1$. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } B_1$$

Then, for $\varepsilon \le r \le 1/2$,

$$\inf_{E \in \mathbb{R}^d, \beta \in \mathbb{R}} \frac{1}{r} \left(\fint_{B_r} |u_\varepsilon - \beta - (x + \varepsilon \chi(x/\varepsilon)) \cdot E|^2 \right)^{1/2}$$
$$\le C r^\alpha \left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2}$$

where C depends only on d, μ, and α.

Interior Lipschitz Estimate

Theorem (M. Avellaneda - F. Lin, 1987)

Assume A is elliptic, periodic, and Hölder continuous. Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } B(x_0, R).$$

Then

$$\|\nabla u_\varepsilon\|_{L^\infty(B(x_0,R/2))} \le C \left\{ \frac{1}{R} \left(\fint_{B(x_0,R)} |u_\varepsilon|^2 \right)^{1/2} + R \left(\fint_{B(x_0,R)} |F|^p \right)^{1/p} \right\},$$

where $p > d$ and C depends only on d, p, μ, and $\|A\|_{C^\alpha(\mathbb{T}^d)}$.

Lipschitz Estimates for Dirichlet Problem

Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega.$$

Then

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C\Big\{\|F\|_{L^p(\Omega)} + \|f\|_{C^{1,\sigma}(\partial\Omega)}\Big\},$$

where $p > d$ and $\sigma > 0$.

Dirichlet Correctors

To use the compactness method, in the place of the corrector χ_k, one introduces the Dirichlet corrector, define by

$$\mathcal{L}_\varepsilon\big(\Phi_{\varepsilon,k}\big) = 0 \quad \text{in } \Omega \quad \text{and} \quad \Phi_{\varepsilon,k} = x_k \quad \text{on } \partial\Omega.$$

Observation

$$\mathcal{L}_\varepsilon(\Phi_{\varepsilon,k} - x_k) = -\mathcal{L}_\varepsilon(x_k) = \mathcal{L}_\varepsilon\big(\varepsilon\chi_k(x/\varepsilon)\big) \quad \text{in } \Omega$$

and

$$\Phi_{\varepsilon,k} - x_k = 0 \quad \text{on } \partial\Omega$$

Lipschitz Estimate for the Dirichlet Corrector

Lemma

Assume that A is elliptic, periodic, and Hölder continuous. Let Ω be $C^{1,\sigma}$. Then

$$\|\nabla \Phi_{\varepsilon,k}\|_{L^\infty(\Omega)} \leq C.$$

Lipschitz Estimate for Dirichlet Corrector

- Step 1. Use the compactness method to establish boundary Hölder estimates
- Step 2. Use the Hölder estimates to show

$$|G_\varepsilon(x,y)| \le \frac{C[\delta(x)]^\alpha[\delta(y)]^\beta}{|x-y|^{d-2+\alpha+\beta}}$$

for $0<\alpha,\beta<1$.

- Step 3. Use the Green function estimate to show if

$$\mathcal{L}_\varepsilon(u_\varepsilon)=0 \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = g \quad \text{on } \partial\Omega,$$

then for $x_0\in\partial\Omega$ and $\varepsilon<r<1$,

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\|\nabla g\|_{L^\infty(\Omega)} + \varepsilon^{-1}\|g\|_{L^\infty(\Omega)}\right\}$$

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- Step 1. Use the compactness method to establish boundary Hölder estimates
- Step 2. Use the Hölder estimates to show

$$|G_\varepsilon(x,y)| \le \frac{C[\delta(x)]^\alpha [\delta(y)]^\beta}{|x-y|^{d-2+\alpha+\beta}}$$

for $0 < \alpha, \beta < 1$.

- Step 3. Use the Green function estimate to show if

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = g \quad \text{on } \partial\Omega,$$

then for $x_0 \in \partial\Omega$ and $\varepsilon < r < 1$,

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2} \le C\left\{ \|\nabla g\|_{L^\infty(\Omega)} + \varepsilon^{-1} \|g\|_{L^\infty(\Omega)} \right\}$$

Lipschitz Estimate for Dirichlet Corrector

- Step 1. Use the compactness method to establish boundary Hölder estimates
- Step 2. Use the Hölder estimates to show

$$|G_\varepsilon(x,y)| \le \frac{C[\delta(x)]^\alpha[\delta(y)]^\beta}{|x-y|^{d-2+\alpha+\beta}}$$

  for $0<\alpha,\beta<1$.
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  then for $x_0\in\partial\Omega$ and $\varepsilon<r<1$,

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\|\nabla g\|_{L^\infty(\Omega)} + \varepsilon^{-1}\|g\|_{L^\infty(\Omega)}\right\}$$

One-Step Improvement

There exists $\varepsilon_0 \in (0,1)$ such that if $0 < \varepsilon < \varepsilon_0$,

$$\mathcal{L}(u_\varepsilon) = 0 \quad \text{in } D_1 \quad \text{and} \quad u_\varepsilon = g \quad \text{on } \Delta_1,$$

where $g(0) = 0$, $\nabla_{tan} g(0) = 0$, $\|\nabla_{tan} g\|_{C^{0,2\sigma}(\Delta_1)} \leq 1$, and

$$\left(\fint_{D_1} |u_\varepsilon|^2 \right)^{1/2} \leq 1,$$

then

$$\left(\fint_{D_\theta} \left| u_\varepsilon - \Phi_{\varepsilon,j} n_j(0) n_i(0) \fint_{D_\theta} \frac{\partial u_\varepsilon}{\partial x_i} \right|^2 \right)^{1/2} \leq \theta^{1+\sigma}$$

Lipschitz Estimates for Neumann Problems

Suppose

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega.$$

Then

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C\Big\{\|F\|_{L^p(\Omega)} + \|g\|_{C^\sigma(\partial\Omega)}\Big\},$$

where $p > d$ and $\sigma > 0$.

Neumann Correctors

To use the compactness method for the Neumann Problem, one introduces the Neumann correctors, defined by

$$\mathcal{L}_\varepsilon(\Psi_{\varepsilon,k}) = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial}{\partial \nu_\varepsilon}(\Psi_{\varepsilon,k}) = \frac{\partial}{\partial \nu_0}(x_k) \quad \text{on } \partial\Omega.$$

Lemma (Kenig - Lin - S., 2013)

Assume A is elliptic, periodic, symmetric, and Hölder continuous. Let Ω be $C^{1,\sigma}$. Then

$$\|\nabla \Psi_{\varepsilon,k}\|_{L^\infty(\Omega)} \leq C.$$

The proof uses Rellich estimates and Hölder estimates for the Neumann functions.

Thank You