3. TRIGONOMETRIC FOURIER SERIES

3.A. Trigonometric Series (Fourier series in $L^2(a,b)$)

Let $[a,b]$ be an interval of length $2\ell = b-a$ and consider the space $L^2(a,b)$ of all square-integrable functions defined on $[a,b]$.

We have already seen that the sequence

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \dots \tag{1}$$

forms a complete O.N. set in $L^2(0,2\pi)$. For convenience, we transform $[a,b]$ to an interval of length 2π by setting

$$x = \frac{\ell\xi}{\pi} + a \quad \text{or} \quad \xi = \frac{\pi}{\ell}(x-a)$$

so that $a \le x \le b \longleftrightarrow 0 \le \xi \le 2\pi$. Then $u(x)$ belongs to $L^2(a,b) \Leftrightarrow \tilde{u}(\xi) := u(x) = u(\frac{\ell\xi}{\pi} + a)$ belongs to $L^2(0,2\pi)$. From now on we will discuss functions defined on $(0,2\pi)$, keeping in mind that this is **not** a restriction.

Let us accept the fact that the sequence (1) forms a *complete* O.N. set in $L^2(0,2\pi)$. Applying our general theory to this Hilbert space with this O.N. set, we see that every $u \in L^2(0,2\pi)$ admits a Fourier expansion

$$u = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n \tag{2}$$

where

$$e_0 = \frac{1}{\sqrt{2\pi}}, \; e_{2n-1} = \frac{\cos nx}{\sqrt{\pi}}, \; e_{2n} = \frac{\sin nx}{\sqrt{\pi}}, \; n = 1, 2, \dots,$$

and

$$\langle u, e_n \rangle = \int_0^{2\pi} u(x) e_n(x) dx.$$

The standard way of writing the series in (2) is

$$u(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \tag{3}$$

with Fourier coefficients

$$a_n = \frac{1}{\pi} \int_0^{2\pi} u(x) \cos nx dx, \; n = 0, 1, 2, \dots \quad b_n = \frac{1}{\pi} \int_0^{2\pi} u(x) \sin nx dx, \; n = 1, 2, \dots.$$

We did not write $=$ in (3) because that might suggest pointwise equality (i.e. that the series converges at each x to $u(x)$) which is **not** true in general. All we know is that the series converges to u in the L^2-sense, i.e.

$$\int_0^{2\pi} \left| u(x) - \frac{a_0}{2} - \sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx) \right|^2 dx \; \to \; 0 \;\; \text{as } m \to \infty \tag{5}$$

and by Parseval

$$\int_0^{2\pi} |u(x)|^2 dx = \|u\|_{L^2}^2 = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 < \infty.$$

Over an interval $[\alpha, \alpha + 2\ell]$, any $\alpha \in \mathbb{R}$, the expression (3) has the form

$$u(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}]$$

with

$$a_n = \frac{1}{\ell} \int_{\alpha}^{\alpha+2\ell} u(x) \cos \frac{n\pi x}{\ell} dx \quad n = 0, 1, 2, \dots, \qquad b_n = \frac{1}{\ell} \int_{\alpha}^{\alpha+2\ell} u(x) \sin \frac{n\pi x}{\ell} dx \quad n = 1, 2, \dots .$$

The following complex notation is more convenient sometimes: Use $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$, $\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$ and combine terms to write (3) as

(6) $$u(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{or} \quad \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}$$

with

(7) $$c_n := \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{inx} dx, \quad n = 0, \pm 1, \pm 2, \dots \text{ or } c_n := \frac{1}{2\ell} \int_{\alpha}^{\alpha+2\ell} u e^{-i\frac{n\pi x}{\ell}} dx$$

where we have set

$$c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \overline{c}_n = \frac{a_n + ib_n}{2}.$$

Given $u \in L^2(0, 2\pi)$, one can compute the Fourier coefficients (4) or (7) and write down the series (3) or (6). We know that this series converges to u in $L^2(0, 2\pi)$, i.e. (5) or

$$\int_0^{2\pi} \left| u(x) - \sum_{n=-m}^{m} c_n e^{inx} \right|^2 dx \to 0 \quad \text{ as } m \to \infty$$

hold and by Parseval

$$\int_0^{2\pi} |u(x)|^2 dx = \|u\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty.$$

It may happen that the Fourier series does not converge in the ordinary (pointwise) sense at all, so pointwise, $u(x)$ and its Fourier series expansion will not be related in that case!

Periodicity

Of course, it may happen that the Fourier series does converge pointwise. Then it defines a **periodic** function of period 2π (in general, of period 2ℓ). So, if we start with a function $u(x)$ defined in $[0, 2\pi]$, compute its Fourier series and find that this series converges pointwise, then the sum of the series is a 2π-periodic function. If it happens to converge pointwise to $u(x)$ (see §2.F), then it converges to a periodic extension of $u(x)$. On the other hand, if we start with a function $u(x)$ defined on all of $\mathbb{R}$, then its Fourier series cannot possibly converge to it pointwise unless u is periodic with period 2π, (or 2ℓ).

As a consequence of the periodicity, the Fourier coefficients of a 2π-periodic (or 2ℓ-periodic) function remain the same no matter over which interval of length 2π (or 2ℓ) they are computed. Thus, if $u(x)$ is 2π-periodic then its Fourier coefficients (7) (and similarly (4)) can be written as

(8) $$c_n = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} u(x) e^{inx} dx, \; n = 0, \pm 1, \pm 2, \dots, \quad \forall \alpha \in \mathbb{R}.$$

Indeed, for a periodic integrand $f(x)$, $f(x + 2\pi) = f(x)$ $\forall x$, we have $\forall \alpha \in \mathbb{R}$

$$\int_{\alpha}^{\alpha+2\pi} f(x) dx = \int_{\alpha}^{2\pi} f dx + \int_{2\pi}^{2\pi+\alpha} f(x) dx = \int_{\alpha}^{2\pi} f(x) dx + \int_0^{\alpha} f(\xi + 2\pi) d\xi = \int_0^{2\pi} f(\xi) d\xi.$$

Expansions of even and odd functions

Consider functions defined on $[-\pi, \pi]$. If $u(x)$ is **even**, $u(x) = u(-x)$, then the coefficients $b_n = 0$, $n = 1, 2, \ldots$, so the Fourier series of u is a cosine series

$$u(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{9}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos nx dx = \frac{1}{\pi}\left[\int_0^{\pi} u(-\xi)\cos n\xi d\xi + \int_0^{\pi} u(x)\cos nx dx\right] = \frac{2}{\pi}\int_0^{\pi} u(x)\cos nx dx.$$

Similarly, if $u(x)$ is **odd**, $u(-x) = -u(x)$, then $a_n = 0$, so

$$u(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \quad \text{with} \quad b_n = \frac{2}{\pi}\int_0^{\pi} u(x)\sin nx dx. \tag{10}$$

Observe that in either case, the coefficients depend only on the values of u on $[0, \pi]$. Hence, even when u is defined only on $[0, \pi]$, the series (9) or (10) can be formed. Therefore,

> Given $u(x)$ on $[0, \pi]$, we can try to expand it into either a cosine or a sine series. If the resulting series converges pointwise to $u(x)$ in $[0, \pi]$ then the cosine series extends u as an even function into $[-\pi, 0]$ and as an even 2π-periodic function for all $x \in \mathbb{R}$. The sine series extends u as an odd function into $[-\pi, 0]$ and as an odd 2π-periodic function for all $x \in \mathbb{R}$.

3.B. Functions of Bounded Variation

A function $u(x)$ defined on $[a, b]$ is said to be a **function of bounded variation**, $u \in BV[a, b]$, if it can be written as the difference of two increasing functions on $[a, b]$.

Facts about functions of bounded variations:

1. The sum, difference and product of two functions of $BV[a, b]$ are of BV.
2. Monotone bounded functions are of BV.
3. A function $u \in BV[a, b]$ need **not** be continuous but the one-sided limits $u(x-0)$, $u(x+0)$ $\forall$ $x \in (a, b)$, and $u(a+0)$, $u(b-0)$ exist. Hence u can have at most jump discontinuities.
4. $u \in BV[a, b]$ implies: u is bounded on $[a, b]$, u is continuous a.e. in $[a, b]$, $|u|$ is integrable on $[a, b]$.
5. $u \in BV[a, b]$ implies: u can have at most finitely many maxima and minima in $[a, b]$, i.e. u cannot oscilate infinitely often. Hence its graph has finite length.
6. Piecewise continuous functions are of BV.
7. The indefinite integral of a (Lebesque) integrable function is a function of BV (and continuous).

3.C. Pointwise Convergence of Fourier Series

Theorem 1. *If u is continuous and if its Fourier series converges uniformly in $[0, 2\pi]$, then the Fourier series represents the function pointwise:*

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \forall x \in [0, 2\pi].$$

Proof. If the series converges uniformly then $\sum c_n e^{inx} =: f(x)$ is a continuous function. But also uniform convergence $\Rightarrow$ convergence in L^2, so $\sum c_n e^{inx} = f(x)$ in $L^2(0, 2\pi)$. On the other hand, $u \in \mathcal{C}[0, 2\pi] \Rightarrow$

$u \in L^2(0,2\pi)$ and $\sum c_n e^{inx} = u(x)$ in $L^2(0,2\pi)$. Since L^2-limits are unique, $u = \sum c_n e^{inx} = f$ in $L^2 \Rightarrow u = f$ a.e. in $[0,2\pi]$, and since u, f are continuous, $u(x) = f(x)\ \forall x \in [0,2\pi]$.

Theorem 2. *If the Fourier coefficients $c_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-inx}dx$ of an $L^2(0,2\pi)$ function u satisfy*

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty \quad \textit{(in addition to } \sum |c_n|^2 < \infty \textit{ due to } u \in L^2\textit{)}$$

then the Fourier series $\sum c_n e^{inx}$ converges absolutely and uniformly to a continuous function which agrees with $u(x)$ a.e. in $[0,2\pi]$.

Proof. $\sum |c_n| < \infty \Rightarrow \sum |c_n e^{inx}| \le \sum |c_n| < \infty$, so by Weirstrass M-test $\sum c_n e^{inx}$ converges absolutely and uniformly to a continuous function $f(x)$, therefore also in L^2. Also $\sum c_n e^{inx} = u$ in L^2, therefore $u = f$ in L^2 $\therefore$ u agrees a.e. in $[0,2\pi]$ with the continuous function $f(x)$ to which the Fourier series converges uniformly. □

Theorem 3. *If $u(x)$ is of $BV[0,2\pi]$ then its Fourier series converges to $\frac{1}{2}[u(x-0)+u(x+0)]$ at each $x \in (0,2\pi)$ and to $\frac{1}{2}[u(0+0)+u(2\pi-0)]$ at the endpoints 0 and 2π.*

The proof is hard and long.

3.D. Properties of the Trigonometric Fourier coefficients

The Fourier coefficients a_n, b_n (or c_n) exist if u is merely in $L^1(0,2\pi)$ i.e. if $\int_0^{2\pi} |u(x)|dx$ exists. Note that $\|u\|_{L^1}^2 = \int_\Omega |u|dx = \langle u, 1\rangle \le \|u\|_{L^2}\|1\|_{L^2} = |\Omega| \cdot \|u\|_{L^2} \Rightarrow L^2(\Omega) \subset L^1(\Omega)$ for Ω bounded.

Riemann-Lebesque Lemma. *If $u \in L^1(0,2\pi)$ then its Fourier coefficients tend to zero as $n \to \infty$. In fact,*

$$c_\lambda = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-i\lambda x}dx \ \to \ 0 \ \textit{as}\ \lambda \to \infty.$$

For the particular case $u \in L^2(0,\pi) \subset L^1(0,2\pi)$, this is a direct consequence of Bessel's inequality $\sum |c_n|^2 \le \|u\|_{L^2}^2$ ($\Rightarrow c_n \to 0$ as terms of a convergent series). The general proof is hard. The rate at which they tend to zero is not known in general (but see below).

Theorem 1. *If $u \in BV[0,2\pi]$ then $a_n = O(\frac{1}{n})$, $b_n = O(\frac{1}{n})$ as $n \to \infty$.*

Proof. Recall the 2nd Mean Value Theorem: If $f(x)$ is monotone and bounded and $g(x)$ is integrable on $[a,b]$ then $\exists \xi \in (a,b)$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx.$$

Now, $u \in BV[0,2\pi] \Rightarrow u$ is the difference of two bounded increasing functions, $f_1 - f_2$, so it is enough to prove the result for such functions $f(x)$. We have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}f(0)\int_0^\xi \cos nx dx + \frac{1}{\pi}f(2\pi)\int_\xi^{2\pi} \cos nx dx \\ &= \frac{1}{\pi}\frac{1}{n}\left[f(0)\sin n\xi - g(2\pi)\sin n\xi\right] \\ &\Rightarrow |a_n| \\ &\le \frac{1}{n}\left(|f(0)| + |f(2\pi)|\right)\cdot\frac{1}{n} = \text{const.}\frac{1}{n}. \quad \text{Similarly for } b_n. \quad \square \end{aligned}$$

Theorem. *If $u \in \mathcal{C}^k[0,2\pi]$, $u, u', \dots, u^{(k)}$ are 2π-periodic, and $u^{(k+1)} \in BV[0,2\pi]$, then*

$$a_n, b_n = O\left(\frac{1}{n^{k+2}}\right) \ \textit{as}\ n \to \infty.$$

Hence, for $k \geq 0$, the Fourier series of u can be differentiated k times term by term and the resulting series will converge uniformly to $u(x), u'(x), \ldots, u^{(k)}(x)$.

Proof. Integrating by parts and using the periodicity $u(0) = u(2\pi), \ldots, u^{(k)}(0) = u^{(k)}(2\pi)$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} u(x)\cos nx dx = -\frac{1}{\pi n}\int_0^{2\pi} u'(x)\sin nx dx = \cdots \\ &= \frac{\pm 1}{\pi n^{k+1}}\int_0^{2\pi} u^{(k+1)}(x) {\sin nx \atop \cos nx} dx = O\left(\frac{1}{n^{k+2}}\right). \end{aligned}$$

For $k \geq 0$, the differentiated series up to order k have majorant $\sum \frac{1}{n^2} < \infty$ so by Weirstrass M-test they all converge uniformly to the respective derivative of $u(x)$. □

3.E. Integration of Trigonometric Fourier Series

We know that uniformly convergent series can be integrated term by term. But the Fourier series of a discontinuous function cannot converge uniformly. Nevertheless, we have

Theorem 1. *The Fourier series of an integrable function can always be integrated term by term and the resulting series converges pointwise to the integral of the function.*

Proof. Let $u \in L^2(0, 2\pi)$ and let $u(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ be its Fourier expansion.

Set $U(x) := \int_0^x u(x)dx$, $0 \leq x \leq 2\pi$. Then $U(x)$ is continuous and of $BV[0, 2\pi]$, so the function $U(x) - \frac{a_0}{2}x$ has a pointwise convergent Fourier series, say

$$U(x) - \frac{a_0}{2}x = \frac{A_0}{2} + \sum A_n \cos nx + B_n \sin nx, \quad 0 \leq x \leq 2\pi.$$

Now,

$$\begin{aligned} \pi A_n &= \int_0^{2\pi}\left[U(x) - \frac{a_0}{2}x\right]\cos nx dx \\ &= \frac{1}{n}\left[U(x) - \frac{a_0}{2}x\right]\sin nx\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\left[U(x) - \frac{a_0}{2}\right]dx \\ &= \frac{1}{n}\frac{a_0}{2}\int_0^{2\pi}\sin nx dx - \frac{1}{n}\int_0^{2\pi} u(x)\sin nx dx \\ &= -\frac{\pi}{n}b_n, \quad n = 1, 2, \ldots \\ \pi B_n &= -\frac{1}{n}\left[U(x) - \frac{a_0}{2}x\right]\cos nx\Big|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\cos nx\left[u(x) - \frac{a_0}{2}\right]dx \\ &= -\frac{1}{n}[F(2\pi) - a_0\pi] - \frac{1}{n}\frac{a_0}{2}\int_0^{2\pi}\cos nx dx + \frac{1}{n}\int_0^{2\pi} u(x)\cos nx dx \\ &= 0 \text{ by definition of } a_0 \ - 0 + \frac{\pi}{n}a_n, \end{aligned}$$

so that we have

$$U(x) - \frac{a_0}{2}x = \frac{A_0}{2} + \sum\left(-\frac{b_n}{n}\cos nx + \frac{a_n}{n}\sin nx\right).$$

But $U(0) = 0 \Rightarrow \frac{A_0}{2} = \sum \frac{b_n}{n}$, therefore

$$U(x) = \frac{a_0}{2}x + \sum(1 - \cos nx)\frac{b_n}{n} + \frac{a_n}{n}\sin nx$$

which is exactly the term by term integration of the series for $u(x)$. □

3.F. Differentiation of Fourier series

The situation is not simple as for integration. The general results for term by term differentiation of a series may be used and we have already seen the

Theorem. *If* $u \in \mathcal{C}^k[0, 2\pi]$, $u, u', \dots, u^{(k)}$ *are* 2π*-periodic and* $u^{(k+1)} \in BV[0, 2\pi]$ *then the Fourier series of* u *can be differentiated* k (≥ 0) *times term by term and the resulting series will converge uniformly to* $u(x), u'(x), \dots, u^{(k)}(x)$.

3.G. Eigenfunctions of Sturm-Liouville problems

Consider the Sturm-Liouville problem

$$\begin{cases} L[v] := -\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + q(x)v = \lambda w(x)v, \quad a < x < b \\ \alpha_1 v(a) + \beta_1 v(b) + \gamma_1 v'(a) + \delta_1 v'(b) = 0 \\ \alpha_2 v(a) + \beta_2 v(b) + \gamma_2 v'(a) + \delta_2 v'(b) = 0 \end{cases}$$

with $p, p', q, w \in \mathcal{C}[a, b]$, $w > 0$, $p > 0$, $q \geq 0$ with **symmetric boundary conditions**:

$$f'g - fg'|_{x=a}^{x=b} = 0$$

for any f, g satisfying the boundary conditions. A λ for which a nontrivial v exists is called an **eigenvalue**, v an **eigenfunction**.

Basic facts about the *regular* Sturm-Liouville problem

1. Eigenfunctions corresponding to distinct eigenvalues are orthogonal in $L^2_w(a, b)$: $\int_a^b w v_i v_j = 0, \quad i \neq j$.
2. The eigenvalues are all real and the eigenfunctions may be chosen real.
3. The eigenvalues are **simple** (of multiplicity one i.e. a single eigenfunction corresponds to each λ_i)
4. There exists a sequence of eigenvalues: $0 \leq \lambda_0 < \lambda_1, \dots < \lambda_n < \cdots$, and $\lambda_n \to \infty$ as $n \to \infty$ so the spectrum is discrete, nonnegative, with a limit point at ∞.
5. The eigenfunctions form a complete O.N. set (an O.N. basis) $\{e_n(x)\}$ for $L^2_w(a, b)$, i.e. w.r.t. the inner product $\langle f, g\rangle_w = \int_a^b w(x) f(x)\overline{g(x)}dx$. Therefore every $u \in L^2_w(a, b)$ can be expanded into a Fourier series (eigenfunction expansion)

$$u(x) \overset{L^2_w}{=} \sum_{n=0}^{\infty} u_n e_n(x), \qquad u_n := \langle u, e_n\rangle$$

Note: The "symmetric boundary conditions" ensure that L is a **self-adjoint** operator in $L^2_w(a, b)$ (with domain $H^2_w(a, b)$)

$$\langle uL[v] - vL^*[u]\rangle_{L^2_w(a,b)} = 0 \quad \forall u, v \in \mathcal{D}(L).$$

Most of the properties listed above are a consequence of this self-adjointness.

4. THE FOURIER METHOD - EIGENFUNCTION EXPANSIONS

4.A. Formal solution

We illustrate the method on the following model problem, describing heat conduction in a finite rod, with Dirichlet boundary conditions.

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \ell, \quad t > 0 \\ u(x,0) = f(x), & 0 \le x \le \ell \\ u(0,t) = 0, \quad u(\ell,t) = 0, & t > 0, \end{cases}$$

which is exactly the problem for which Fourier invented the formal method in the early 1800's.

Setting $u(x,t) = X(x) \cdot T(t)$, we must have $T' + \lambda^2 T = 0$, and the eigenvalue (Sturm-biouville) problem $X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad X(\ell) = 0$. The eigenvalues are $\lambda_n = \frac{(n\pi)}{\ell}, \quad n = 1, 2, \ldots$ with eigenfunctions $X_n(x) = \sin \frac{n\pi x}{\ell}, \quad n = 1, 2, \ldots$. Hence, for $n = 1, 2, \ldots$

$$u_n(x,t) := e^{-\lambda_n^2 t} \sin(\lambda_n x)$$

is a solution of the PDE and satisfies the BCs. Following Fourier, we seek u in the form $u(x,t) := \sum_{n=1}^{\infty} b_n u_n(x,t)$, with b_n to be found so that the IC be satisfied:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \overset{\text{want}}{=} f(x).$$

Thanks to the orthogonality of the eigenfunctions $\{\sin \frac{n\pi x}{\ell}\}$ in $L^2(0,\ell)$, it is easy to find the b_n's:

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx, \quad n = 1, 2, \ldots.$$

Thus, we have found a **formal** solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right),$$

in the form of a Fourier sine-series.

In what sense is this a solution of the problem? We need: series to converge in some sense, in fact to define a continuous function, so need uniform convergence on $\overline{D^T} = [0,\ell] \times [0,T]$; $u_t, \ u_{xx}$ to exist and be obtainable via term-by-term differentiation; $u(x,t)$ to converge to $f(x)$ as $t -> 0$.

Note that very little is needed for the b_n's to exist: $f \in L^1(0,\ell)$ suffices.

4.B. Justification of the Fourier Method

We found the **formal** solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\lambda_n)^2 t} \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{\ell}$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin(\lambda_n x) dx, \quad n = 1, 2, \ldots$$

are the Fourier coefficients of $f(x)$ on $[0,\ell]$. When is this an actual solution?

The very least we need, for the b_n's to even exist, is $f \in L^1(0,\ell)$. For such initial values, the series can be written down, at least formally. Let's examine its properties:

Convergence: If $f \in L^1(0,\ell)$, then the Riemann-Lebesque Lemma implies $b_n \to 0$ as $n \to \infty$, so the sequence $\{b_n\}$ is bounded: $|b_n| \le Q \quad \forall n = 1, 2, \ldots$. Hence, $\sum_1^\infty \left| b_n e^{-\lambda_n^2 t} \sin \lambda_n x \right| \le Q \sum_1^\infty e^{-\lambda_n^2 t}$, which converges (by Ratio test) for $t > 0$. Therefore, for $(x,t) \in \overline{D_\tau^T}$, for any $\tau > 0$, by the Weirstrass M-test, the Fourier series converges uniformly, so $u \in \mathcal{C}(\overline{D_\tau^T})$ and satisfies the BC's.

Satisfying the Heat Equation: We need to justify term-by-term differentiation for u_t, u_{xx}. It will be valid if the differentiated series converges uniformly: we have

$$\left| - \sum b_n \lambda_n^2 e^{-\lambda_n^2 t} \sin \lambda_n x \right| \le Q \sum \lambda_n^2 e^{-\lambda_n^2 t} \le c \sum \lambda_n^2 e^{-\lambda_n^2 \tau} \text{ on } \overline{D_\tau^T}, \quad \tau > 0,$$

and the last series (of constants) converges (by Ratio-Test), so by Weierstrass M-test the differentiated series converges uniformly in D_τ^T, $\forall \tau > 0$, and since $u_t = u_{xx}$, the Heat equation is satisfied in D_τ^T, any $\tau > 0$. Note that we didn't need to assume anything more about f, just $f \in L^1(0,\ell)$. So, for such f, $u(x,t)$ satisfies the Heat equation in D_τ^T, $\tau > 0$, and the BCs, thanks to the $e^{-\lambda_n^2 t}$ term!

Moreover, we observe that the Fourier series differentiated **any** number of times still converges uniformly in $D_\tau^T \quad \forall \tau > 0$, so u is $\mathcal{C}^\infty$ for $t > 0$ even though f may be just $L^1(0,\ell)$! This is the (infinitely) smoothing action of the heat operator!

Satisfying the IC: As $t \searrow 0$, we loose the help from the exponential and for $\sum b_n \sin \lambda_n x$ to converge, the Fourier coefficients themselves must decay to zero fast enough so that $\sum |b_n| < \infty$.

We know that a sufficient condition for this is

$$f \in \mathcal{C}[0,\ell], \quad f(0) = f(\ell), \quad f' \in BV[0,\ell],$$

because then $b_n = \mathcal{O}\left(\frac{1}{n^2}\right)$ as $n \to \infty$, so $\sum |b_n| \le$ (const.) $\sum \frac{1}{n^2} < \infty$. For such f, by Weierstrass M-test, since $|\sum b_n e^{-\lambda_n^2 t} \sin \lambda_n x| \le \sum |b_n| < \infty$ in $\overline{D_0^T}$, we have uniform convergence on $\overline{D_0^T}$, so $u(x,t)$ will be continuous even down to $t = 0$, provided $f(0) = f(\ell) = 0$ and then $u(x,0) = \sum b_n \sin \lambda_n x$ will converge to $f(x)$ uniformly on $[0,\ell]$, so the IC will be satisfied continuously pointwise.

We have proved the

Theorem. *If $f \in L^1(0,\ell)$ then the series $\sum b_n e^{-\lambda_n^2 x} \sin \lambda_n x$ defines a function in $\mathcal{C}(\overline{D_\tau^T}) \;\; \forall \tau > 0$ which satisfies the Heat Equation in D_τ^T and the BC's. Moreover, $u \in \mathcal{C}^\infty(D_\tau^T), \quad \tau > 0$.*

If $f \in \mathcal{C}[0,\ell]$, $f(0) = f(\ell) = 0$, and $f' \in BV[0,\ell]$ (say, if $f'(x)$ is piecewise continuous) then the Fourier series defines a classical solution of the full First Fourier Problem in $\mathcal{C}(\overline{D_0^T}) \cap \mathcal{C}^{2,1}(D_0^T)$.