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# On the Modeling of Traffic and Crowds: A Survey of Models, Speculations, and Perspectives*

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**Abstract.** This paper presents a review and critical analysis of the mathematical literature concerning the modeling of vehicular traffic and crowd phenomena. The survey of models deals with the representation scales and the mathematical frameworks that are used for the modeling approach. The paper also considers the challenging objective of modeling complex systems consisting of large systems of individuals interacting in a nonlinear manner, where one of the modeling difficulties is the fact that these systems are difficult to model at a global level when based only on the description of the dynamics of individual elements. The review is concluded with a critical analysis focused on research perspectives that consider the development of a unified modeling strategy.

**Key words.** vehicular traffic, crowds and swarm dynamics, complexity, scaling, living systems

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**Contents.**



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**1. Introduction.** The optimization and control of traffic flow along a road or network of roads, whether highways or urban streets, is an interesting and challenging field of interactions between mathematics and applied sciences. Several economic and social motivations can be related to the need to minimize the time spent in vehicles for transportation and consequently their related pollution problems. An additional problem worth mentioning is the need to reduce traffic accidents, a human and social cost that is related not only to inadequate driving, but also to the planning of the flow conditions. Analogous reasoning can be applied to pedestrian flows, where optimization of the movement can possibly decrease the time spent uselessly along nonoptimal paths and also reduce damage related to panic situations.

Due to the above motivations, the literature on traffic phenomena is already vast and characterized by contributions covering modeling aspects, statement of problems, qualitative analysis, and simulations generated by applications. Less developed is the literature on crowd dynamics, perhaps due to the fact that motivations have only recently been recognized. The mathematical literature on traffic flow modeling has been developed following the pioneering book by Prigogine and Hermann [185], which focused on kinetic type models, and Lighthill and Whitham [151], Payne [179], and Richards [186], [187], who proposed a modeling approach based on classical methods of continuum mechanics. The literature on crowd dynamics was arguably initiated by Henderson [109], [112] and subsequently developed by various authors (see [202], [75], [76]), as we shall see in the following sections. Crowds need to be interpreted in a broad sense, namely, not only as an assembly of pedestrians, but also of individuals who aggregate or disaggregate according to specific strategies [193]. Recent literature is reported in the special issue [15], while an interesting source of information is the website [25].

Applied mathematicians generally agree that modeling has not yet reached a satisfying level. Therefore, a great deal of additional work is still necessary to reach a good mathematical theory suitable to reproduce, using equations, the non predictable complexity of traffic and crowd phenomena. Bearing this in mind, it is worth providing a preliminary analysis of the conceptual difficulties in the modeling of vehicular traffic and crowds. Let us first anticipate the scaling problem; this matter is extensively treated in the following section. As is well known, there are several different approaches that are typical of mathematical modeling. Three approaches can be related to scaling. One is microscopic modeling, which consists in deriving, in a framework close to Newtonian mechanics, a differential equation for the dynamics of each vehicle under the action of the surrounding vehicles. The solution of a large system of ordinary differential equations can provide the desired description of the flow conditions on the road. The second approach corresponds to the macroscopic description, analogous to that of hydrodynamics, which consists in deriving evolution equations for the mass density and linear momentum regarded as macroscopic observables of the flow, which is assumed to be continuous. Mathematical models are stated in terms of nonlinear partial differential equations derived from conservation equations, and phenomenological models are used for their closure. The third approach is based on a statistical description, in a framework close to that of the kinetic theory of gases, consisting in the derivation of a Boltzmann-type evolution equation for the statistical distribution function of the position and velocity of the vehicle along the road.

Different classes of equations correspond to each type of representation, while different mathematical structures can be used for each class of equations. Certainly, each of the modeling representations that has been outlined above is characterized by advantages and disadvantages and, in any case, none of them is satisfactory. Therefore, the present state of the art does not allow us to establish the validity of one class of models with respect to the others, and so research activity in the field should look for new approaches able to overcome the technical difficulties outlined above.

Further critical analysis from an engineer's point of view on traffic phenomena modeling is given by the sharp paper of Daganzo [54]. A few sentences can be extracted from this paper:

(i) *Shock waves and particle flows in fluid dynamics refer to thousands of particles, while only a few vehicles are involved in traffic jams.*

(ii) *A vehicle is not a particle but a system linking driver and mechanics, so that one has to take into account the reaction of the driver, who may be aggressive, timid, prompt, etc. This criticism also applies to kinetic type models.*

(iii) *Increasing the complexity of the model increases the number of parameters to be identified.*

The above criticisms can be straightforwardly extended to crowd phenomena. Particularly interesting is the comment concerning the heterogeneous distribution of the quality of drivers that motivates the development of methods focused on modeling complex systems of living matter [10]. Indeed, the interest of mathematical scientists in the challenging objective of modeling complex systems, namely, large systems of individuals interacting in a nonlinear manner, has seen, in recent years, a remarkable increase. The attraction is also related to the fact that these systems are difficult to model or understand at a global level based only on the description of the dynamics of individual elements. In general, a complex living system is a large ensemble of entities that interact by rules that follow specific strategies and that have the ability to communicate with the other entities and to organize their own dynamics according to both their own strategy and their interpretation of those of the others [13].

Traffic and crowd systems are of this type of system, where knowledge of the dynamics and interaction of a few entities is not enough to describe the collective dynamics of the overall system. A further difficulty is generated by the fact that individual dynamics cannot be observed, while the overall behavior can be observed and geometrically interpreted. Applied mathematicians are also strongly attracted to modeling the complex systems considered due to the conceptual difficulties outlined above.

Vehicles on roads or networks of roads and crowds are systems possibly linked by common features although characterized by remarkable differences. The common feature is that individuals belonging to the above systems communicate, although in different ways, and have a common strategy. On the other hand, a traffic flow is one-directional in one space dimension (or multilane) and over well-defined networks, while the dynamics of crowds is in two or three space dimensions, either in bounded domains or in the whole space. Crowds may be constrained by particular geometries that generate different aggregation rules. Moreover, in traffic flows all drivers have approximately the same target, which is not consistently modified by road and environmental conditions, although signals can modify both the mean speed and the style of driving. On the other hand, in crowds the dynamics of the interactions and the overall strategy are modified according to specific situations, for instance, the presence of panic can change them consistently [97]. The modeling approach should capture both analogies and differences.

This paper provides a review of models of vehicular traffic and crowd phenomena. The survey covers the modeling approaches related to the different representation scales and is regularly referred to a critical analysis focused on the identification of research perspectives and hints to deal with them. The review of mathematical models deals with the representation scales and, specifically, the relative mathematical structures corresponding to each scale. Some sample simulations are included (or cited in the existing literature) to show the predictive ability of models.

The survey refers to fundamental issues to provide the conceptual basis for specific applications such as the analysis of networks or crowd structure interactions. The contents are organized through seven other sections. Although some aspects of macroscopic modeling are dealt with, a major part of the contents is devoted to methods from mathematical kinetic theory.

Section 2 introduces the concepts of scaling and representation of traffic and crowds according to the microscopic and macroscopic scales and to generalized kinetic theory, as well as to the mathematical kinetic theory for active particles. The statistical description has to be properly related to the granular nature of traffic flow.

Section 3 presents a critical analysis of the empirical data obtained by experiments that can be used to design and validate models. The main difficulty is that data are obtained at the macroscopic scale, while individual behaviors are not generally observed. It is worth stressing that the modeling approach should reproduce empirical data without a priori including them in the model using ad hoc assumptions.

Section 4 presents a review of traffic and crowd models derived at the microscopic scale, generally stated in terms of large systems of ordinary differential equations. The technical difficulties of dealing with multiple interactions and averaging the solution to obtain macroscopic information are discussed.

Section 5 is focused on the modeling approach according to the macroscopic hydrodynamic description. First, the conservation and equilibrium equations for mass and momentum (or suitable invariants corresponding to momentum) are introduced as a general mathematical framework. Subsequently, specific models obtained by the suitable closure of these equations are reviewed. The contents also critically analyze the mathematical properties of models, which are hyperbolic rather than parabolic.

Section 6 develops the analogous overview according to the framework of the generalized mathematical kinetic theory, following the approach for traffic flow initiated by Prigogine and subsequently developed by various authors. As in the previous section, it is shown how different mathematical structures can be used for modeling, while a critical analysis looks forward to developing a modeling strategy to be generalized to the case of crowd dynamics.

Section 7 focuses on the approach of kinetic theory that refers to the granular essence of traffic and crowd flows and reports on mathematical methods based on the discretization of phase space aimed at modeling granular flow phenomena of systems that do not satisfy either the continuity paradigms of continuum mechanics or kinetic theory.

Section 8 finally presents a critical analysis devoted mainly to open problems and research perspectives concerning both modeling and analytic issues. Specifically, some guidelines are proposed on the modeling of swarms by suitable development of the approach used to model crowd dynamics. Moreover, a brief overview of the methods of active particles is given, where the entity *driver-vehicle* is modeled as an active particle able to develop a specific strategy that is heterogeneously distributed among vehicles.

The first six sections report on the existing literature, while the last two sections mainly consider research perspectives that are focused, due to the authors' expertise, on recent developments in the approach of the mathematical kinetic theory. This paper does not include the modeling of networks. However, some concise reasoning and references are given in section 5.

## 2. The Scaling Problem.

This section presents the identification of the observation and modeling scales. Subsequently, for each scale the parameters and variables to be used for modeling need to be identified. Classically, the following descriptions can be considered.

*Microscopic description* refers to entities individually identified. In this case, their position and velocity identify, as variables dependent on time, the state of the whole system. Mathematical models are generally stated using systems of ordinary differential equations.

*Macroscopic description* is used when the state of the system is described by averaged gross quantities, namely, density, linear momentum, and kinetic energy, regarded as variables dependent on time and space. Mathematical models describe the evolution of the above variables using systems of partial differential equations.

*Kinetic theory description* is used when the state of the system is still identified by the position and velocity of the microscopic entities, but their representation is given by a suitable probability distribution over the microscopic state. Mathematical models generally describe the evolution of this distribution function using nonlinear integrodifferential equations.

This section provides a detailed analysis of the mathematical descriptions corresponding to the above scalings, first in the case of traffic flow modeling and subsequently in the case of crowds. However, the modeling approach has to face the additional difficulty that none of the usual representation scales is effectively consistent with the physics of the complex systems under consideration. A further problem is that the description by purely mechanical variables does not take into account the behavioral heterogeneity of the individuals composing the overall system.

### 2.1. Scaling and Representation of Traffic Flow.

Let us then consider, with reference to Figure 2.1, a one-directional flow of vehicles along a road with length $\ell$ and with one or more lanes, each labeled by the superscript $r$, where $r = 1, \dots, R$; see [18] and [21].

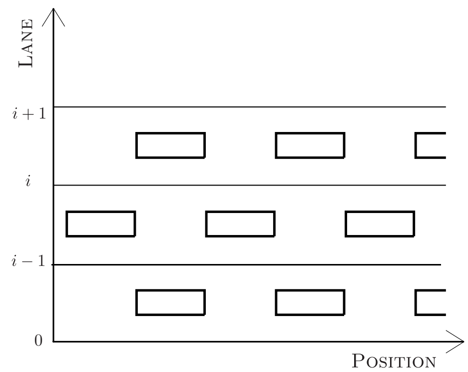


**Fig. 2.1** *Multilane flow.*

Time and space are, for all lanes, the independent variables:

- $t$ is the dimensionless time variable obtained by referring the real time to a suitable critical time $T_c$ to be properly defined by a qualitative analysis of the differential model. Generally, it is convenient to identify the critical time $T_c$ as the ratio between $\ell$ and $V_M$.
- $x$ is the dimensionless space variable obtained by dividing the real space by the length $\ell$ of the lane.

Moreover, suitable reference variables can be introduced to define the dependent variables for each representation scale in a suitable dimensionless form:

- $n_M$ is the maximum density of vehicles corresponding to bumper-to-bumper traffic jam.
- $V_M$ is the maximum admissible mean velocity that can be reached, on average, by vehicles running in free flow conditions, while a fast isolated vehicle can reach velocities larger than $V_M$. Specifically, a limit velocity can be defined as

$$V_\ell = (1+\mu)V_M, \qquad \mu > 0\,, \tag{2.1}$$

such that no vehicle can reach, simply for mechanical reasons, a velocity higher than $V_\ell$.

The *microscopic scale* corresponds to the identification of all vehicles individually. Therefore, the state of the whole system is defined, for each lane, by the dimensionless position and velocity of the vehicles, which can be regarded, neglecting their dimensions, as

$$x_i = x_i(t), \qquad v_i = v_i(t), \qquad i = 1, \dots, N\,, \tag{2.2}$$

where the subscript refers to each vehicle, and $x_i \in [0,1]$ and $v_i \in [0, 1+\mu]$ are dimensionless variables referred to $\ell$ and $V_M$, respectively.

In the case of a multilane lane flow with $R$ lanes, a superscript is necessary to identify the lane. In this case, the variables $x_i^r$ and $v_i^r$ are defined for $i = 1, \dots, N$ and $r = 1, \dots, R$.

Knowledge of the above quantities can provide, by suitable averaging processes, gross quantities such as density and mass velocity. However, this is a delicate problem related to the fact that the real discrete system made up of single vehicles has been approximated by a continuous flow. Therefore, an averaging needs to be performed; see Darbha and Rajagopal [60] and Tyagi, Darbha, and Rajagopal [208].

In principle, macroscopic quantities can be averaged either at fixed time over a certain space domain or at fixed location over a certain time range. For instance, the number density is given, for each lane, by the number of vehicles $n(t;x)$ which at time $t$ are found in the tract $[x-\Delta, x+\Delta]$:

$$\rho(t;x) \cong \frac{1}{2\Delta}\frac{n(t;x)}{n_M}. \tag{2.3}$$

Similar calculations can be applied for the mass (mean) velocity

$$\xi(t;x) \cong \frac{1}{\rho(t;x)} \sum_{i=1}^{n(t;x)} v_i(t;x), \tag{2.4}$$

where $v_i(t;x)$ denotes the velocity of the $i$th vehicle at time $t$ in the tract $[x-\Delta, x+\Delta]$ and $n(t;x)$ is the number of vehicles in the tract. This representation can be particularized for each $r$-lane.

However, the choice of the space interval is a critical problem and fluctuations may be generated by different choices of $\Delta$. The averaging can be developed in a time interval rather than in a space interval. This is, in some cases, practically related to experimental measurements. Therefore, fluctuations cannot be avoided.

Mathematical models at the microscopic scale have a structure analogous to that of Newtonian dynamics. The model describes the acceleration of vehicles as the output of the action of surrounding vehicles. However, due to the complexity of the mathematical description of the acceleration of vehicles related to the presence of other vehicles, specific models simply relate the acceleration to the action of the *leader*, namely, the vehicle ahead of the *test* vehicle. The above assumption already demonstrates the need to simplify the technical difficulties of the modeling process.

As we have seen, macroscopic quantities can be recovered by local averages of the microscopic state of active particles, in our case vehicles. The representation at the *macroscopic scale* uses the above quantities directly. For instance,

- $\rho = \rho(t,x)$ is the dimensionless density obtained by dividing the number density $n(t,x)$ by the maximum density $n_M$ of vehicles;
- $\xi = \xi(t,x)$ is the dimensionless mean velocity obtained by dividing the mean velocity by the maximum velocity $V_M$.

The mean velocity can be replaced, if the case, by the flow

- $q = q(t,x) = \rho(t,x)\,\xi(t,x)$, that is, the linear momentum referred to $q_M = n_M\,V_M$.

These quantities can be particularized for each lane simply by characterizing them by the superscript related to that lane. Therefore, the density and mean velocity for each $r$-lane are, respectively, $\rho^r = \rho^r(t,x)$ and $\xi^r = \xi^r(t,x)$. The total density is the sum with respect to all lanes,

$$\rho(t,x) = \sum_{r=1}^{R} \rho^r(t,x)\,, \tag{2.5}$$

while the total flow is

$$q(t,x) = \sum_{r=1}^{R} q^r(t,x) = \sum_{r=1}^{R} \rho^r(t,x)\xi^r(t,x)\,. \tag{2.6}$$

Let us now consider the *representation by kinetic theory methods*, where the state of the whole system is defined, for each lane, by the statistical distribution of position and velocity of the vehicles. Specifically, consider, for a one-lane road, the following distribution over the dimensionless microscopic state:

$$f = f(t, x, v) : \quad \mathbf{R}_+ \times [0, 1] \times [0, 1] \to \mathbf{R}_+ \,, \tag{2.7}$$

where $f(t, x, v)dxdv$ gives the number of vehicles which, at time $t$, are in the phase space domain $[x, x + dx] \times [v, v + dv]$. The distribution function $f$ is normalized with respect to $n_M$ so that all derived variables can be given in a dimensionless form.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, by moments of the above distribution function. In particular, the *dimensionless local density* is given by

$$\rho(t, x) = \int_0^{1+\mu} f(t, x, v)\, dv \,, \tag{2.8}$$

while the *total number of vehicles* at time $t$ is computed as follows:

$$N(t) = \int_0^1 \int_0^{1+\mu} f(t, x, v)\, dv\, dx \,. \tag{2.9}$$

In the same way, the local *mean velocity* can be computed as

$$\xi(t, x) = \frac{q(t, x)}{\rho(t, x)} = \frac{1}{\rho(t, x)} \int_0^{1+\mu} v\, f(t, x, v)\, dv \,, \tag{2.10}$$

while the local *speed variance* is given by

$$\sigma(t, x) = \frac{1}{\rho(t, x)} \int_0^{1+\mu} [v - \xi(t, x)]^2\, f(t, x, v)\, dv \,. \tag{2.11}$$

Moreover, the *speed pressure* is defined by the speed variance multiplied by the local density: $p(t, x) = \sigma(t, x)\, \rho(t, x)$.

Additional technical notations are needed in the case of multilane flows, where the superscript $r$ to the distribution function identifies the lane. When needed, one can obtain quantities which are averaged over all lanes.

**2.2. On the Representation of Crowds.** The representation of crowds is analogous to that of traffic with the difference that the dynamics is in more than one space dimension. Let us consider the system in two space dimensions (see Figure 2.2) and let $\Omega$ be the domain occupied by the crowd; in the case of a crowd such a domain can be bounded, while the generalization to three space dimensions can be formally obtained by straightforward calculations.

The reference quantities $\ell$, $n_M$, and $V_M$ can still be used, but they have a slightly different meaning. Specifically, $\ell$ is the largest dimension of the domain $\Omega$; $n_M$ is the maximum density corresponding to the highest admissible packing; $V_M$ is the maximum admissible mean velocity that may be reached, on average, in free flow conditions; the maximum admissible velocity for an isolated individual is larger than $V_M$ and is denoted by $(1+\mu)V_M$, $\mu > 0$. The assessment of the independent variables is as follows: $t = t_r/T_C$ is the dimensionless time variable referred to the critical time $T_C = V_M/\ell$; $x = x_r/\ell$ and $y = y_r/\ell$ are the dimensionless space variables.

After these preliminary definitions, it is possible to describe the system at the various scales.

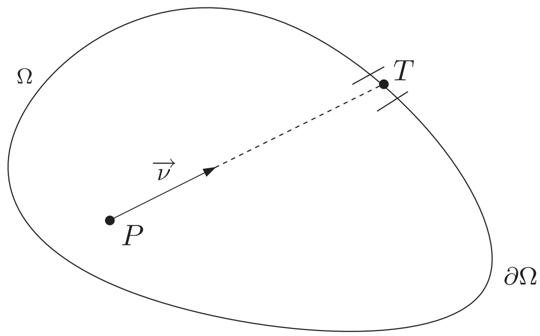


**Fig. 2.2** *Geometry of the domain occupied by the crowd.*

The *microscopic representation* is defined by the following variables:

- $\mathbf{x}_i(t) = \{x, y\}_i$ is, for $i = 1, \dots, N$, the position in $\Omega$ of $i$th individual of a crowd of $N$ individuals;
- $\mathbf{v}_i(t) = \{v_x, v_y\}_i$ is the dimensionless velocity of $i$th individual.

Mathematical models are generally stated as a system of $N$ ordinary differential equations, where $\mathbf{v}_i$ and $\mathbf{x}_i$ are the dependent variables.

The *macroscopic representation* of a system consisting of a large number of interacting individuals concerns groups of pedestrians rather than individual units. The macroscopic description may be selected for high density, large-scale systems in which the local behavior of groups is sufficient. In detail, the macroscopic description is defined by the following variables:

- $\rho = \rho(t, \mathbf{x})$, the *dimensionless density* referred to the maximum density $n_M$ of pedestrians;
- $\vec{\xi} = \vec{\xi}(t, \mathbf{x})$, the *dimensionless mean velocity*, referred to $V_M$, that in two space dimensions gives

$$\vec{\xi}(t, \mathbf{x}) = \xi_x(t, \mathbf{x})\vec{i} + \xi_y(t, \mathbf{x})\vec{j}\,, \tag{2.12}$$

  where $\mathbf{x} = \{x, y\}$, $\vec{i}$ and $\vec{j}$ denote the unit vectors of the coordinate axes, and the relationship between the flow rate, the mean velocity, and the pedestrian density is given, in a dimensionless form, as $\vec{q} = \rho\,\vec{\xi}$.

The *kinetic (statistical) representation* of a system consisting of a large number of interacting individuals is defined by the statistical distribution of their position and velocity,

$$f = f(t, \mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in \Omega, \quad \mathbf{v} \in D_{\mathbf{v}}, \tag{2.13}$$

where $f$ is normalized with respect to $n_M$. If $f$ is locally integrable, $f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x} d\mathbf{v}$ denotes the number of individuals which at time $t$ are in the elementary domain of the microscopic states $[x, x + dx] \times [y, y + dy] \times [\mathbf{v}, \mathbf{v} + d\mathbf{v}]$.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, by moments of the distribution. In particular, the *dimensionless local density* is given by

$$\rho(t, \mathbf{x}) = \int_{D_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}\,. \tag{2.14}$$

The total number of individuals in $\Omega$ at time $t$ is given by

$$N(t) = \int_{\Omega} \rho(t, \mathbf{x})\, d\mathbf{x}\,, \tag{2.15}$$

which depends on time in the presence of entry and/or departure of pedestrians.

Analogously, the mean velocity can be computed as

$$\vec{\xi}(t, \mathbf{x}) = E[\mathbf{v}](t, \mathbf{x}) = \frac{1}{\rho(t, \mathbf{x})} \int_{D_{\mathbf{v}}} \mathbf{v}\, f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\,, \tag{2.16}$$

and, similarly, the speed variance provides a measure of the stochastic behavior of the system with respect to the deterministic macroscopic description.

**2.3. Further Analysis on the Selection of the Representation Scale.** The various representation schemes given in the preceding subsections have been referred to in the usual three scales from the microscopic to the macroscopic, through the statistical description. On the other hand, some simple reasoning shows that none of them can be regarded as effectively consistent with the complex systems under consideration.

The analysis offered by Daganzo [54] clearly stresses the above considerations. In fact, it is plain that the number of vehicles in traffic flow conditions is not, even in congested traffic, large enough to ensure the validity of continuity of matter which is necessary to approach hydrodynamic-type modeling. It is not even sufficient for a statistical description of the kinetic theory. Therefore, the distribution function cannot be regarded as continuous with respect to the variables describing the microscopic state. The above reasoning can be straightforwardly extended to crowds.

Therefore, new ideas different from those ones we have seen above should be looked for. Moreover, the scaling problem should be related to the sources of complexity which appear in the modeling approach. A list of these sources, not exhaustive, is reported below:

1. The system is definitely discrete, with finite degrees of freedom. However, it is necessary for practical purposes that the model allow the computation of macroscopic quantities.
2. The flow is not continuous, hence models derived at the macroscopic scale are not consistent with the classical paradigms of continuum mechanics. Moreover, it is difficult, if not impossible, to evaluate their approximation with respect to physical reality.
3. The number of individual entities is not large enough to allow the use of continuous distribution functions within the framework of the mathematical kinetic theory. Moreover, interactions are not localized, as in the case of classical particles, considering that drivers adapt the dynamics of their vehicles to the flow conditions ahead. The same reasoning can be applied to pedestrian crowds.
4. Individual entities are regarded not as classical particles but as active particles due to their ability to modify their dynamics according to specific strategies.
5. The self-organizing ability and the overall strategy in crowds and traffic are substantially modified by environmental conditions, such as the appearance of panic situations.

The selection of the proper reference scale needs to be related to the complexity problems listed above (others may be added). The existing literature only addressses the above topics at a preliminary level, while the sections which follow attempt to develop new ideas to tackle the abovementioned issues, focusing on the objectives of the modeling approach.

In general, the following objectives should be pursued:

1. Mathematical models should include a limited number of parameters related to well-defined physical phenomena, which should be technically identified by experiments.
2. Empirical data should not be artificially plugged into mathematical models, which instead should reproduce the data after a suitable choice of parameters.
3. Mathematical models are required to reproduce, at least at a qualitative level, emerging phenomena that are observed in real flow conditions, such as queuing, jam formation at bottleneck or ramps, interactions of groups of fast and slow vehicles, stop and go, and so on.

A technical difficulty is that empirical data are properly collected with quantitative results only in uniform flow conditions, as we shall see in the following section. However, emerging phenomena are observed at a qualitative level and properly interpreted as documented in the book by Kerner [127], which offers a panorama of physical phenomena which constitute well-defined objectives to be pursued by models.

**3. On the Derivation and Use of Empirical Data.** Empirical data can be, or ought to be, used to validate mathematical models. On the other hand, the difficulty in obtaining these data, due to the great variation in the environment and in the individuals belonging to the system, reduces the amount of available data useful for validating theoretical models.

Experimental data can be roughly classified into two main categories, namely, quantitative results, for instance, focused on steady flow conditions or outlet flows, and the qualitative description of emerging behaviors related to the collective self-organizing ability of human beings and animals. Both types of data can and should be used to validate models, while artificially plugging them into models should be avoided. Such data are available in the case of both vehicular and pedestrian flows, while experimental data on swarms, besides the individual observation of the beautiful shapes designed in nature by birds, bees, and other beings of the living world, are mainly focused on emerging behaviors and on understanding the different ways in which the animal world organizes its dynamics depending on environmental conditions including the localization and state of the surrounding individuals.

This section focuses on vehicular and pedestrian flows, while the analogous analysis on swarms is postponed to section 8. It is worth stressing that the contents are not claimed to be exhaustive, but simply aim to extract out of a broad literature some selected information to be used by applied mathematicians to derive new improved models. Referring to vehicular traffic, measurements are obtained by sophisticated devices which provide rather accurate macroscopic data quantities such as number density, mean velocity, and flow. Some technical difficulties can immediately be stressed:

(i) Empirical data provide macroscopic quantities, while the dynamics is ruled at the microscopic scale.

(ii) The averaging process which leads to macroscopic quantities from measurements on a system with finite degrees of freedom introduces unavoidable fluctuations due not only to measurement errors, but also to the stochastic nature of the flow, where deceleration and acceleration of vehicles are observed even in flow conditions unform in space.

(iii) Generally, experimental results refer to steady state conditions, while traffic conditions rarely reach such a state.

(iv) Data are very sensitive to the quality of the road as well as to environmental conditions. Therefore, it is impossible to identify a unique deterministic representation.

(v) Various models, mainly at the macroscopic scale, use analytic approximation of empirical data by inserting them artificially in the structure of the models. On the other hand, experimental results—a typical example is the trend to steady uniform conditions—should not be forced a priori into models. A correct use of empirical data means validation of models by observing whether flow conditions that are experimentally observed are effectively described by them.

Interesting information on the above topics is given in the book by Kerner [127], who reports and critically analyzes various aspects of the physics of traffic and, in particular, emerging phenomena that should, at least in principle, be reproduced by models. The ETH report by Buchmueller and Weidmann [37] provides a rich source of data and parameters concerning pedestrian traffic related to walking facilities. The report is very detailed on specific parameters such as pedestrian dimension and weight, viewed as heterogeneously distributed variables, energy consumption on different types of pathways, walking speed, lateral oscillation, flow rates, and other similar data. The report also analyzes the heterogeneous behavior of individuals related to age, handicaps, and so on.

Particularly important in both cases is the information delivered by the so-called velocity and fundamental diagrams that are dealt with in the following two subsections, while the third one is focused on the analysis of emerging behaviors and on a related critical analysis.

**3.1. Velocity and Fundamental Diagrams for Traffic and Crowd Flow.** The representation of empirical data concerning vehicular traffic is documented in various books by Prigogine and Hermann [185], Leutzback [149], Daganzo [57], and Kerner [127]. Experiments report the mean velocity or the flux versus the local density. Data may also give information on the spread of the measured quantities. The books by Daganzo [57] and Kerner [127] have the additional advantage of a critical analysis of an interesting variety of traffic flow phenomena.

Experimental data concerning the mean dimensionless velocity $v_e = v_e(\rho)$ versus the dimensionless density $\rho$ show that $v_e$ reaches its maximum value for $\rho = 0$ and tends monotonically to zero for $\rho \to 1$, where the subscript $e$ is used to identify the equilibrium conditions. Correspondingly, the flow $q_e = v_e(\rho)\,\rho = q_e(\rho)$ starts from the value $q_e(0) = 0$, first increases, and then decreases to the value $q_e(1) = 0$. The above representations are often called, respectively, the *velocity diagram* and the *fundamental diagram*. The graphical representation of the relations between the macroscopic characteristics of a flow and density raises special points. In particular,

- the *free speed* $\xi = 1$ corresponds to $q = 0$ and $\rho = 0$;
- the *capacity* $q_c$ is the maximal flow, also called the *critical flow*. Due to the relation between density and speed, the maximum flow is not achieved at the maximum mean speed;
- the *capacity density* or *critical density* $\rho_c$ is the density corresponding to $q = q_c$;
- the *capacity speed* $\xi_c$ is the mean speed if $q = q_c$;
- the *jam density* corresponds to $\rho = 1$ and $q = 0$.

The part of $q(\rho)$ characterized by a constant speed corresponds to a *stable region*, while as speed decreases with increasing density, the *unstable region* enters. The

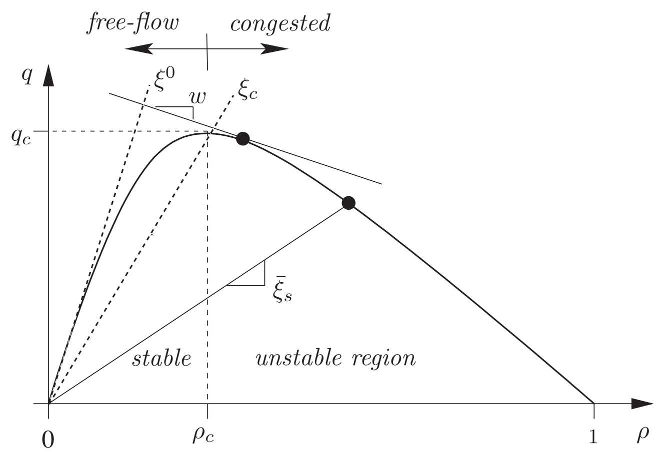


**Fig. 3.1** *Flow-density relation for pedestrian traffic. The capacity flow $q_c$ is reached at the critical density $\rho_c$. The space mean velocity $\bar{\xi}_s$ for any point on the curve is defined as the slope of the line through that point and the origin. Taking the slope of the tangent to points on the curve gives the total derivative with respect to q, also known as the wave speed or characteristic speed, denoted here by w.*

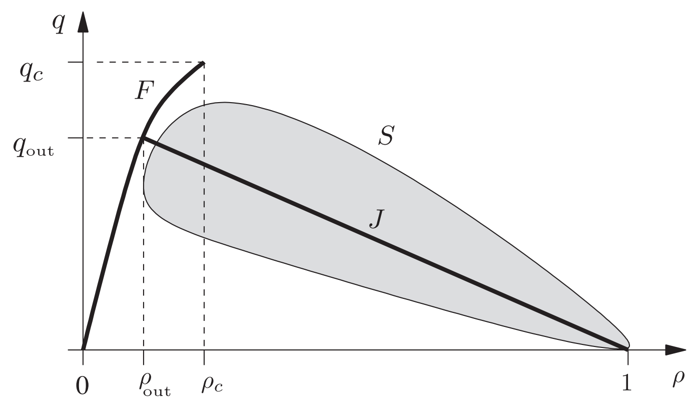


**Fig. 3.2** *Schematic form of the fundamental diagram according to Kerner's three-phase traffic theory* [128]. *F denotes the free flow branch and line J is determined by the properties of wide moving jams. As a result of Kerner's fundamental hypothesis, the region of synchronized flow (denoted by S) covers a large two-dimensional part of the density flow phase space. The line J also intersects the free flow branch in the outflow from a jam $q_{out} \ll q_c$ at the associated density $\rho_{out}$.*

region in which densities are greater than the capacity density is called the *congestion region*, whereas the region with densities lower than the capacity is called the *free flow region*. See Figures 3.1 and 3.2 for examples of the fundamental diagram of vehicular traffic flow; the pedestrian flow shows analogous behavior. The scale of the density axis is not the same as that of the velocity or flow axis, otherwise the initial slope would correspond to the bisecting axis.

The results on pedestrian flows are qualitatively analogous; see [37] and the review paper by Venuti and Bruno [210]. However, in the case of pedestrian flows additional phenomena have to be taken into account such as lateral oscillations, larger heterogeneity of the parameters, and the influence of pathways.

**3.2. Analytic Interpretations.** The modeling problem consists in looking for analytic expressions of $v_e$ and $q_e$ which, subsequently, can be used for the derivation of hydrodynamical equations. For instance, the formula

$$v_e = (1-\rho^{1+a})^{1+b}\,, \qquad q_e = \rho(1-\rho^{1+a})^{1+b}\,, \tag{3.1}$$

with $a, b \geq 0$, is due to Kühne and Rödinger, as reported in the review by Klar, Kühne, and Wegener [137], where various alternative models are discussed.

Relatively more recent experiments (see Kerner [127] and also [132], [133], [134], [136]) have shown that, at low density, vehicles generally keep to the maximum velocity $v_e = 1$ until a critical value $\rho_c$ of the density is reached. Then, for $\rho \geq \rho_c$, the velocity $v_e$ starts decaying with increasing $\rho$. An additional analysis focused on congested flow is reported in [196].

It is worth stressing that modeling steady flow conditions by analytic formulae should attempt to relate only one parameter to each specific phenomenon. This avoids the ambiguity that the same event may be described by different pairs of parameters; see Figure 3.1.

As an alternative to (3.1), the following model was proposed by Bonzani and Mussone [32]:

$$\begin{cases} \rho \leq \rho_c : & v_e = 1, \\[2ex] \rho \geq \rho_c : & v_e = \exp\left\{ -\alpha\, \dfrac{\rho-\rho_c}{1-\rho} \right\}. \end{cases} \tag{3.2}$$

The above model is characterized by two free parameters, $\rho_c$ related to the transition from free to congested flow, and $\alpha$ related to the decay of the equilibrium velocity with increasing density. The model is able to capture, with the same number of parameters, additional phenomena with respect to that captured in (3.1). The analysis of experimental data given in the above-cited paper suggests the following ranges for the admissible domains of the parameters above:

$$\rho_c \in D_c = [0, 0.2]\,, \qquad \alpha \in D_\alpha = [1, 2.5]\,, \tag{3.3}$$

depending on hour, weather, seasonal conditions, etc. The whole set of outer conditions is called, in what follows, *environmental conditions*.

Moreover, Kerner [127] remarks that additional transitions can be observed when the flow is congested; however, empirical data have to be interpreted carefully due to their variability with the *environmental conditions*. A careful classification followed by a sharp interpretation of traffic phenomena is offered in [106]. The interesting aspect of this paper is that congested traffic phenomena are related to real traffic phenomena, e.g., small bottlenecks on- and off-ramps.

The interest in experiments concerning pedestrian flow is more recent, motivated by safety problems such as evacuation in case of danger or structural collapses. Experiments due to various authors are reported and critically analyzed in the papers by Venuti et al. [211], [209], [210] that deal with a coupling of the pedestrian traffic with vibration. Particularly interesting is the problem of the crowd synchrony with respect to lateral vibrations observed on the Millennium Bridge [72], [153], [197].

Experiments show behavior technically analogous to that of traffic flow, while the modeling of the velocity diagram is delivered by the formula

$$v_e = 1 - \exp\left\{ -\gamma\left(\frac{1}{\rho} - 1\right)\right\}, \tag{3.4}$$

where $\gamma$ is a parameter which accounts for environmental conditions such as the pedestrian travel purpose or biometrics.

The above model can be technically generalized to include the transition from free to congested flow. This remarkable analogy is, however, valid only in the case of steady uniform flow, while pedestrian flow phenomena in dynamic conditions show remarkable differences with respect to vehicular traffic flow.

**3.3. Empirical Data on Emerging Behaviors and Critical Analysis.** The various data that have been reviewed in the previous subsections offer quantitative results that can be used to validate models. It is particularly important, at least according to the authors' bias, that the derivation of models is based on a detailed analysis of interactions at the microscopic level that leads to models which have the ability to describe the above data.

This is definitely possible as we shall see from some models reported in the following section. Indeed, Bonzani and Mussone [34] have shown how the fundamental diagram can be reconstructed by a detailed analysis of the drivers' braking reaction to other vehicles. This approach identifies in the critical density $\rho_c$ the parameter that can refer to the quality of the environmental conditions.

Additional experiments can be used referring, for instance, to multilane flows [139], [164]. Particularly important is the synchronization of motion in different lanes, observed and critically analyzed by Kerner [130].

An important test for validation is the analysis of the ability of models to reproduce emerging behaviors that can be qualitatively observed in some cases by rather sophisticated devices. Let us mention, among others, the spontaneous appearance of traffic jams in slightly inhomogeneous traffic flows [135], congestion at heavy bottlenecks [129], jamming transitions induced by slow vehicles [156], and heterogeneity in the response of drivers'reactions [35].

Particularly important are the experiments planned to show the influence of the behavior of the driver (or pedestrian) on the dynamics of the system. This aspect was stressed in [54] and was carefully taken into account in [58] to model multilane dynamics. Section 8 reports on this specific topic.

Experimental activities on crowd emerging phenomena are focused on the self-organizing ability of pedestrians as documented in [103], where a variety of emerging behaviors are visualized. This type of investigation has some analogy with that related to swarms that we shall see later. A particularly relevant issue refers to experiments concerning panic phenomena [97] to capture the substantial difference with respect to flow under normal conditions.

**4. A Survey of Models at the Microscopic Scale.** The representation of vehicles and pedestrians at the microscopic scale is delivered, as we have seen in section 2, by position and velocity of individual entities. Mathematical models at the microscopic scale have a structure analogous to that of Newtonian dynamics. For instance, in the case of vehicular traffic, the model describes the acceleration of vehicles as the output of the action of surrounding vehicles. In general, for a one-lane flow, the structure of models is as follows:

$$\text{(4.1)} \qquad \begin{cases} \dfrac{dx_i}{dt} = v_i \,, \\[2ex] \dfrac{dv_i}{dt} = F_i(x_i, \dots, x_N, v_i, \dots, v_N)\,, \end{cases}$$

where $i = 1, \dots, N$ and $F_i$ is the acceleration acting on the $i$th vehicle. In general, $F_i$ depends on the position and velocity of all vehicles.

Due to the complexity of the mathematical description of $F_i$, various models simply relate the acceleration of the vehicle to the action of the *leader*, namely, the vehicle ahead of the *test* vehicle. The above assumption shows the need to simplify the technical difficulties of the modeling process. As a matter of fact, drivers organize the dynamics of their vehicles in their visibility zones. Moreover, the difficulty in dealing with a large system of ordinary differential equations obliges us to reduce the complexity of the dynamics of each vehicle. If the flow is on a multilane road, the dynamics should also consider the passage from one lane to another. However, this specific modeling aspect is not systematically treated in the literature, and it is achieved by heuristic approaches that do not directly refer to a specific class of equations.

The case of crowds is analogous with the difference that positions, velocity, and acceleration are two-dimensional vectors, or vectors on a surface. The structure, with the obvious meaning of notation, is as follows:

$$\text{(4.2)} \qquad \begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i \,, \\[2ex] \dfrac{d\mathbf{v}_i}{dt} = \mathbf{F}_i(\mathbf{x}_i, \dots, \mathbf{x}_N, \mathbf{v}_i, \dots, \mathbf{v}_N)\,. \end{cases}$$

In this case, generally, the modeling of the acceleration term refers to normal flow conditions, considering only a few surrounding individuals, while in the case of panic conditions individuals also consider stimuli far from their pathways.

The solution of (4.1) or (4.2) provides the time evolution of position and velocity of vehicles or pedestrians. Macroscopic quantities are obtained by suitable averaging performed either at fixed time over a suitable space domain or at fixed location over a suitable time interval. In both cases, uncertainties and fluctuations cannot be avoided. The use of suitable filters, such as those proposed in [208], can be applied to average the individual behavior for both vehicles and pedestrians. Furthermore, the assessment of the parameters of the model needs a remarkable amount of empirical data that are very difficult and expensive to obtain also to the dependence of the dynamics on environmental conditions.

**4.1. Modeling Traffic Flow.** Traffic flow models are called *microscopic* if they describe traffic flow on the level of individual vehicles. In contrast to macroscopic models, microscopic models attempt to define the behavior of a traffic stream by describing the behavior of individual drivers in different driving situations. In general, drivers have two basic tasks:

(i) controlling the vehicle's position along the direction of motion;
(ii) controlling the vehicle's position across the width of the road or lane.

The first task is referred to as longitudinal control and is achieved by adjusting the vehicle's speed (i.e., through acceleration/deceleration). The second task, which refers to lateral control, is achieved through the proper choice of steering angles. In reality, both these activities are interdependent. In contrast to macroscopic models, microscopic traffic flow models are derived on the basis of individual driver behavior specifications. A typical example is the so-called *car-following* theories [36], [184].

Various different microscopic models are known in the literature. They differ in their assumptions concerning the behavior interactions of vehicles. Detailed information can be found in the book by May [157]. One of the main aims is to understand the nature of the steady states of the system. An excellent interpretation of the physics of traffic at the microscopic scale is offered in the review by Helbing [93] that is aimed at showing how different improvements have been proposed in the literature based on a detailed analysis of drivers' behaviors.

The main difficulty consists in reducing the size of (4.1) by selecting an appropriate number of representative vehicles. Moreover, models should be characterized by a limited number of parameters to be identified by suitable empirical data. Therefore, an exhaustive review of all models in the literature is not given in what follows, while a critical analysis is developed for a selection of examples with the aim of stimulating research perspectives.

Mathematical models, according to the *car-following theories of vehicular traffic*, are based on equations of motion analogous to Newton's equations for each individual particle in a system of interacting classical particles [82], [113], [188]. The driver adjusts velocity and acceleration of the vehicle according to the conditions ahead. This approach is defined by Holland [115] as the natural way to model traffic.

Car-following theory relates the acceleration of a vehicle-driver unit to motivational or perceived stimuli such as desired speed, speed difference, and distance to the predecessor. The equation of the dynamics is the following *ordinary differential equation* for single-lane traffic:

$$\frac{dv_i}{dt}(t) = \lambda_1 (v_{i+1} - v_i)(t), \tag{4.3}$$

where $v_i(t)$ and $v_{i+1}(t)$ are the speeds of the following and, respectively, leading vehicle at time $t$, and $\lambda_1$ is a parameter inversely proportional to the relaxation time. The underlying assumption/justification is that the $i$-vehicle (the follower) tries to achieve the speed $v_{i+1}(t)$ of the $i+1$-vehicle (its leader), while a certain relaxation time is introduced to identify the rate of such a trend.

An alternative (specialization) of (4.3) was proposed by Chandler, Herman, and Montroli [40]. The model is as follows:

$$\frac{dv_i}{dt}(t+\tau) = \lambda_2 \, (v_{i+1} - v_i)(t), \tag{4.4}$$

where $\tau$ is the time delay and $\lambda$ is a sensitivity coefficient, constant and independent of $i$. Drivers receive a stimulus at the time $t$ and respond with a certain lag time corresponding to their reaction time, $\tau$. Equation (4.4) is a *delay differential equation*, which, in this case, is known to behave in an unstable manner, even resulting in collisions under certain initial conditions.

A further improvement (see Gazis, Herman, and Rothery [83]) has been proposed that includes the effect of the distance between pairs of vehicles:

$$\frac{dv_i}{dt}(t+\tau) = \lambda_3 \, v_i^m(t) \frac{v_{i+1}(t) - v_i(t)}{(x_{i+1}(t) - x_i(t))^\ell}, \tag{4.5}$$

where $\ell$ and $m$ are additional parameters to be properly identified.

The car-following theory has been developed and gradually improved since the 1950s by various authors, including Helly [108], Pipes [184], and Todosiev and Barbosa [199]. The reader interested in an extensive overview of the car-following model can refer to Rothery [188], Brackstone and McDonald [36], and mainly the review [95].

Some recent extensions to the classic car-following theory can be mentioned. Specifically, Treiber and Helbing [203], [204], [205] developed an intelligent driver model, which includes in the acceleration term several effects such as a specific limit to the maximum allowed acceleration and a natural trend to a suitable desired speed of vehicles, see section A.3 of [95].

An additional example of a car-following model is the "simple" model of Newell [172], who proposes his theory in terms of vehicle trajectories whereby the trajectory of a following vehicle is essentially the same as that of its leader. Remarkable properties include the facts that the model has no driver reaction time and that it corresponds to some first-order macroscopic traffic flow model that will be critically discussed in the next section.

Furthermore, one can also mention the model of Zhang's car-following theory [221] which is based on a multiphase vehicular traffic flow. This means that the model is able to reproduce both the capacity drop and hysteresis phenomena.

*Optimal velocity (OV) models* may be interpreted as a technical variant of the car-following approach, where the acceleration is determined by the difference between the velocity of the vehicle $v_i(t)$ and an optimal velocity $v_{\text{opt}}$, which depends on the preceding distance $\Delta x_i = x_{i+1} - x_i$, the difference in coordinates between the vehicle $i$ and its heading vehicle $i+1$. In general, $V_{\text{opt}}(\Delta x) \to 0$ for $\Delta x \to 0$ in order to avoid accidents. For $\Delta x \to \infty$, cars should not interact.

Among others, Bando et al. [8] proposed an OV model in which the optimal velocity $V_{\text{opt}}(\Delta x)$ increases monotonically to its maximal value and has a turning point (i.e., critical point), that corresponds to the safety distance. In this model, the acceleration of every car is determined by its velocity $v_i$ and an optimal speed depending on the headway $\Delta x_i$; in fact, they considered an approximation of the Newell [172] model at order 1 and obtained

$$
\begin{cases}
\dfrac{dv_i}{dt}(t) = \lambda_4 \left[ v_{\text{opt}}(\Delta x_i)(t) - v_i(t) \right], \\[2ex]
v_{\text{opt}}(\Delta x_i)(t) = \tanh(\Delta x_i(t)) - d_c + \tanh(d_c),
\end{cases}
\tag{4.6}
$$

where $\lambda_4$ represents the driver's sensitivity, which equals the opposite of the driver's reaction time, and $d_c$ is the safety distance. This type of driver sensitivity with respect to velocity differences was taken into account by Treiber, Henneche, and Helbing [203].

Various alternative models have been subsequently proposed that are not reported here. Among them, Newell [173] used only the velocity-headway function to describe the dynamics of the flow.

A technical difficulty related to the use of microscopic models, especially in their applications to the analysis of the dynamics of road networks, consists in dealing with a large number of differential equations. *Traffic cellular automata models* can be used to overcome, at least in part, these computational difficulties. Cellular automata can be interpreted as microscopic models discrete in space and time. This feature makes them ideally suited for high-performance computer simulations. Simple rules dictate when and how a vehicle goes from one cell to another. A typical example is a probabilistic model by Nagel and Schreckenberg [165]. However, microscopic methods are computationally expensive, as each vehicle has a differential equation to be solved at each step governing its behavior, so as the number of vehicles increases so does the size of the (coupled) system. Cellular automata appear to be more efficient in the case of macroscopic models, as we shall see in section 5.

**4.2. Modeling Crowd Dynamics.** The mathematical structure to be used for the modeling of crowd dynamics is given by (4.2), since it is analogous to that of vehicular traffic, but in a vector form to take into account the fact that pedestrians move in more than one space dimension. Therefore, different modeling approaches correspond to different ways of describing the acceleration term on the basis of a detailed interpretation of individual behaviors.

In general, pedestrian dynamics has not been studied as extensively as vehicular traffic, although the literature in the field is rapidly developing, as documented in the papers by Helbing and Molnár [102], Hoogendoorn et al. [117], [118], [121], Huang et al. [122], Schelhorn et al. [189], and Willis et al. [214]. The aim of modeling is analogous, namely, describing interesting collective effects and self-organization phenomena (jamming, lane formation, oscillation, etc.) from a detailed analysis of the dynamics at the microscopic scale.

A brief description of different approaches to microscopic pedestrian models is proposed in what follows. The description is limited to the conceptual guidelines, leaving to the interested reader the derivation of the acceleration terms according to the various cited papers or, possibly, after having improved the current approaches.

*Cellular automata models* have recently been used in the simulation of pedestrian flows (see, e.g., [26], [27], [28], [70], [78], [162], [163]). The models simulate pedestrians as entities (automata) in cells. The walkway is modeled as grid cells and a pedestrian is represented as a circle that occupies a cell. Most cellular automata models for pedestrian dynamics proposed so far are rather simple.

*Magnetic force models* are based on the idea of describing the dynamics of each pedestrian as a positive pole in a magnetic field which causes its movement [174]. Obstacles like walls, columns, and handrails also have positive poles and negative poles are said to be located at the goal of pedestrians.

*Social force models*, introduced in [102], are based on the assumption that interactions among pedestrians are implemented by using the concept of a *social force* or *social field* [150]. Important examples of social force models can be found in [92], [102], [107], [161], and references therein. The model proposed by Seyfried et al. [190] deals with pedestrians treated as particles subject to long-range forces induced by the social behavior of the individuals.

The social force model is able to cover several natural phenomena which occur during pedestrian movements. For instance,

(i) pedestrians normally choose the fastest route and chase a well-defined objective as visualized in Figure 2.2 of section 2;

(ii) the concept of desired speed can be introduced to reflect the motivation of pedestrians to reach the desired goal with the desired speed;

(iii) pedestrians move with an individual speed, taking into account the situation, sex, age, handicaps, surroundings, and so on. The speed can be assumed, in this case, to be Gaussian distributed [109];

(iv) pedestrians keep a certain distance from other pedestrians. The distance is dependent on the pedestrian density and the walking speed. Suitable repulsive, short-range potentials can be introduced to describe these phenomena;

(v) the interaction potential can be attractive, for long-range interactions, to model the aggregation phenomena of pedestrians, who often display a tendency to walk in groups. Once separated (for instance, if a pedestrian has to avoid an obstacle), the individual pedestrians try to reform the group.

The above guidelines define a methodological approach to be followed in the derivation of specific models such as those cited above. In general, one has to maintain, within the modeling approach, the simplicity of the structure of the model. In other words, the design of models should include a small number of parameters to be identified by experimental data.

Various technical improvements to the above approach are available in the literature. For instance, *models of crowd turbulence* have been designed (see Yu and Johansson [215]) to describe the motion of pedestrians when the crowd is extremely dense and people attempt to gain space by pushing others, which leads to irregular movements or even to people falling. If the fallen pedestrians do not manage to stand up quickly enough, they will become obstacles and cause others to fall as well. Such dynamics can eventually spread over a large area and result in a crowd disaster.

Technical developments are occasionally represented by the so-called *OV models*, originally introduced to describe highway traffic and subsequently generalized to higher dimensions [167] with an application to pedestrian dynamics. In this approach, the OV is identified by its *desired velocity* and interactions with other particles; in other words, a particle moves with the desired velocity, if it is alone, otherwise the acceleration is the result of short-range repulsive and long-range attractive actions.

**4.3. Critical Analysis.** One of the crucial problems in modeling vehicular and crowd dynamics at the microscopic scale lies in dealing with a large number of equations and in transferring the microscopic information to the macroscopic level, namely, to physical quantities which can be possibly observed and measured.

This aspect, which requires the development of suitable computational schemes and use of filters [208], needs to be related to the identification of the parameters characterizing the models. Generally, this is not an easy task considering that empirical data refer to macroscopic events rather than to microscopic ones.

A further aspect to be considered, as is well remarked in Daganzo's criticisms [54], is the heterogeneous behavior of drivers and pedestrians. This aspect is even more critical in the physics of crowds where changes in environmental conditions can introduce substantial modifications in the individual behaviors. Indeed, this is the case in transition from normal to panic conditions, where individuals lose their trend to the target and are attracted by directions which they identify, correctly or incorrectly, as leading to escape from danger.

Still referring to the heterogeneous behavior, some models include social behaviors (education) in the modeling of the self-ability of pedestrians. The report [37] provides various empirical data that can possibly be used for such a modeling objective.

Modeling also has to take into account the fact that the human interpretation of danger is not, at least in some cases, correct. For instance, escaping a danger can lead to the localization of overcrowded areas, which constitute additional danger, and a subsequent additional panic. This is an interesting topic (see [97]) which is definitely worth future research in order to properly relate it to well-defined models.

**5. A Survey of Models at the Macroscopic Scale.** Vehicular traffic models at the macroscopic scale generally refer to a mathematical structure consisting of a two-dimensional system of partial differential equations that define the time and space evolution of the density and mean velocity of the flow of vehicles (and pedestrians), assumed to be continuous with respect to both dependent variables. Specifically, the mathematical framework is identified by mass and momentum conservation equations, namely, two independent coupled equations, which, in the absence of source terms, can be written, using dimensionless variables, as

$$\begin{cases} \partial_t \rho + \partial_x(\rho \xi) = 0\,, \\ \partial_t \xi + \xi\, \partial_x \xi = \mathcal{A}[\rho, \xi]\,, \end{cases} \tag{5.1}$$

where $\partial_t$ and $\partial_x$ correspond to the partial derivatives with respect to time and space, respectively, $\mathcal{A}[\rho, \xi]$ models the component of the mean acceleration, and the square brackets indicate that it may be a functional of its arguments. According to the representation of section 2, which we again summarize here, the above structure uses dimensionless variables, where $\rho$ is the ratio between the real number density and the maximal density $n_M$ corresponding to jam conditions, $\xi$ is the ratio between the real mean velocity and the maximal velocity $V_M$, and the independent variable space and time are referred, respectively, to the length of the road $\ell$ and the critical time $T_c$ defined as follows:

$$\frac{T_c\, V_M}{\ell} = 1 \quad \Rightarrow \quad T_c = \frac{\ell}{V_M}\cdot$$

In other words, $T_c$ is the time necessary to travel the length of the road at the maximal velocity. It has a physical meaning as it represents the minimal observation time.

The above framework can be technically generalized. For instance, Aw and Rascle [6] suggest, using heuristic arguments based on the critical analysis of Daganzo [54], an alternative model stated in terms of two coupled conservation laws: the first one is mass conservation, and the second one corresponds to a suitable Riemann invariant. Specifically, the mathematical framework is

$$\begin{cases} \partial_t \rho + \partial_x(\rho \xi) = 0\,, \\ \partial_t(\xi + p(\rho)) + \xi\, \partial_x(\xi + p(\rho)) = 0\,, \end{cases} \tag{5.2}$$

where $p$ is a heuristic expression of the pressure modeled by a suitable constitutive relation, for instance, $p = p(\rho)$, where $p$ is an increasing function of $\rho$. The simplest model is linear, $p = c\rho$. This dimensional variable has the same dimension as the velocity variable; therefore, in the above dimensionless structure, it is divided by the maximal mean velocity $V_M$. This topic will be critically analyzed in section 5.3.

This particular framework [6] was subsequently used and developed by various authors, among others Aw and Rascle [6], Chakroborty, Agrawal, and Vasishtha [39], Kerner and Klenov [131], Nagel, Wayner, and Woesler [166], Berthelin et al. [23], [24], Colombo [45], and Degond and Delitala [62].

The same structure applies to crowd modeling in two or more space dimensions. The generalization is immediate as both equations can be written in two space dimensions (see Hughes [123], [124]) as

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \vec{\xi}) = 0\,, \\ \partial_t \vec{\xi} + (\vec{\xi} \cdot \nabla_{\mathbf{x}})\vec{\xi} = \vec{\mathcal{A}}[\rho, \vec{\xi}]\,, \end{cases} \tag{5.3}$$

where the *dot product* denotes an inner product of vectors and, as already mentioned, all above equations involve, according to sections 2 and 3, dimensionless variables.

It is worth stressing that the review of this section refers to the above dimensionless frameworks, although the greater part of the literature proposes mathematical models based on conservation equations involving dimensional variables. Indeed, dimensionless structures provide, as is usual in mathematical physics, a unified framework useful both in developing computational schemes and in comparing analogous, but quantitatively different, phenomena.

Specific models can be designed referring to the above frameworks. For instance, first-order models are obtained from the first equation only with a closure $\vec{\xi} = \vec{v}_e[\rho]$ [186], [187]. Therefore, these models need experimental data to be plugged into the model. However, first-order models may be useful for some specific applications, although the abovementioned lack of descriptive ability needs to be underlined. The interested reader is referred to the review paper [16] for traffic flow modeling and to the paper [49] for the modeling of a pedestrian flow.

Moreover, it is worth mentioning that a class of traffic flow models exists that is given in the form of discrete space domain and continuous time domain equations, such as those given by Cremer and Papageorgiou [51], Papageorgiou, Blosseville, and Hadj-Salem [177], and Daganzo [56]. These models are similar to the continuous models above but they use space discretization and incorporate other modifications such as nonlinear density saturation functions. They are also useful for dealing with modeling traffic on networks [44], [59], [79].

The survey proposed in the following subsections is limited to second-order models, which are obtained from both equations with the addition of a phenomenological relation describing the psychomechanic acceleration $\mathcal{A} = \mathcal{A}[\rho, \xi]$ on the vehicles or the pseudopressure $p(\rho)$. Some authors have proposed higher-order models that include an evolution equation for the energy. However, it is quite difficult to provide a correct identification of the energy for a system, where the overall amount of available energy also depends on the individual's driving style. These models need additional parameters that cannot be easily related to empirical data.

It is worth stressing, thus anticipating the critical analysis of section 5.3, that models should reproduce the velocity and fundamental diagrams, while emerging collective behaviors should be reproduced at least at a qualitative level.

**5.1. Second-Order Traffic Models.** This section reports a survey of second-order models which refer to both (5.1) and (5.2). Focusing on the first framework, it is necessary, to close system (5.1), to specify the model of the acceleration $\mathcal{A}[\rho, \xi]$; it is plain that different assumptions lead to different models.

A significant number of papers in the existing literature decompose $\mathcal{A}[\rho, \xi]$ as the sum of a term modeling relaxation toward the equilibrium velocity $v_e(\rho)$ and a term expressing braking and acceleration actions related to density gradients. The qualitative analysis of these models needs to be carefully developed considering that the model should, at least in principle, reproduce equilibrium conditions for individual behavior without plugging them artificially into the model. Moreover, it is crucial to assess their consistency with the physical reality. For instance, parabolic models show unrealistic propagation velocity of perturbations, while it is well understood that propagation should be comparable with the mean velocity of vehicles.

The first second-order model was proposed and well argued by Payne [179] and Whitham [213], where the basic assumption refers to the similarity between the traffic flow on roads and an incompressible fluid. The authors introduced an equation for the speed of Navier–Stokes type (for incompressible fluids). The acceleration of the

Payne–Whitham (PW) model is constructed as the sum of two terms,

$$\mathcal{A}[\rho,\xi] = \mathcal{A}_r[\rho,\xi] + \mathcal{A}_a[\rho,\partial_x\rho]\,, \tag{5.4}$$

where $\mathcal{A}_r$ is called the *relaxation term* and models the tendency of traffic flow to relax to an equilibrium velocity,

$$\mathcal{A}_r[\rho,\xi] = \frac{\mu_1}{\tau}\,(v_e(\rho)-\xi), \tag{5.5}$$

where $\mu_1$ is a constant, $v_e(\rho)$ is the equilibrium velocity, and $\tau$ a relaxation time.

The second term, $\mathcal{A}_a[\rho,\partial_x\rho]$, called the *anticipation term*, is similar to the pressure term in fluids and accounts for the reaction of drivers to the variations in the traffic conditions ahead of them:

$$\mathcal{A}_a[\rho,\xi] = -\,\frac{\mu_2}{\rho}\,\partial_x\mathcal{P}(\rho)\,, \tag{5.6}$$

where $\mathcal{P}(\rho)$ is analogous to the "*pressure*" in the fluid of the traffic.

Using (5.5) and (5.6), the resulting form of the PW model is as follows:

$$\begin{cases} \partial_t\rho + \partial_x(\rho\xi) = 0\,, \\[2mm] \partial_t\xi + \xi\,\partial_x\xi = \dfrac{\mu_1}{\tau}\,(v_e(\rho)-\xi) - \dfrac{\mu_2}{\rho}\,\partial_x\mathcal{P}(\rho)\cdot \end{cases} \tag{5.7}$$

The PW-like models are distinguished from each other by the form of the function $\mathcal{P}(\rho)$. Specifically, Whitham [213] proposed taking $\mathcal{P}(\rho)$ to be simply proportional to $\rho$, while Payne [179] suggested

$$\mathcal{P}'(\rho) = \frac{1}{2\tau}|v_e'(\rho)|\,, \tag{5.8}$$

where the prime denotes the derivative with respect to the argument of the function.

A disadvantage of (5.7) lies in the stability of the linear approximation of the stationary uniform solution to smaller perturbations for all values of density. Analysis of the empirical data shows, however, that for higher values of density the laminar motion of the traffic flow becomes unstable and small perturbations lead to *phantom congestions* or waves of the stop-and-go movement. This problem can be tackled (see, e.g., Günter et al. [90] and Klar and Wegener [138]) by the following modification of the anticipation term: $\mathcal{P}(\rho) = \rho\,\Theta_e(\rho)$, where $\Theta_e$ denotes the equilibrium value in homogeneous traffic conditions of the variance of the microscopic velocity of cars. Among various technical developments, Phillips [181] proposed using a density-dependent relaxation time $\tau = \tau(x)$. Moreover, he approximated the traffic pressure using $p(\rho) = \rho\,\theta_e(\rho)$, with $\theta_e(\rho) = \theta_0(1-\rho)$.

A substantial revision was proposed by Kühne [143] to soften the abrupt variations of the density and velocity by including a dissipative velocity viscosity term proportional to $\partial_{xx}\xi$ in (5.7), which is similar to the term describing viscosity in classical hydrodynamics equations. Consequently, the acceleration equation becomes

$$\partial_t\xi + \xi\,\partial_x\xi = \frac{\mu_1}{\tau}\,(v_e(\rho)-\xi) - \frac{\mu_2}{\rho}\,\partial_x\mathcal{P}(\rho) + \mu_3\,\partial_{xx}\xi\,, \tag{5.9}$$

where the third term on the right-hand side of (5.9) shows that in regions of spatial accelerations, $\partial_{xx}\xi > 0$. This diffusion term gives an increase in the velocity of the moving observer; that is, when the moving observer drives in a region of spatial acceleration, its driving *goes along with that of the other drivers*.

For various technical specializations see, among others, Kerner and Konhäuser [133], [134] and Kühne [144], [145]. In particular, the authors suggest $\mathcal{P}'(\rho) = c_0$, where $c_0$ is a constant that is used instead of the constant $\nu$ in order to improve compatibility with the Navier–Stokes equation of classical hydrodynamics. It is plain that for both of the models above, the diffusion term $\partial_{xx}\xi$, while it smooths the solution, also induces the aforementioned inconsistencies in the propagation velocity. Moreover, the assumption of linear diffusion is in contrast to flow conditions where the density ranges from vacuum $\rho = 0$ to jam conditions $\rho = 1$. This aspect is critically analyzed by Bonzani [30], where different types of nonlinearities are considered focusing on the analysis of the hyperbolic structure of the equations.

Further technical variations are due to Berg, Mason, and Woods [22] and Gupta and Katiyar [91], who introduced phenomenological models of nonlinear diffusion, while heuristic models have been proposed to describe anisotropic and hyperbolic feature of traffic flows, among others Zhang [218] and Michalopoulos, Yi, and Lyrintzis [158]. A relevant contribution is offered in the paper by Zhang [216], where several ideas give a sharp interpretation of experimental results, while Nelson [169] suggests a modification to compute traveling wave solutions, and Marasco [154] models a $2 \times 2$ system where the presence of tollgates is properly taken into account.

However, all the aforementioned models lack the ability to describe some basic qualitative properties of the flow. For instance, a property of PW-like models (and other similar models, e.g., [65]) is that small perturbations are transported along two (curved) characteristic lines (one moving faster and one moving slower than the average vehicular speed), which is unlike first-order models that contain one (straight) characteristic. This means that in PW-like models the information is transported downstream at a higher speed than traffic velocity. Indeed, this is a violation of the anisotropic property of traffic flow that refers to the fact that drivers respond only to traffic conditions ahead. In addition, models should be able to describe both equilibrium conditions without artificially inserting them into the model and also those phase transitions that are well documented by Kerner.

A successful attempt to incorporate the anisotropic property and to tackle the above inconsistencies is due to Aw and Rascle [6], who suggested an alternative second equation to couple with the mass continuity equation. Specifically, the authors propose a conservation equation for an interaction invariant as given in (5.2). The simplest expression for the pseudopressure term is simply a monotone increasing function of the type $p(\rho) = \rho^\gamma$, with $\gamma > 0$. The authors show that the corresponding model is strictly hyperbolic with distinct propagation velocities except for $\gamma = 0$, when the eigenvalues coalesce. Indeed, the propagation velocities are consistent with the physics of the systems.

This approach has generated new research in the field and various authors have proposed qualitative analysis and simulations as well as further improvements. Among others, the qualitative analysis and simulations by Goatin [84] have shown how the model can predict Kerner's phase transitions.

As already mentioned, several modifications or similar derivations have been developed; see Colombo [45], [46], who proposes a model with a transition from free to congested flow; Berthelin et al. [23], [24]; Degond and Delitala [62], who focus on the identification of the pressure term; Zhang [217]; Jiang, Wu, and Zhu [125]; and Siebel and Mauser [195]. Some of the above modifications include a relaxation term modeling the trend to equilibrium, while interesting improvements, also due to the authors of the original model, focus on a detailed modeling of the pseudopressure term that ensures that the maximal velocity does not exceed $\xi = 1$.

However, rather than analyzing a variety of technical modifications generally based on heuristic reasonings, it is worth focusing on the modeling of the pseudo-pressure term. In fact, recent literature has shown that a careful modeling of such a term based on the complex physics of the system can possibly lead to models with the ability to describe the essential feature of traffic dynamics.

In particular, the model proposed in [23] is called the macroscopic *"Aw–Rascle model with density constraint"* or "modified Aw–Rascle (MAR) model." It is based on the idea that the velocity offset becomes infinite as the density of cars approaches this maximal density.

The model uses the framework (5.2), which after a suitable change of variables [23] becomes

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho\,\xi) = 0\,, \\[2ex] \partial_t \rho + \xi\,\partial_x \xi = \varepsilon \rho\, p'(\rho)\,\partial_x \xi\,, \end{array}\right. \tag{5.10}$$

where $\varepsilon$ is a constant and

$$p(\rho) = \left(\frac{\rho}{1-\rho}\right)^{\gamma}, \qquad \gamma > 0\,, \tag{5.11}$$

such that

$$\left\{\begin{array}{lll} p(\rho) \to \infty & \text{as} & \rho \to 1\,, \\ p(\rho) \sim \rho^{\gamma} & \text{as} & \rho \to 0\,. \end{array}\right. \tag{5.12}$$

The modification of the offset term does not significantly alter the analytical properties of the model, and most of the properties of the Aw–Rascle model remain true for the MAR model (for instance, the form of the conservation equations and the Riemann invariants). In addition, with no substantial modification with respect to the calculations developed in Aw et al. [5], the MAR model can be derived as the macroscopic limit of a modified follow-the-leader (MFL) model. Therefore, the paper constructs an asymptotic limit in which the density can span from vacuum to jam conditions. It is shown that the model obtained is useful for the description of the formation and the evolution of jams or car clusters.

Finally let us focus on an important modification introduced by Degond and Delitala [62] to tackle the fact that the MAR model shows a built-in density constraint, with a fixed maximal distance, corresponding to a "bumper-to-bumper" situation. As experimentally measured by Kerner [127], the safety distance between the vehicles is a function related to the reaction braking time and linearly correlated with the velocity of the vehicle; only in a stopped situation, or a "jam" of vehicles, is the minimal distance correctly the "bumper-to-bumper" distance.

The safety distance $d$ between the vehicles is related to the reaction time of braking, $\tau$, and is assumed to be linearly correlated to the velocity:

$$d(\xi) = \delta + \tau\xi\,, \tag{5.13}$$

where $\delta$ is the minimal distance between the vehicles, corresponding to a bumper-to-bumper situation, as in the previous section. More generally, $d$ can be assumed to be a monotonically increasing function of the velocity. Correspondingly, the maximal

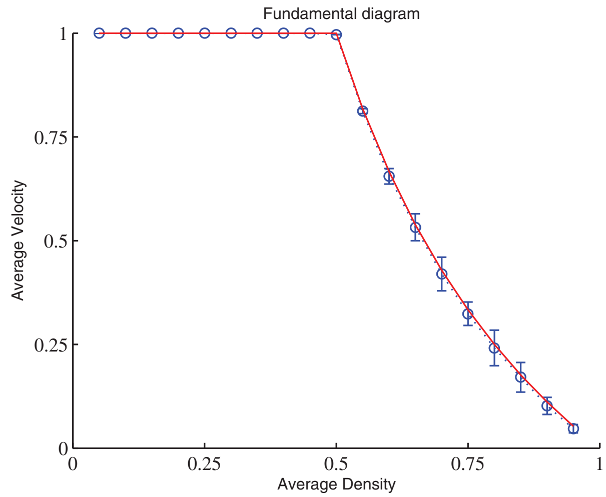


**Fig. 5.1** *Mean speed versus density at equilibrium.*

density is inversely related to the velocity. The MAR model, in the "large space scale limit," has no acceleration term. To overcome this deficiency without losing the well-established microscopic basis of the model, an *acceleration term* is added in the evolution equations. So drivers may accelerate at a constant rate $\eta$ until their velocity reaches a maximal (or desired) velocity, supposed to be uniform for all vehicles.

This feature is shown in the velocity diagram reported in Figure 5.1, which shows the transition from free to congested flow. Specifically, the velocity is constant in free flow conditions and subsequently decays for larger densities as shown in Figure 5.1. This aspect was exploited by Bonzani and Mussone [33] to identify the parameters of a kinetic-type model by measuring the critical density that corresponds to the aforementioned transition. The same reasoning can be applied to identify the parameters of the above model.

Various simulations have been developed to study traffic flow phenomena such as the Riemann problem, merging groups of fast vehicles into slow ones, or vacuum (very slow density) formation, when a group of fast vehicles leaves a group of slow vehicles. Some sample simulations focused on the above phenomena are reported in Figures 5.2 and 5.3.

**5.2. From Traffic to Crowd Modeling.** Modeling of crowd dynamics at the macroscopic scale is far less developed than that of vehicular traffic. However, the interest in this specific field is rapidly growing due to various motivations such as engineering applications, for instance, interactions of crowds and the structures of lively bridges [211] and [210], or the modeling of panic situations and related safety initiatives. For instance, Coscia and Canavesio [49] develop calculations related to crowd dynamics on the pilgrim Jamaral bridge, where every year safety conditions are violated.

The modeling, as shown in [64], can refer to the structure (5.3) in more than one space dimension. Although the dynamics is not precisely the same, models simply extend the one-dimensional approach proposed in vehicular traffic by adding the dynamics of individuals who overtake each other and, for example, move across a room toward a doorway.

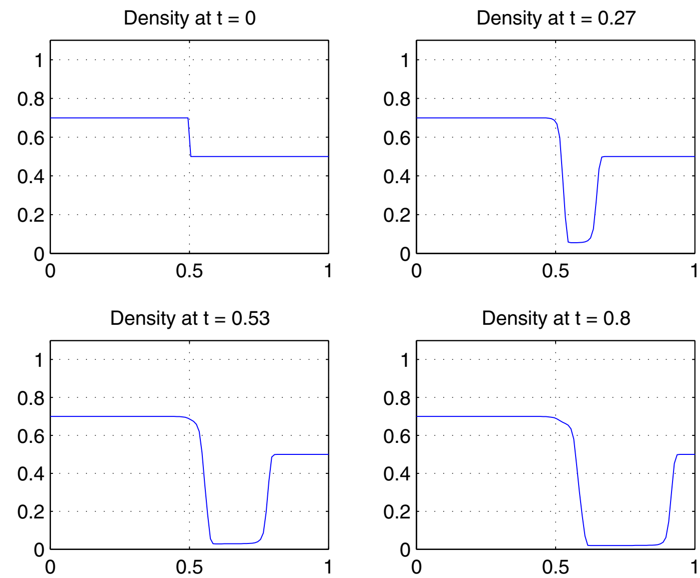


**Fig. 5.2** *Vacuum formation.*

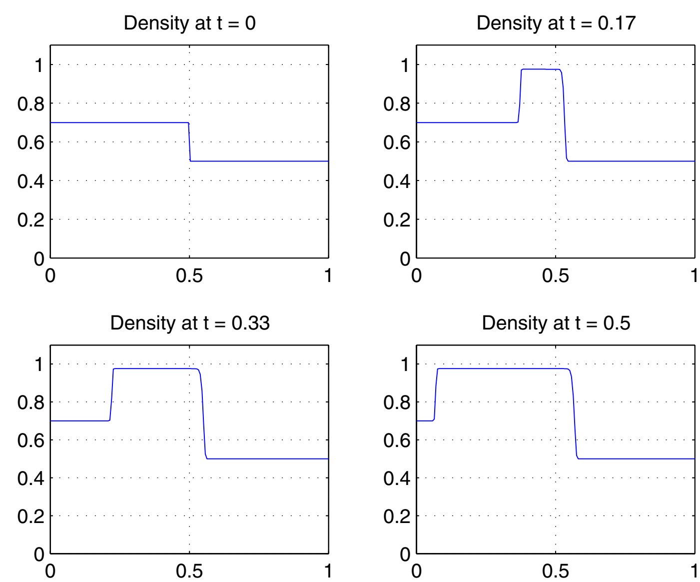


**Fig. 5.3** *Jam formation.*

Macroscopic modeling was initiated by Henderson's pioneering works [109], [112]. He proposed a modeling approach using equations related to the kinetic theory of a homogeneous gas constituted of statistically independent particles in equilibrium in a two-dimensional space. This approach was also documented in Henderson and

Lyons [110], Henderson and Jenkins [111], AlGadhi and Mahmassani [2], and AlGadhi, Mahmassani, and Herman [3]. Subsequently, Hughes [123], [124] extended Henderson's fluid dynamics approach to allow for factors of human decision and interaction. Hughes developed a model that represents pedestrians as a continuous density field, where an evolving potential function models the guide to the density field optimally toward its goal. His approach can be classified as related to first-order models. The interested reader can find further information on the subject in the papers by Coscia and Canavesio [49], which has the merit of developing several interesting simulations, and by Venuti and Bruno [209], [210], focused on coupling crowd dynamics to structures. Further analysis is given in [48], [206].

Additional developments of pedestrian flow modeling at the macroscopic scale are due to Huang et al. [122], who generalized Hughes models to model the case of multiple types of pedestrians with different walking characteristics and destinations. Hoogendoorn and Bovy [119], [120] developed a pedestrian flow model formulated by assigning an optimal—for the user—equilibrium dynamics.

Finally, let us stress that, as already mentioned, the modeling approach refers to the structure (5.3), not (5.2), in more than one space dimension, although this approach has given, as we have seen, several interesting results in vehicular traffic modeling.

The main additions to the models of pedestrian dynamics are a *desired velocity vector* field that makes the actual velocity follow some movement profile and the expression of a strategy to reach a well-defined target (the exit or a meeting point). The modeling problem consists in the mathematical description of the accelerations which depends also on the local flow conditions.

This acceleration can be viewed, according to a simple approach, as the superposition of two contributions due to an adaptation to the mean flow velocity measured in steady uniform flow conditions $\xi_e$ and to local density gradients. Both contributions are supposed, in a first approximation, to act along the unit vector $\vec{\nu} = \vec{\nu}(x, y)$ directed from $P$ to the target $T$, as shown in Figure 2.2. In other words, individuals at the point $(x, y)$ aim to reach a destination $(x_T, y_T)$ along their intended direction of movement given by the unit vector $\vec{\nu}$.

Bearing all the above ideas in mind, a brief review of some second-order macroscopic models of crowd is now given, referring specifically to [19]. In detail, the acceleration can be modeled as

$$\vec{\mathcal{A}}[\rho, \vec{\xi}] = \mathcal{A}_F[\rho, \vec{\xi}]\, \vec{\nu}(x, y) + \mathcal{A}_P[\rho, \vec{\xi}]\, \vec{\nu}(x, y)\,, \tag{5.14}$$

which is based on two assumptions:

(i) the frictional acceleration $\mathcal{A}_F$ is proportional to the deficit in velocity (from what is typical of pedestrians in a crowd of the same density):

$$\mathcal{A}_F = c_F\, (v_e(\rho) - \xi); \tag{5.15}$$

(ii) the acceleration $\mathcal{A}_P$ between pedestrians is normal and determined only by the gradient of pedestrian density:

$$\mathcal{A}_P = -c_P\, \nabla_s \rho\,, \tag{5.16}$$

where $c_F$ and $c_P$ are constants, $s$ is a scalar coordinate along the direction of $\vec{\nu}$, $\xi = |\vec{\xi}|$, and $v_e$ is the equilibrium velocity defined in section 2.

Assuming that the density $\rho$ is a smooth function of the space everywhere, the acceleration equation for pedestrian flows (5.3) is as follows:

$$\partial_t \vec{\xi} + (\vec{\xi} \cdot \nabla_{\mathbf{x}})\vec{\xi} = c_F \, (\xi_e(\rho) - \xi)\vec{\nu} - c_P \, \nabla_s \rho \, \vec{\nu} \,. \tag{5.17}$$

The above expression aims to capture the way in which individuals adjust their velocity toward an optimal value given by the velocity function, and how they react to density gradients. As in traffic modeling, one can take $c_F$ to be inversely proportional to the relaxation time. Moreover, the density gradients can be substituted by those of the "pseudopressure" that more precisely takes into account the inconvenience experienced by the pedestrians in high density conditions. The simplest assumption consists in $p = \rho$, while, more generally, one has

$$\mathcal{A}_\rho = -c_\rho \frac{1}{\rho} \, K^2(\rho) \, \nabla_\nu \rho \cdot \tag{5.18}$$

Several different assumptions concerning the above terms are proposed in [19], where a qualitative analysis focused on the property of hyperbolic structures is developed. The interested reader is referred to [19], [71] for technical details. Among various improvements, models can be further refined by taking the gradients in the computation of the local mean velocity. For instance, by computing it at a density higher than the real one in the presence of positive gradients and lower than the real one for negative gradients. This type of reasoning is proposed in [61]. Moreover, models can be designed that add a linear or nonlinear diffusion term of the velocity that corresponds to a viscous-like dissipation of the fluid.

On the other hand, the mathematical structures corresponding to (5.2), adapted to crowds in several space dimensions, have not yet been used in crowd modeling, as is critically examined in the following section.

**5.3. Critical Analysis.** Macroscopic models of traffic and crowds, although the flow of vehicles does not fulfill the paradigms of continuum mechanics, can provide interesting information on the time-space evolution of the macroscopic variables which describe the real system. This survey has been limited to one-lane dynamics. The generalization to multilane flow can be obtained by phenomenological models of the segregation or movement from one lane to another as documented, among others, in [94], [146], and [198]. However, it is worth stressing that macroscopic models are quite cumbersome to study and apply to this specific target. Hence, they lose their simplicity in comparison with microscopic models that appear to be well suited for such a purpose. An additional problem related to multilane flows is that the shape of the equilibrium flow versus the density fundamental diagram can be significantly modified by the number of lanes and their geometry. Moreover, [219] shows contradictions in the presence of several lanes to the criticisms raised in [54] and [6]. This issue will be further commented on in what follows.

In general, it is expected that macroscopic models have the ability to reproduce experimental data concerning the fundamental and velocity diagrams reported in section 3 and to depict the emerging behaviors that are qualitatively observed in various traffic conditions. Particularly important is the transition from free to congested flow. Indeed, recent developments have shown that the modeling approach introduced in [62] has been properly developed to describe a variety of interesting peculiarities of traffic dynamics such as phase transitions and the trend to equilibrium velocity, on the whole consistent with anisotropic traffic behavior. Moreover, the simulations developed in [62] have shown that models can be sufficiently refined to describe various emerging collective behaviors that have been observed in experimental data. A systematic analysis using simulations and qualitative analysis of the solutions was developed by Goatin et al. [38], [47], [84] to show the existence of phase transitions. Phase transitions can be related to instability problems [99], [104].

The reader interested in dealing with the effective validation of models is referred to [106], which provides sharp classification and prediction of congested traffic states followed by their theoretical interpretation. Indeed, validation of models has to be based on their ability to depict emerging collective behaviors in different flow circumstances. The paper [106] offers a significant contribution with its analysis of collective dynamics in several flow conditions such as bottlenecks, ramps, junctions, etc.

Still focusing on validation, we have previously stated that the model has to reproduce fundamental diagrams. However, at least in principle, models should also depict different types of diagrams due not only to the environmental conditions, but also to the different geometry of the flow. Particularly important is the derivation of such diagrams in urban traffic flow conditions [96].

A further problem to be carefully considered is the statement of boundary conditions at junctions [59]. This topic is well documented in the book [79] devoted to modeling traffic phenomena in large networks. This book reports the pertinent literature in the field. The analysis of networks shows interesting mathematical problems such as Riemann solvers for conservation laws [81] and challenging applications [80]. Particularly important for engineering applications is the analysis of optimization and control problems [89], [77], [114]. However, this topic has to be further investigated, possibly by taking advantage of the various hints offered by [101].

In general, models should be characterized by hyperbolic, rather than parabolic, structures. This aspect has been stressed in paper [6]. Accordingly, qualitative analysis of the hyperbolic properties of the model can play an important role in its validation, as documented in the paper by Goatin [84]. The analysis is developed using the classical approach based on writing the equations in a suitable conservation form and subsequently computing the eigenvalues and eigenvectors. Additional work has to be developed to prove the existence of solutions to the initial value problems, as documented in [23] and [24], to identify the onset of shock waves. Various authors critically analyze the validity of models based on their hyperbolic structure, specifically on the speed of perturbation referred to the mean speed of vehicles. Criticisms in [54] and [6] are related to the mechanical behavior of vehicles, while the presence of drivers, namely, their ability to look both a distance in front of vehicles and also to the rear, needs further examination, as documented in [99] and further discussed in [220].

Of course, macroscopic models do not take into account the heterogeneous behavior of the driver-vehicle subsystem, although the heterogeneity related to different types of vehicles and environmental conditions can be considered as in [105]. Methods of mathematical kinetic theory can be developed toward this target. Therefore, the dispersion of data observed by Kerner [136] cannot be described by macroscopic models, while stability analysis is often used to identify stop-and-go phenomena. Models with a variety of different types of vehicles have been proposed by various authors who considered multiclass systems; see, e.g., [42], [116]. However, a deeper analysis of the heterogeneous behavior of the driver-vehicle subsystem, which is influenced by local traffic conditions, can be developed using the multiscale approach that is reviewed in the last section.

On the other hand, pedestrian dynamics has not been studied as extensively as vehicular traffic. Up to now, models have been developed simply by generalizing vehicular traffic models by considering the multidimensional nature of the dynamics and the trend of pedestrians toward specific targets such as exit areas. The reasoning on the hyperbolic structure that we have seen for traffic models can be extended to crowd modeling. Detailed calculations need to be further developed to consider the fact that dependent variables are defined over space domains in more than two dimensions.

Arguably, further improvements can be obtained using the modeling approach developed in [62] for the case of crowds. Indeed, pedestrians, like vehicles, have a braking ability and a finite dimension and an ability to defend themselves from jams of individuals. Further experimental work could definitely improve the models. The ETH report [37] is a useful reference on empirical data on crowd dynamics. In general, the modeling approach has to take into account not only local interactions, but also long-range interactions which can be identified either by specific targets, such as an exit zone, or by attraction or repulsion from groups of individuals. It is worth mentioning, along these lines, that Piccoli and Tosin [182], [183] have proposed a model of evolution of probability measures occupied by the crowd. It is an interesting approach, which, although developed within the framework of the macroscopic scale, refers to the dynamics at the lower scale, namely, to the strategy developed by pedestrians.

A further aspect, to be carefully taken into account, is the sensitivity of the behavior of the system to panic conditions which introduce several additional modifications to both individual and collective behaviors.

As already mentioned, macroscopic modeling needs the a priori assumption of validity of the paradigms of continuum mechanics, while these models do not include the heterogeneous behavior of driver-vehicle subsystems. This specific aspect can be treated by methods of kinetic theory, as we shall see in the following sections.

## 6. Models of the Generalized Kinetic Theory.

This section presents a survey and critical analysis of the existing literature on the modeling of vehicular traffic and crowd dynamics taking the approach of generalized mathematical kinetic theory.

The development takes advantage of some review papers in the literature [18], [95], [137], which cover specific topics of the overall subject; the final aim is analogous to that of the preceding section, namely, to extract new modeling perspectives. The contents are divided into three subsections that follow this brief introduction. The first one describes various mathematical structures that can be used for the modeling approach; the second reports on various models that use those structures; the third shows how some ideas on modeling vehicular traffic can be developed toward the description of crowd dynamics. The contents are focused on a one-lane flow, while various generalizations have been proposed to include the dynamics of the passage from one lane to another [98], [152], [194], all of which are obtained by a straightforward development of one lane models by adding the modeling of the dynamics by which vehicles shift from one lane to another.

Before dealing with the above topics, it is worth anticipating two relevant complexity problems in the modeling of the systems under consideration. We focus on the assumption of continuity of the distribution function and the assumption of homogeneity of the behavior of the driver, both of which have been criticized by Daganzo [54]. It is plain that the number of vehicles is not large enough to justify the continuity assumption. Moreover, behavior is not the same for all car drivers or pedestrians.

Therefore, new methodological approaches to modeling should take into account the various criticisms raised on the models reviewed in this section. The last two sections deal with these complexity problems. Moreover, the analysis of this section motivates a search for *unified mathematical structures* able to capture the most significant features of several different approaches.

**6.1. Mathematical Structures.** Let us look at the *mathematical structures* that have been proposed in the literature for modeling traffic phenomena. The conceptual background for the derivation of kinetic-type models of traffic flow is classical mathematical kinetic theory [180]. Different models correspond to different ways of modeling interactions: among particles, which may be localized, as in the case of the Boltzmann equation, or long range, as for the Vlasov equation. Modeling may also include the role of the dimension of particles in the dynamics of interactions, as in the case of Enskog-type equations [20]. In some cases, the difficulty of modeling interactions at the microscopic level has suggested phenomenological models, for instance, the BGK models [180], which describe the trend of the distribution function to the Maxwellian equilibrium distribution.

Similarly, the modeling of traffic flow has been developed using different mathematical frameworks. The difference with respect to the classical theory is that interactions do not follow the rules of classical mechanics, but instead the driving strategy is expressed by the vehicle-driver subsystems.

• Let us first consider *phenomenological models* that are characterized by the structure

$$\partial_t f + v\,\partial_x f + F(t,x)\,\partial_v f = Q[f;\rho]\,, \tag{6.1}$$

where $f = f(t,x,v)$, $F(t,x)$ denotes the acceleration applied to it by the environment, and $Q[f;\rho](t,x,v)$ is a suitable function of $f$ to be derived on a phenomenological basis that can be parameterized by local macroscopic quantities.

A simple way to model the term $Q$ consists in describing a trend to equilibrium analogous to the BGK model in kinetic theory,

$$Q(t,x,v) = c_r(\rho)\,(f_e(v;\rho) - f(t,x,v))\,, \tag{6.2}$$

where the rate of convergence $c_r$ depends on the local density and $f_e$ denotes the equilibrium distribution function that may be parameterized by the local density.

Models generally assume $F = 0$; however, acceleration terms may be imposed by the environment, for instance, signaling to accelerate or decelerate. A question that arises naturally refers to the possibility that a vehicle may be subject to an acceleration due to the vehicles ahead, similar to that characterizing hydrodynamical models at the macroscopic scale.

• *Localized binary interaction models* are based on microscopic modeling, which assumes binary interactions between the test and the field vehicles localized at point $x$ of the field vehicle. Interactions, similarly to the Enskog equation, can be localized at a fixed distance $d$ from the test vehicles ahead. Moreover, similarly to the Boltzmann equation, a factorization of the joint probability related to the two vehicles is assumed. For both types of interaction, the formal structure of the evolution equation is as follows:

$$\partial_t f(t,x,v) + v\,\partial_x f(t,x,v) = J[f](t,x,v)\,, \tag{6.3}$$

where $J[f]$ can be written as the difference between the inflow (*gain*) and outflow (*loss*) of vehicles in the elementary volume of the phase space. The structure of this operator depends on the methods used to model the interaction at the microscopic level.

The paper by Delitala [67] suggests the following formal structure:

$$\begin{aligned}\partial_t f(t,x,v) &+ v\,\partial_x f(t,x,v)\\ &= \int_0^{1+\mu}\int_0^{1+\mu} \eta(v_*,v^*)A(v_*,v^*;v)f(t,x,v_*)f(t,x,v^*)\,dv_*\,dv^*\\ &- f(t,x,v)\int_0^{1+\mu}\eta(v,v^*)f(t,x,v^*)dv^*\,,\end{aligned}\tag{6.4}$$

where the right-hand side gives the difference between the inflow (gain) and outflow (loss) of vehicles in the control volume of the phase space. $\eta(v_*,v^*)$, or $\eta(v,v^*)$, is the *encounter rate*, the number of interactions between pairs of vehicles per unit time in the unit space; generally it is assumed to be proportional to the relative velocities $|v_*-v^*|$ and $|v-v^*|$. $A(v_*,v^*;v)$ is the *transition probability density* that a *candidate* or *test* vehicle with velocity $v_*$ interacting with a vehicle with velocity $v^*$ ends up with velocity $v$. The density $A$ must be equal to zero for $v \geq 1+\mu$.

The above structure can be immediately generalized to the case of interactions localized at a fixed distance ahead of the test vehicle. Then Enskog-type models are obtained. The technical difference is that the field vehicle is not localized in the same place $x$ as the candidate or test vehicle, but at a certain distance from $x$ that can be chosen depending on the velocities of the interacting pair. Moreover, the Enskog-type modeling introduces a pair correlation function depending on the local densities in the positions of the interacting pairs.

• *Models with weighted binary interactions.* The paper [67] also introduces structures for models where a suitable function $\varphi(x,y)$ models the weight of the action on the driver of the test or candidate vehicle at $x$ due to the field vehicle at $y$ within the *visibility area* $D=[x-\Delta_r, x+\Delta_f]$ of the vehicle at $x$. $\Delta_r$ and $\Delta_f$ are, respectively, the rear and frontal visibility distance. For $y\in D$ the weight $\varphi(x,y)$ must be such that $|x-y|\uparrow\Rightarrow\varphi\downarrow$, and its integral in $dy$ over the domain $D$ is equal to 1.

The corresponding mathematical structure is as follows:

$$\begin{aligned}&\partial_t f(t,x,v)+v\,\partial_x f(t,x,v)\\ &\quad=\int_D\int_0^{1+\mu}\int_0^{1+\mu}\varphi(x,y)\eta(v_*,v^*)A(v_*,v^*;v)f(t,x,v_*)\,f(t,y,v^*)\,dv_*\,dv^*\,dy\\ &-f(t,x,v)\int_D\int_0^{1+\mu}\varphi(x,y)\eta(v,v^*)f(t,y,v^*)\,dv^*\,dy\,.\end{aligned}\tag{6.5}$$

It is immediate to show that (6.4) is derived from (6.5) simply by assuming $\varphi(x,y)=\delta(y-x)$, where $\delta$ denotes the Dirac delta function, while Enskog-type models are obtained by assuming that $\varphi$ is a delta function over the length of vehicles.

A substantial difference with respect to (6.4) and (6.5) was introduced in [68] using a mathematical structure that corresponds to so-called *averaged stochastic games*. It is based on the assumption that the driver identified by the variables $x$, position, and $v_*$, velocity, plays a game based on a weighted vision of the distribution of vehicles in its visibility zone. Both the table of interaction rates and the table of games depend on the local density. Considering that this structure has been proposed to model granular flow by discrete velocity models, it will be discussed in detail in the following section.

• *Models with long-range interactions* need the definition of the quantity: $\mathcal{F}(x, y, v, v^*)$, the positional acceleration applied to the vehicle at $x$ with velocity $v$ by the vehicle at $y$ with velocity $v^*$. The corresponding structure is

$$\partial_t f(t, x, v) + v\, \partial_x f(t, x, v) + \partial_v \big(\mathcal{A}[f] f\big)(t, x, v) = 0\,, \tag{6.6}$$

where $\mathcal{A} = \mathcal{A}(t, x, v)$ is given by summing all actions in the visibility zone of the test vehicle:

$$\mathcal{A}[f](t, x, v) = \int_D \int_0^{1+\mu} \mathcal{F}(x, y, v, w^*) f(t, y, w^*)\, dy\, dw^*\,. \tag{6.7}$$

The survey of traffic flow models proposed in the following subsection refers to the above structures, where models correspond to specific versions of the various interaction terms that we have seen above.

**6.2. A Survey of Traffic Flow Models.** Some classical models can be referred to the schemes above. Specifically, the derivation of the pioneering model by Prigogine and Hermann [185] is based on the contribution of various frameworks. In detail, it is derived assuming that the driver is willing to adjust the vehicle's velocity, either increasing or decreasing it, toward a certain desired velocity distribution. In addition, the velocity can also change due to its interaction with the heading vehicle. In both cases, the rate of change depends on the density of vehicles. The flow is assumed to be one-directional, and each vehicle is modeled as a point, i.e., the length of each vehicle is negligible with respect to the length of the road, although a maximal density $n = n_M$, namely, $\rho = 1$, is considered. The evolution of $f$ is ruled by a balance equation, generated by vehicle interactions, according to the scheme

$$\partial_t f + v\, \partial_x f = J[f] = J_r[f] + J_i[f]\,, \tag{6.8}$$

where $J_r$ is called the relaxation term and accounts for the speed change toward a certain *program* of velocities independent of the local concentration, and $J_i$ is the term reflecting the *slowing down* interaction between vehicles.

The term $J_r$ is modeled assuming that drivers have a desired velocity described by the distribution $f_e = f_e(x, v)$, the *desired-velocity distribution function.* Moreover, the driver desires reach this velocity distribution within a certain relaxation time $\tau$, related to the normalized density and equal for all drivers.

The term $J_i$ models the interaction between a *test* vehicle and its heading *field* vehicle. It accounts for the changes in $f(t, x, v)$ caused by a braking of the test vehicle due to an interaction with a field vehicle, and reflects the braking when the test vehicle has velocity $v < w$ and the acceleration when the field vehicle has velocity $w < v$. Moreover, $J_i$ is proportional to the probability $P$ that the fast car may pass the slower one, which may be related to the normalized density assumed to be equal for all drivers. Technical details of the model above are reported in [185] as well as in the review papers [18] and [137]. The model needs empirical data able to provide an expression for the desired velocity distribution function that may be related to the distribution corresponding to uniform equilibrium.

It is worth mentioning the relevant modification of the above model proposed by Paveri Fontana [178], who criticizes the relaxation term by showing that it has some unacceptable consequences, for instance, it becomes meaningless for low densities. To overcome such a problem, the desired velocity $v^*$ is assumed to be an independent variable of the problem, and a generalized one-vehicle distribution function $g = g(t, x, v; v^*)$ is introduced to describe the distribution of vehicles at $(t, x)$ with speed $v$ and desired speed $v^*$. Hence, the distributions $f_e$ and $f$ concern, respectively, the distribution over the desired and real speeds,

$$f_e(t, x, v^*) = \int_0^{1+\mu} g(t, x, v; v^*)\, dv \tag{6.9}$$

and

$$f(t, x, v) = \int_0^{1+\mu} g(t, x, v; v^*)\, dv^* \,. \tag{6.10}$$

The evolution equation, which now refers to the generalized distribution function $g$, is again determined by equating the transport term on $g$ with the sum of the interaction term and the relaxation term:

$$\partial_t g + v \partial_x g \;=\; J_P[g] \;= J_r[g] + J_i[g] \,. \tag{6.11}$$

Detailed calculations are reported in the review papers [18] and [137].

The above models have the disadvantage that the equilibrium velocity distribution has to be introduced into the interaction operator, while detailed modeling of interactions at the microscopic level should naturally generate, as direct predictions of the model, stationary solutions that are of great relevance in the analysis of traffic flow.

Models stemming from the microscopic description of pair interactions have the advantage that comparisons with experimental data (and organization of suitable experiments) can be arranged related only to the microscopic behaviors. This approach was introduced and developed by various authors, first Nelson [168] and Nelson and Sopasakis [171], and subsequently Klar and Wegener [138], [140], who were able to exploit the advantages of modeling based on short-range interactions including the case of Enskog-type localizations.

However, modeling microscopic interactions is certainly not a simple task as it requires a detailed analysis of vehicle dynamics and driver reactions, together with the organization of related experiments. The problem consists in finding suitable expressions for the postinteraction velocities which, in the microscopic modeling, are directly related to the preinteraction velocities. Furthermore, if high densities must be taken into account, modifications of the interaction frequency should be included in a manner similar to that used in deriving the Enskog equation. The paper by Klar and Wegener [141] shows how equilibrium distributions are obtained corresponding to each local density, with a distribution that tends to a Dirac delta function over the null velocity when the density of vehicles tends to its maximal admissible value, while the variance of the velocity distribution increases with decreasing density.

**6.3. From Vehicular Traffic to Crowd Modeling.** The existing literature on crowd modeling is limited to the approaches at the microscopic and macroscopic scales, while the application of methods of the generalized kinetic theory is still in progress. Therefore, the presentation of this topic is limited to focusing on some guidelines to the modeling approach.

The reference frameworks are identified by (6.5) for localized interactions and (6.6)–(6.7) for mean field interactions, where space and velocity variables are defined in more than one space dimension. These structures allow the derivation of kinetic-type models simply by modeling the dynamics at the microscopic level.

The localized interaction method needs the identification of the interaction rate $\eta$ and the transition probability density $\mathcal{A}$. Interactions are weighted over a visibility zone. The modeling of the term $\mathcal{A}$ also needs to consider the trend of pedestrians toward the exit area in the same way as for the macroscopic modeling approach. The modeling of $\mathcal{A}$ is not an easy task considering that the dynamics of pedestrians is remarkably influenced by the local density distribution.

The reasoning in the case of long-range interactions is analogous. The modeling approach needs the identification of the acceleration term $\mathcal{F}$ and hence of $\mathcal{A}$ within a suitable visibility zone. Also, in this case, the trend of pedestrians toward the exit area has to be taken into account, while the local density distribution affects the aforementioned actions.

Long-range interactions are important in the modeling of pedestrian dynamics, considering that individuals develop their strategy by taking into account not only nearby pedestrians, but also long-distance interactions. Generally, long-distance effects are qualitatively different from short-distance effects and play a relevant role in the dynamics.

The modeling, as we have already seen in the case of the macroscopic approach, needs to be adjusted to distinguish normal conditions from panic situations [100], [170].

**7. Modeling Granular Flows.** The contents of this section are motivated, as previously mentioned, by the observation criticisms of Daganzo [54] that the assumption of continuity of the distribution function over the microscopic state of vehicles (or pedestrians) can be criticized on the basis that the number of interacting entities is not large enough to justify this assumption. Therefore, not only the assumption of continuity of matter has to be questioned, but also the assumption concerning the distribution function. The lack of continuity is also analyzed in [191], which focuses on the fractal aspects of traffic phenomena.

Two recent papers, Coscia, Delitala, and Frasca [50] and Delitala and Tosin [68], have proposed that kinetic-type models with discrete velocities take into account the granular nature of traffic. Therefore, the overall state of the system is described by a discrete probability distribution over groups of vehicles with velocity within a certain range. The model [68] has also been generalized to the case of multilane flow [31].

The authors observe that vehicles traveling along a road do not continuously span the whole set of admissible velocities; rather, they tend to move in clusters which can be identified and distinguished from each other by a discrete set of velocity values. The discretization creates cells in the velocity space for vehicles whose velocity belongs to these cells. The approach is based on different motivations with respect to the so-called discrete Boltzmann equation, which is a crude approximation of physical reality. Here, the discretization of the velocity space is used to simulate the noncontinuous behavior of the distribution function; in other words, it is used to simulate granular flow.

This section provides a description and critical analysis of the above models and subsequently shows how the whole phase space can be discretized. Finally, it is shown how these ideas can be properly developed to model crowd dynamics.

**7.1. Discrete Velocity Models.** Let us consider, referring to [68], the derivation of a mathematical structure corresponding to *discrete velocity models*. The velocity variable belongs to the set

$$I_v = \{v_1 = 0\,, \dots\,, v_i\,, \dots\,, v_n = 1\}\,, \tag{7.1}$$

where velocities have been divided by the maximal admitted velocity $V_\ell$.

The corresponding discrete representation is obtained by linking the discrete distribution functions to each $v_i$:

$$f(t,x,v) = \sum_{i=1}^{n} f_i(t,x)\,\delta(v - v_i)\,. \tag{7.2}$$

The above discrete velocity approach naturally implies that vehicles with velocity larger than $V_\ell$ can be disregarded. In other words, it is technically assumed that the presence of vehicles with velocity much larger than the maximum mean velocity corresponding to the given density is negligible. However, in a discrete velocity framework such a detail is actually not very relevant, since vehicles are grouped and classified on the basis of velocity classes $\{v_i\}_{i=1}^n$, so that those which travel at speeds higher than $V_M$ are simply included in the extreme class $v_n$.

According to the mathematical representation above, the following macroscopic quantities are obtained by weighted sums. In particular, mass density and flow are given by

$$\rho(t,x) = \sum_{i=1}^{n} f_i(t,x) \tag{7.3}$$

and

$$q(t,x) = \sum_{i=1}^{n} v_i f_i(t,x) = \rho(t,x)\,\xi(t,x)\,, \tag{7.4}$$

where $\xi$ is the mean velocity.

Moreover, as in the gas kinetic theory, the speed variance and the $H$ functional can be defined as follows:

$$\sigma(t,x) = \frac{1}{u(t,x)} \sum_{i=1}^{n} \left(v_i - \xi(t,x)\right)^2 f_i(t,x)\,, \quad u(t,x) = \frac{\sum_{i=1}^{n} v_i f_i(t,x)}{\sum_{i=1}^{n} f_i(t,x)}\,, \tag{7.5}$$

and

$$H(t,x) = \sum_{i=1}^{n} f_i(t,x)\log f_i(t,x)\,. \tag{7.6}$$

The model consists in a set of evolution equations for the densities $f_i$ derived according to the following structure:

$$\begin{aligned}
\partial_t f_i(t,x) + v_i\,\partial_x f_i(t,x) &= J_i[\mathbf{f};\alpha](t,x) \\
&= \sum_{h=1}^{n}\sum_{k=1}^{n} \int_{D_w} \eta[\mathbf{f}](t,\,y) A_{hk}^i[\mathbf{f};\alpha](t,\,y) f_h(t,\,x) f_k(t,\,y) w(x,\,y)\,dy \\
&\quad - f_i(t,\,x) \sum_{h=1}^{n} \int_{D_w} \eta[\mathbf{f}](t,\,y) f_k(t,\,y) w(x,\,y)\,dy
\end{aligned} \tag{7.7}$$

for $i = 1, \ldots, n$, where $\mathbf{f} = \{f_i\}_i^n$, $\eta[\mathbf{f}]$ is the *interaction rate*, which gives the number of interactions per unit time among the vehicles, and $A_{hk}^i[\mathbf{f};\alpha]$ defines the so-called *table of games*, which models the microscopic interactions among the vehicles by giving the probability that a vehicle with speed $v_h$ adjusts its velocity to $v_i$ after an interaction with a vehicle traveling at speed $v_k$.

The authors assume that both terms depend on the local density $\rho$ with the additional requirement

$$A^i_{hk}[\rho,\alpha] \geq 0, \qquad \sum_{i=1}^{n} A^i_{hk}[\rho,\alpha] = 1 \quad \forall\, h,\, k \in \{1,\, \dots,\, n\}\,, \tag{7.8}$$

and for all $\rho \in [0,1]$ and $\alpha \in [0,1]$, where $\alpha$ is a parameter depending on the quality of the road: $\alpha = 0$ and $\alpha = 1$ correspond, respectively, to the best and worst conditions.

We summarize the table of games as follows:

• *Interaction with a faster vehicle* $(h < k)$. The candidate vehicle either maintains its current speed or shifts, in probability, to a higher velocity:

$$A^i_{hk}[\rho] = \begin{cases} 1 - \alpha\,(1-\rho) & \text{if } \; i = h, \\ \alpha\,(1-\rho) & \text{if } \; i = h+1 \\ 0 & \text{otherwise.} \end{cases} \qquad (h,\, k = 1,\, \dots,\, n),$$

• *Interaction with a slower vehicle* $(h > k)$. The candidate vehicle does not accelerate and either it is forced to queue, reducing its speed to that of the leading vehicle, or it maintains its current speed because it has enough free space to overtake:

$$A^i_{hk}[\rho] = \begin{cases} 1 - \alpha\,(1-\rho) & \text{if } \; i = k, \\ \alpha\,(1-\rho) & \text{if } \; i = h \\ 0 & \text{otherwise.} \end{cases} \qquad (h,\, k = 1,\, \dots,\, n),$$

• *Interaction with an equally fast vehicle* $(h = k)$. The interacting vehicles are unlikely to strictly preserve their speed during the motion, for this would imply they do not interact, behaving as if they were alone along the road:

$$A^i_{hh}[\rho] = \begin{cases} \alpha\,\rho & \text{if } \; i = h-1, \\ 1-\alpha & \text{if } \; i = h \\ \alpha\,(1-\rho) & \text{if } \; i = h+1, \\ 0 & \text{otherwise.} \end{cases} \qquad (h = 2,\, \dots,\, n-1),$$

• The form of $A^i_{hh}[\rho]$ above applies only if $h \neq 1,\, n$; when $h = 1$ or $h = n$ the candidate vehicle cannot brake or accelerate, respectively, due to the lack of further lower or higher velocity classes:

$$A^i_{11}[\rho] = \begin{cases} 1 - \alpha\,(1-\rho) & \text{if } \; i = 1, \\ \alpha\,(1-\rho) & \text{if } \; i = 2, \\ 0 & \text{otherwise,} \end{cases} \qquad A^i_{nn}[\rho] = \begin{cases} \alpha\,\rho & \text{if } \; i = n-1, \\ 1 - \alpha\,\rho & \text{if } \; i = n, \\ 0 & \text{otherwise.} \end{cases}$$

One might observe that the table of games is very simple, being based only on one parameter, namely, $\alpha$, which models the quality of the road and environment, while $(1-\rho)$ models the increasing, with $\rho$, difficulty in maneuvering.

Moreover, $w(x,\, y)$ represents the function weighting the interactions over the visibility zone in front of the driver and is required to satisfy

$$w(x,\, y) \geq 0, \qquad \int_x^{x+\xi} w(x,\, y)\, dy = 1\,, \tag{7.9}$$

while, borrowing some ideas from the Enskog kinetic theory of dense gases (see [20]), the rate $\eta$ of the interactions among the vehicles is assumed to behave as

$$\eta[\rho] \simeq \frac{1}{1-\rho} \tag{7.10}$$

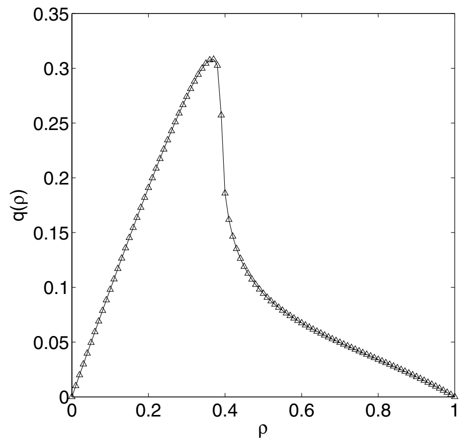


**Fig. 7.1** *Macroscopic flux as a function of the macroscopic density.*

for $\rho \in [0, 1)$. This function is monotonically increasing with respect to $\rho$ in the interval $[0, 1)$, which implies that the local interaction rate becomes higher and higher as the density increases toward its limit threshold fixed by the road capacity.

The model above, despite its simplicity, as shown in the simulations reported in [68], has the ability to describe several interesting phenomena experimentally observed in traffic flow such as propagation of perturbations due to bottlenecks, interactions of groups of fast vehicles with slow vehicles, and so on. Moreover, in the spatially homogeneous case, where the distribution function depends only on time and the model can be written as

$$\partial_t f_i(t,x) = \sum_{h=1}^{n}\sum_{k=1}^{n} \eta[\rho] A_{hk}^{i}[\rho;\alpha] f_h(t) f_k(t) - f_i(t) \sum_{h=1}^{n} \eta[\rho] f_h(t)\,, \tag{7.11}$$

an interesting dynamics is observed: the fundamental velocity diagram described in section 2 is obtained. Specifically, the diagram shows phase transition from free to congested flow according to the quality parameter $\alpha$. Increasing values of critical density where the transition occurs correspond to increasing values of $\alpha$. This behavior is visualized in Figure 7.1, which shows the flux versus density corresponding to specific values of the quality parameter.

A sample simulation is shown in Figure 7.2, which shows how a group of vehicles interacts with an initially empty bottleneck, whose representation is denoted by the thin line; the figure shows how the maximal velocity is progressively reduced. The vehicles in the group are obliged to slow down their velocity leading to a locally increasing density, as shown by the thick line in each of four sequential steps. Finally, considering that the bottleneck is initially empty, the vehicles adjust their density further along the road to lower values.

This is a simple application that, however, shows the flexibility of the model to describe real flow conditions. Moreover, the use of dimensionless coordinates contributes to a rapid interpretation of the phenomena as well as to the efficiency of the computational scheme. Further improvements will be analyzed in the following section.

Technically different is the approach proposed in [50], where it is assumed that the velocity grid depends on the local density. Specifically, the grid $I_v$ has a variable

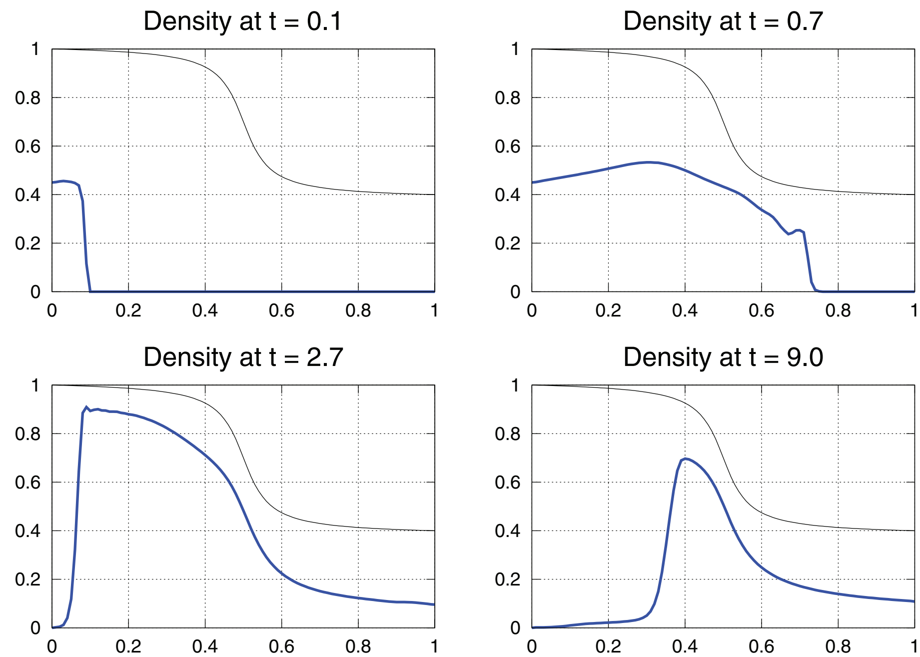


**Fig. 7.2** *Interaction of crowded vehicles with a bottleneck.*

step $\Delta v$, which tends to zero for high vehicle concentrations (*adaptive velocity grid*). The following discretization is adopted:

$$I_v = \{v_1 = 0, \dots, v_i, \dots, v_n = v_e(\rho), \dots, v_{2n-1} = 2v_e(\rho)\}\,, \tag{7.12}$$

where $i = 1, \dots, 2n-1$, with

$$v_i(\rho) = \frac{i-1}{n-1}\, v_e(\rho)\,,$$

and where $v_e$ is the mean velocity in uniform steady state flow conditions delivered by experimental data and approximated as indicated in section 2.

The modeling of the terms $A^i_{hk}$, which define the *table of games*, needs a mathematical interpretation of the microscopic phenomenology of the system. The table of games reported in [68] is designed assuming that vehicles with velocity $v < v_e$ have a natural trend to increase their velocity, while if $v > v_e$, the natural trend is to decrease the velocity. Moreover, slow vehicles have a tendency to increase their velocity to follow fast vehicles, while the opposite behavior is observed in interactions with slow vehicles.

The model is characterized by the following parameters: $n$ defines the number of nodes; $\varepsilon_a \in [0,1]$ corresponds to the slowing-down tendency; and $\varepsilon_b \in [0,1]$ corresponds to the acceleration tendency. The formal structure of the model is as follows:

$$\begin{aligned} \partial_t f_i(t,x) + \partial_x(v_i(\rho) f_i(t,x)) = \sum_{h=1}^{2n-1}\sum_{k=1}^{2n-1} |v_h - v_k| A^i_{hk}[\varepsilon_a, \varepsilon_b] f_h(t,\,x) f_k(t,\,x) \\ - f_i(t,\,x) \sum_{h=1}^{2n-1} |v_i - v_k| f_h(t,\,x)\,. \end{aligned} \tag{7.13}$$

Equation (7.13) corresponds to a nonhomogeneous system of hyperbolic first-order equations with a quadratic right-hand-side term in the unknowns. Moreover, since the velocity $v_i$ depends on $\rho$, namely, on $f_i$, the left term is nonlinear too.

Various simulations are given in the above-cited paper that show how the model has the ability to provide a qualitative description of various phenomena which are observed in physical reality. An interesting feature of the model is the grid with variable size that naturally adapts the intensity of the interaction to the local density: the higher the density, the lower the rate of the interactions. On the other hand, the model needs a suitable assumption on the behavior of $v_e$ versus $\rho$. In other words, the model does not reproduce the velocity diagram.

**7.2. Discretization of the Phase Space.** The contents of this section focus on some very recent results on the modeling approach using discrete kinetic methods developed to model the granular aspect of traffic phenomena. A critical analysis is proposed in this subsection focused on research perspectives.

A first criticism refers to the fact that granular behavior of vehicular flow has to be modeled in the whole phase space. Therefore, following Chapter 6 of [10], an additional framework can be designed by a double discretization including the space variable. Let us consider an equally spaced grid in the space variable of the type

$$I_x = \{x_1 = 0\,, \dots, x_j\,, \dots, x_m = 1\}\,, \tag{7.14}$$

where the space interval $d_j = x_j - x_{j-1}$, which identifies volume cells, should be less than the visibility zone. Moreover, let $f_{ij}(t) = f(t, v_i, x_j)$; it follows that the model consists in an evolution equation for the discrete distribution function.

The derivation of the model can be developed by approximating the space derivative by a conservative scheme using the values of $f$ on the nodes $x_j$ and $x_{j+1}$. Similarly, the interaction term can be properly weighted by its value in the cells in front of $x_j$; see [55] and [56].

Therefore, the formal structure is as follows:

$$\frac{df_{ij}}{dt} + v_i \mathcal{D}_{ij}[f_{ij}, \dots, f_{i,j+r}] = \sum_{p=j}^{p=j+r} \overline{w}_p \, J_{ip}[\mathbf{f}]\,, \tag{7.15}$$

where $\mathcal{D}_{ij}$ denote the approximation of the space derivative of $f_{ij}$, and the weights are such that

$$\sum_{p=j}^{p=j+r} \overline{w}_p = 1\,.$$

Specific models can be obtained referring to the above structure after a detailed modeling of the interaction term $J_{ij}[\mathbf{f}]$. Reasoning analogous to that in the preceding sections can also be developed in the case of the doubly discrete model.

**7.3. Granular Models of Crowds.** The approach reviewed in this section, which has been focused on vehicular traffic, can be generalized to crowd modeling by straightforward calculations. Indeed, the motivations to discretize the velocity or the phase space are the same as those that have generated vehicular granular models.

A technical problem is that both space and velocity are in two dimensions, and therefore appropriate schemes have to be used. For instance, polar coordinates can be used for the velocity so that not only velocity modules, but also angular directions have to be discretized. The mathematical structures (7.7) and (7.15) appear to be

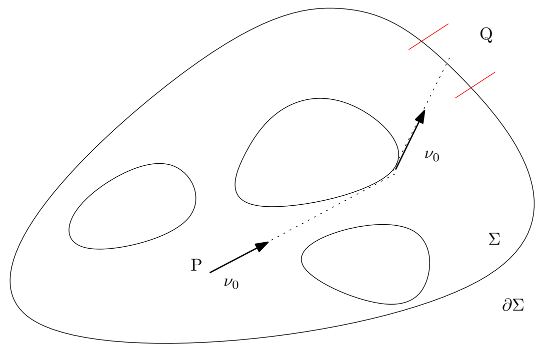


**Fig. 7.3** *Geometry of the crowd domain with obstacles.*

appropriate to model pedestrian dynamics, considering that pedestrians have the ability to look ahead and design a walking strategy based on the distribution of individuals in their visibility zone. The generalization in the case of continuous space and velocity variables is as follows:

$$\begin{aligned}\partial_t f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\partial_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) &= J[f](t,\mathbf{x},\mathbf{v}) \\ &= \int_\Lambda \eta[\rho](t,\mathbf{x}^*)w(\mathbf{x},\mathbf{x}^*)\,\mathcal{A}(\mathbf{v}_*\to\mathbf{v}|\mathbf{v}_*,\mathbf{v}^*,\rho(t,\mathbf{x}^*)) \\ &\quad\times f(t,\mathbf{x}_*,\mathbf{v}_*)f(t,\mathbf{x}^*,\mathbf{v}^*)\,d\mathbf{v}_*\,d\mathbf{v}^*\,d\mathbf{x}^* \\ &\quad - f(t,\mathbf{x},\mathbf{v})\int_\Gamma \eta[\rho](t,\mathbf{x}^*)w(\mathbf{x},\mathbf{x}^*)f(t,\mathbf{x}^*,\mathbf{v}^*)\,d\mathbf{v}^*\,d\mathbf{x}^*\,, \end{aligned}\tag{7.16}$$

where $\mathcal{A}$ is assumed to depend on the local density while $\Lambda = D_{\mathbf{v}}\times D_{\mathbf{v}}\times\Sigma$ and $\Gamma = D_{\mathbf{v}}\times\Sigma$. The discretization of space and/or velocity needs additional straightforward calculations.

Let us now look, with reference to the hints in [11], at modeling aspects. A necessary preliminary observation is that a crowd generally has a target, therefore some geometrical notations are necessary to identify it. For instance, referring to Figure 2.2, given a target point $T = (x_T, y_T)$ on the boundary of $\Sigma$, the walking direction is identified by the unit vector from $P = (x,y)$ to the target $T$:

$$\vec{\nu}_0(x,y) = \frac{x - x_T}{\sqrt{(x-x_T)^2 + (y-y_T)^2}}\,\vec{i} + \frac{y - y_T}{\sqrt{(x-x_T)^2 + (y-y_T)^2}}\,\vec{j}\,,\tag{7.17}$$

where $\vec{i}$ and $\vec{j}$ are two unit orthogonal vectors in a two-dimensional domain. The geometry can be further modified by inserting internal obstacles and an inlet zone, as shown in Figure 7.3.

The modeling problem consists in the characterization of the terms $\eta$, $w$, and $\mathcal{A}$, while the mathematical problem needs, in addition to the initial condition, the statement of boundary conditions, unless the modeling refers to crowds in unbounded domains. Some differences between traffic and crowd modeling and the implications of the onset of panic conditions are now considered.

- The modeling of the interaction rates can be developed similarly to the case of vehicular traffic, namely, by increasing the interaction rate with increasing local density.

- Panic conditions modify the dynamics of interactions in various ways, for instance, by increasing quantitatively the interaction rate and by disregarding the target in favor of clustering.
- The assumption that the weight $w$ decays with distance is not a general rule. Indeed, different contexts (e.g., the escape from fire, etc.) lead to different pedestrian behavior. For instance, clustering phenomena do not always follow the same rules as attraction, and repulsion depends on the specific strategies expressed by the individuals composing the crowd.
- The statement of boundary conditions must be carefully developed, taking into account the active particles leaving $\partial\Sigma$ and those moving into $\partial\Sigma$. This topic is dealt with in [66] by suitable development of methods of classical kinetic theory.

The above essential points have to be regarded, as already mentioned, as preliminary hints to develop a specific research program. A simple model is described in [11], where it is supposed that pedestrians move by only one velocity modulus along a fixed number of directions. A table of games models the passage from one direction to the other by taking into account the trend toward the target and the streaming effect of the other pedestrians.

**8. Research Perspectives.** The review and critical analysis proposed in the preceding sections were focused on various aspects of vehicular traffic and crowd modeling at different representation scales. Some introductory ideas on research perspectives were also outlined. This final section is devoted to some specific research perspectives selected according to the personal biases and intellectual engagement of the authors of this paper. Specifically, the following two topics are considered:

(i) modeling according to the kinetic theory of active particles;

(ii) some introductory ideas on the modeling of swarm dynamics.

These topics are treated in the two subsections that follow, while a conclusion in the final subsection closes this paper. Some preliminary results already available in the literature are described, while ideas for research perspectives are subsequently offered.

**8.1. Modeling by the Kinetic Theory of Active Particles.** The various kinetic models of vehicular traffic reported in the preceding two sections have been derived according to the so-called *generalized kinetic theory*, where interactions at the microscopic level do not follow laws of classical mechanics; however, they are the same for all interacting vehicles. In other words, the behavior of the vehicle-driver subsystem follows specific strategies that modify classical mechanical rules, but they are not heterogeneously distributed among the vehicles.

This aspect is considered in a recent paper by Gramani [85], which has shown that even in space uniform flow the above phenomena play an interesting role in the description of fluid patterns in agreement with the experimental observations of Kerner [127]. For instance, Figure 8.1 shows the velocity diagram and the speed variance versus density for a constant distribution of a variable called *activity* that models the ability of drivers, as depicted by model [68]. The largest values of fluctuations are localized corresponding to the transition from free to congested flow.

The analysis of [85] is limited to the case of constant (with respect to time) probability distribution of the activity variable. On the other hand, such a distribution is modified by interactions among vehicles and depends on local density conditions. Specifically, when the density increases, the behavioral differences decrease, while in jam conditions all vehicles show the same behavior.

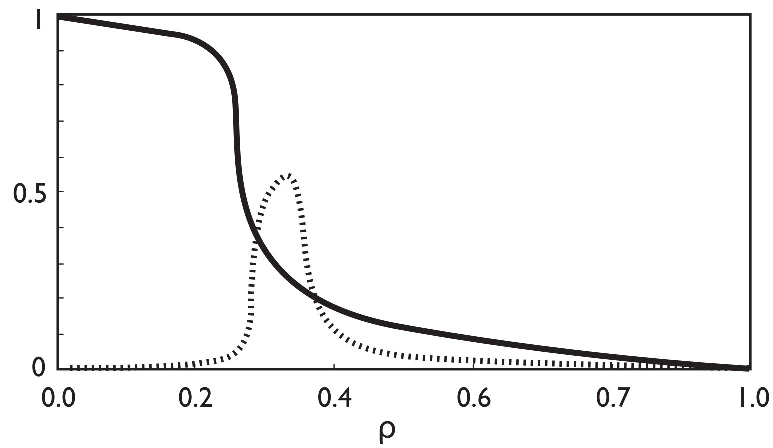


**Fig. 8.1** *Velocity and speed variance versus density.*

This topic can be referred to kinetic-type models, where a parameter corresponding to the quality of the road is included in the interaction operator. Subsequently, the interaction dynamics involves the aforementioned activity variable. The mathematical approach is that of the kinetic theory for active particles, developed starting from [4] to model complex systems in life sciences as documented in [14].

In particular, borrowing some ideas in [14], the following dynamics can be considered in the spatially homogeneous case:

(i) The overall system is described at two scales: the higher of the vehicles and the lower of the drivers.

(ii) The evolution of the system at the higher scale is determined by the interaction between vehicles regarded as active particles, whose activity variable is linked to a time-dependent probability density.

(iii) The evolution of the system at the lower scale is determined by the interaction between active particles among drivers conditioned by the local density.

(iv) The two systems are coupled by the density and the activity variable.

(v) Heterogeneity decreases with increasing density and disappears in traffic jam conditions, when drivers can no longer express their own driving ability.

The modeling and coupling with the lower scale are based on the idea of replacing $\alpha$ by a new parameter $\beta = u\,\alpha$, for $u \in [0,1]$, where $u$ models the quality of the driver-vehicle subsystem. This simple assumption amounts to stating that the quality $\alpha$ of the road corresponds to the best quality driver, while $\alpha$ is reduced to $\beta$ in the case of lower quality drivers. Moreover, $u$ is regarded as a random variable linked to the probability density $\varphi(t,u)$.

The modeling of the dynamics of $\varphi$ can be approached by assuming that vehicles with activities whose distance is greater than a critical value $d_c$ do not modify their state, while they approach their state when the distance is smaller than $d_c$. A reasonable choice of the critical distance is $d_c = \rho$, which means that the trend to mixing increases with increasing density of the vehicles. Simulations confirm the qualitative behavior of Figure 8.1. These ideas have been generalized in [11] to model crowd dynamics.

**8.2. From Crowds to Swarm Dynamics.** This review has been focused on modeling aspects of vehicular traffic and pedestrian crowds. Modeling of swarms is an

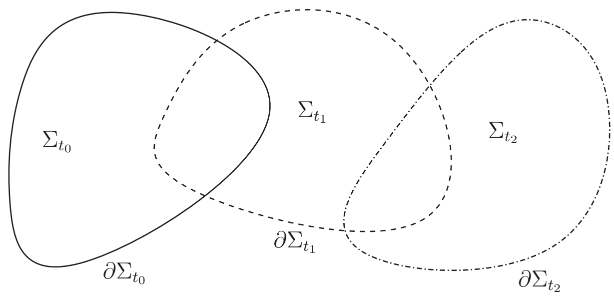


**Fig. 8.2** *Time evolution of the swarm's shape.*

attractive research perspective which is partly motivated by the observation of the beauty of the shapes formed by birds which appear in the sky during spring and autumn periods. Analogous phenomena are, however, observed in other systems such as fish which try to escape the attack of a predator or cells which aggregate forming particular patterns.

The mathematical literature on swarm modeling is very limited compared to that related to traffic and crowds. Moreover, although the scaling and representation are analogous, different modeling approaches have been proposed, such as stochastic differential equations [1], macroscopic equations derived from stochastic perturbation of individual dynamics [41], [63], modeling swarming patterns [25], [201], and flocking phenomena [52], [53], [159], [160], [175], [176], [200]. Deep insight into emerging strategies depends on the type of individuals composing the swarm [9], [69], [73], [74], [88]. The dynamics is occasionally modified by the strategy of the swarm aimed at its survival [176]. Heterogeneity of individual behaviors plays an important role in the dynamics of swarms of cells in biology [17], [43], where several complex events such as proliferative/destructive events or mutations may arise in short time intervals. A specific characteristic is that the swarm has the ability to express a collective intelligence related to the environmental conditions [29], [73], which can evolve by learning processes. This feature is used to drive learning processes in the modern technology of robots [126], [148].

The experimental activity on swarms differs from that in the case of crowds and traffic. In fact, it is mainly focused on understanding the dynamics (and topology) of the interactions corresponding to different animal species. Further, experiments aim to understand emerging behaviors, such as flocking phenomena and break up and aggregation of swarms, which should be depicted by models. Therefore, the collection of empirical data is generally focused on qualitative, rather than quantitative, aspects. The papers cited in this subsection are generally related to specific experiments.

The above reasoning does not claim to be exhaustive, but simply shows how interest in the field is rapidly growing. This should stimulate the interest of applied mathematicians in this challenging research field. A few guidelines are given to support modeling projects:

- Interactions between active particles of a swarm are in three space coordinates, while those of particles of a crowd are defined over two space coordinates.
- Mathematical problems are stated in unbounded domains with initial conditions of compact support. The solution of problems should provide the evolution in time of the domain of the initial conditions. See Figure 8.2.

- Generally, swarms refer to animal behaviors, which differ from population to population and can be modified by external actions that can induce panic. A swarm in normal conditions has a well-defined objective, for instance, reaching a certain zone starting from a localization. However, panic conditions can modify the overall strategy to pursue this objective, which is consequently modified.
- The swarm has the ability to express a common strategy, which is a nonlinear combination of all individual contributions generated by each individual based on the microscopic state of all other individuals. In general, a swarm has the ability to express a collective intelligence that is generated by a cooperative strategy [29], [159], [192].
- The dynamics of interactions differs in the various zones of the swarm, for instance, from the border to the center of $\Sigma$. Stochastic behaviors are an essential characteristic of the dynamics.
- The abovementioned strategy includes a clustering ability (flocking) that prevents the fragmentation of $\Sigma_t$. Moreover, when a fragmentation of $\Sigma_t$ occurs, the clustering ability induces an aggregation.
- The concept of swarm can be extended to various types of microorganisms and ultimately to cells in a multicellular system. In this case, the strategy expressed by the interacting entities depends on the biological functions that characterize the population. Moreover, the modeling approach should include proliferative and/or destructive events.
- The modeling approach should take into account the fact that fluctuations are an intrinsic feature of the systems under consideration. Moreover, it is worth mentioning that recent studies [7] conjecture, on the basis of empirical data, that some systems of the animal world develop a common strategy based on interactions depending on topological rather than metric distances. This is definitely a valuable suggestion for modeling.
- The insight on emerging strategies needs to be specifically referred to the type of individuals composing the swarm [87] and the specific applications considered in [9], [69]. Heterogeneity of individual behaviors plays an important role in the dynamics of swarms of cells in biology [17], where several complex events, such as proliferative/destructive events or mutations, may arise in short time intervals.

Finally, let us mention that although far from the specific contents of this paper, various interesting papers investigate crowding and swarming phenomena at the low scale (molecular and cellular) in biology, such as [155], [207], [212]. Interesting analogies and differences can be observed in nature that should motivate further research in the field.

The ideas above need to be regarded as a very preliminary step toward the development of models suitable to describe the complex dynamics of the system under consideration. Possibly, research projects could be developed using the kinetic theory methods reviewed in the preceding sections, and the modeling of the interaction terms should take into account the qualitative indications given above.

**8.3. Conclusion.** The contents of this review paper have focused on the modeling of vehicular and pedestrian traffic, with very few ideas given on the modeling of swarms. The survey has concentrated its modeling strategy on one-way roads or simple plane geometry. The interested reader is referred to the specialized literature for modeling networks of roads [44], [79], [147], in the case of vehicular traffic, or complex geometries in the case of crowd dynamics [49]. In this case, the problem requires detailed modeling of boundary conditions at junctions [47], [59] or on obstacles and boundaries; see [101].

The complex systems dealt with in this paper, namely, vehicles on roads or networks of roads, crowds, and swarms, possibly share common features, although they are characterized by remarkable differences. A common feature is that individuals belonging to the above systems have the ability to communicate, although in different ways, and have a common strategy. On the other hand, it can be observed that traffic flow is directional in one space dimension (or multilane) and over well-defined networks, while the dynamics of crowds and swarms is in two or three space dimensions, either in bounded domains or in the whole space. Crowds may be constrained by particular geometries that generate different aggregation rules.

Moreover, in traffic flows all drivers have approximately the same strategy, which is not significantly modified by outer conditions, while in crowds the dynamics of the interactions and the overall strategy is modified according to specific situations; for instance, the presence of panic may lead to quite significant changes. The modeling approach should capture both analogies and differences. In all cases, the behavior of the system is difficult to understand, no matter how simple the behavior of its parts, even though a global pattern or structure certainly occurs.

An examination of research perspectives on modeling aspects should consider the rising awareness that many systems in nature resemble those dealt with in this paper, for instance, in fields such as communication, social, and economics interchanges. These systems cannot be successfully modeled by the traditional methods used for inert matter. The common feature is that the overall dynamics is determined by individual interactions, while modeling of individual dynamics does not straightforwardly lead to a mathematical description of the collective dynamics.

A relevant mathematical problem is the modeling of the heterogeneous behavior of individuals that includes their ability to organize interactions according to well-defined objectives or strategies. This self-organizing ability is not the same for all individuals, and has to be regarded as a random variable linked to a probability distribution that might have a local nature and may be modified by several types of interactions at the microscopic level. Therefore, the modeling should include heterogeneity and stochastic features. For instance, it could include the closure of macroscopic equations by material models including a random behavior. Similarly, when methods of kinetic theory are applied, interactions should be modeled by stochastic games rather than by deterministic rules. The approach outlined in section 8.1 can be regarded as a first step toward this challenging goal.

As we have seen in some recent developments of the modeling approach, models have reached the ability to reproduce flow capacity, the correct transition point between free flow and congested traffic, fluctuations after transition to the congested phase, and the correct jam wave speed. On the other hand, it should be expected that stop-and-go dynamics should be included in the predictive ability of models.

Therefore, rather than selecting one only modeling scale, the mathematical approach should also consider the simultaneous interaction of two scales, where the lower scale modifies the heterogeneous behavior at the higher scale. This aspect should possibly also refer, as in the case of multicellular systems, to the derivation of macroscopic equations from the underlying microscopic description [5], [12], [41]. The lack of analytic expressions for equilibrium configurations and of an entropy functional constitutes a remarkable difficulty.

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