$u_t + Vu_x = 0$ preserves positivity (and monotonicity): solution is $u(x,t) = u_0(x-Vt)$, so $u_0 \geq 0 \Rightarrow u \geq 0$

and so does $u_t + (vu)_x = 0$ for any v. Want monotone scheme.

Stability of linear advection schemes:

$$U_i^{n+1} = U_i^n + \frac{\Delta t_n}{\Delta x_i}\left[ADV_{i-1/2}^{n+\theta} - ADV_{i+1/2}^{n+\theta}\right] + \Delta t_n S_i^{n}$$

upwind flux:

$$ADV_{i-1/2}^{n+\theta} = \begin{cases} V_{i-1/2}^{n+\theta}\, U_{i-1}^{n+\theta} & \text{if } V>0 \\ V_{i-1/2}^{n+\theta}\, U_i^{n+\theta} & \text{if } V<0 \end{cases}$$

Let's examine the case $V > 0$:

$$U_i^{n+1} = U_i^n + \frac{\Delta t_n}{\Delta x_i}\left[V U_{i-1}^{n+\theta} - V U_{i}^{n+\theta}\right]$$

$$= U_i^n + \underbrace{\frac{V\Delta t_n}{\Delta x_i}}_{\mu}\left[(1-\theta)U_{i-1}^n + \theta U_{i-1}^{n+1} - (1-\theta)U_i^n - \theta \cdot U_i^{n+1}\right]$$

$$= \left[1-(1-\theta)\mu\right]U_i^n + (1-\theta)\mu U_{i-1}^n + \theta\mu U_{i-1}^{n+1} - \theta\mu U_i^{n+1}\Big]$$

$$\Rightarrow \left[1+\theta\mu\right]U_i^{n+1} = \left[1-(1-\theta)\mu\right]U_i^n + (1-\theta)\mu U_{i-1}^n + \theta\mu U_{i-1}^{n+1}$$

Positive Coefficient Rule for a "monotone scheme": $1-(1-\theta)\mu > 0$

$$0 < \frac{V\Delta t}{\Delta x} =: \mu < \frac{1}{1-\theta}, \quad 0 \leq \theta \leq 1$$

$\therefore$ CFL condition for explicit scheme $(\theta = 0)$:

$$\boxed{0 < \frac{V\Delta t}{\Delta x} < 1} \quad \text{or} \quad \Delta t < \frac{\Delta x}{V}$$ much milder than in parabolic case

Repeating with $V<0$, we find the same but $\mu = |V|\frac{\Delta t}{\Delta x}$, so in all cases: $0 < \Delta t < \frac{\Delta x}{|V|}$

in general: $$0 < \Delta t_n < \frac{\min_i \Delta x_i}{\max_i |V_{i-1/2}^n|}$$

CFL for Crank-Nicolson: $\frac{|V|\Delta t}{\Delta x} < 2$ i.e. $\Delta t < 2\frac{\Delta x}{|V|}$, twice Δt_{expl}, but implicit, must solve system at each t_n

CFL for fully implicit: none! unconditionally stable

Consistency of upwind for advection: $u_t + V u_x = 0$ $\;= [D u_{xx}]$

scheme: $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left[F_{i-1/2} - F_{i+1/2}\right]$, $\quad F_{i-1/2} = \begin{cases} V U_{i-1}, & V>0 \\ V U_i, & V<0 \end{cases}$

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left\{ V U_{i-1} - V U_i \right\} \qquad i = 1, 2, \dots, M$$

For $V>0 \Rightarrow$ $\text{FDE}[U] = \dfrac{U_i^{n+1} - U_i^n}{\Delta t} + V \dfrac{U_i^n - U_{i-1}^n}{\Delta x} = 0$ to resemble the PDE $\left[- D \dfrac{U_{i-1} - 2U_i + U_{i+1}}{\Delta x^2}\right] = 0$

Set $U_i^n, U_i^{n+1} \to u(x_i, t_n)$ = exact sol of PDE

Consistency error $= \text{FDE}[u] = \dfrac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + V \dfrac{u(x_i, t_n) - u(x_{i-1}, t_n)}{\Delta x}$ $\left[- D \dfrac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{\Delta x^2}\right]$

Taylor expansion about (x_i, t_n): $u(x_i, t_n + \Delta t) = u(x_i, t_n) + \Delta t\, u_t(\;) + \frac{\Delta t^2}{2} u_{tt}(\;) + O(\Delta t^3)$

$$u(x_i - \Delta x, t_n) = u(\;) - \Delta x\, u_x(\;) + \frac{\Delta x^2}{2} u_{xx} + O(\Delta x^3)$$

$\Rightarrow$

$$\text{FDE}[u] = u_t + \frac{\Delta t}{2} u_{tt} + V\left[u_x - \frac{\Delta x}{2} u_{xx}\right] + O(\Delta t^2) + O(\Delta x^2)$$

all at (x_i, t_n)

$$= \underbrace{u_t + V u_x}_{\text{PDE}[u] = 0} + \frac{1}{2}\left[\Delta t \cdot u_{tt} - V \Delta x\, u_{xx}\right] + O(\cdots)$$

and $u_t = -V u_x \Rightarrow u_{tt} = -V u_{tx} = -V(-V) u_{xx} = +V^2 u_{xx}$

$$= u_t + V u_x + \frac{1}{2}\left[V^2 \Delta t - V \Delta x\right] u_{xx} + O(\Delta x^2) \quad \left[- D u_{xx} + \frac{\Delta x^2}{12} u_{xxxx}\right]$$

CFL: $\mu = \dfrac{V \Delta t}{\Delta x} \Rightarrow \Delta t = \dfrac{\mu \Delta x}{V}$

$$= u_t + V u_x + \frac{1}{2}\left[V\mu - V\right] \Delta x\, u_{xx} + O(\Delta x^2) \quad \left[- D u_{xx}\right]$$

$$= \underbrace{u_t + V u_x - \overbrace{\frac{1}{2} V (1 - \mu) \Delta x \cdot u_{xx}(x_i, t_n)}^{\text{num. diffusion}}}_{\text{to } O(\Delta x^2)} + \left[- D u_{xx}\right] + \overbrace{\frac{V \Delta x^2}{6}(3\mu - 2\mu^2 - 1) u_{xxx}}^{\text{num. dispersion}} + O(\Delta x^3)$$

$\therefore$ upwind scheme actually approximates to $O(\Delta x^2)$ an advection-diffusion eqn: $u_t + V u_x = D_{num} u_{xx}$ $\left[- D u_{xx}\right]$

with numerical diffusivity $D_{num} = \dfrac{\Delta x}{2} V \cdot (1 - \mu)$

and approximates a diffusion-dispersion with dispersion $\frac{V \Delta x^2}{6}(3\mu - 2\mu^2 - 1) \cdot u_{xxx}$ to $O(\Delta x^3)$!

For fixed μ (≤ 1 for stability), $D_{num} \to 0$ as $\Delta x \to 0$ $\therefore$ scheme is consistent

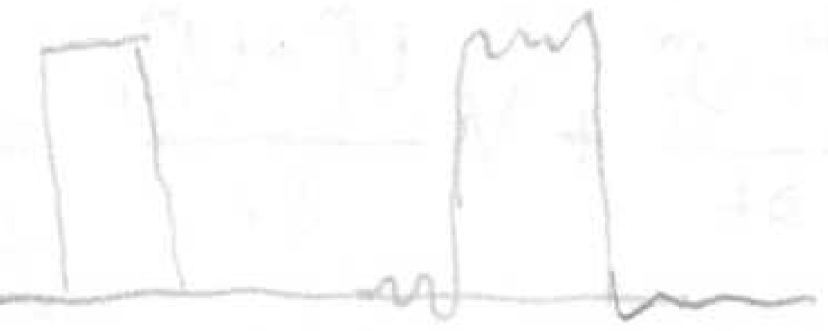
$\mu > 1 \Rightarrow$ backward diffusion ($D_{num} < 0$!), ill-posed

but for any nonzero Δx, it exhibits numerical diffusion unless $\mu = 1$!!!

For $\mu \lneq 1$, D_{num} grows with Δx and with V, i.e. for coarse grid and/or high V will be worse!

Note: even order derivative terms cause numerical dissipation (num. diffusion), leading to amplitude errors:

odd order derivative terms cause num. dispersion, leading to phase errors



num. diffusion: $\frac{V\Delta x}{2}(1-\mu)u_{xx}$
$\Rightarrow$ amplitude errors

num. dispersion: $\frac{V\Delta x^2}{6}(3\mu-2\mu^2-1)u_{xxx}$
$\Rightarrow$ phase errors

$\boxed{\text{For } \mu=1 \text{ the upwind scheme is exact!}}$

$$U_i^{n+1} = U_i^n + \underbrace{\frac{V\Delta t}{\Delta x}}_{=\mu}\left[U_{i-1} - U_i\right] = (1-\mu)U_i + \mu U_{i-1}$$

$$\overset{\mu=1}{=\!=} U_{i-1}\ , \quad U \text{ moves from } x_{i-1} \text{ to } x_i \text{ exactly!}$$