About ODEs - core ideas

General <u>n-th order ODE</u> for $y(t)$: $F(t, y, y', y'', \dots, y^{(n)}) = 0$

Useful form: can solve for highest order derivative

most important (by far...): 1^{st} and 2^{nd} order in "standard form":

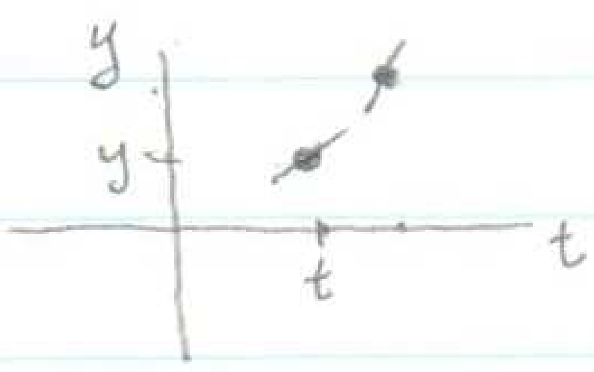
$$y' = f(t,y) \quad , \quad y'' = f(t, y, y')$$

and 1^{st} order systems:

$$\begin{cases} y_1' = f_1(t, y_1, \dots, y_n) \\ y_2' = f_2(t, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(t, y_1, \dots, y_n) \end{cases} \quad \text{or} \quad \vec{y}\,' = \vec{f}(t, \vec{y})$$

<u>1^{st} order</u>: $y' = f(t,y)$: find a <u>curve</u> $y = y(t)$ given its slope $y'(t)$

At each point (t, y) the ODE specifies the direction (slope) of $y = y(t)$



To solve it: pick a starting point (t_0, y_0)
and follow the <u>direction field</u>

To get a unique solution curve, need to specify a starting point
so need an <u>Initial Condition</u> (IC): $y(t_0) = y_0$

Operationally, must somehow "integrate" to find $y(t)$ from $y'(t)$
so there will be an arbitrary constant of integration,
so need an IC to find the constant.

standard form of 1^{st} order (IVP): $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ find solution curve through (t_0, y_0)

<u>Well-posed problem</u> if
1. existence: a solution exists
2. uniqueness: only one solution
3. continuous dependence on data
(small change in data $\Rightarrow$ small change in solution)
(stability under perturbations)

About ODEs ...

Well-posedness Theorem: Consider the (IVP) $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

If (i) f is defined & continuous in some region $\mathcal{R}$ containing (t_0, y_0)

$$\mathcal{R} = \{(t,y) : |t-t_0| \le a, \ |y-y_0| \le b\}$$

(ii) f is bounded on $\mathcal{R}$ by some M: $|f(t,y)| \le M$ for $(t,y) \in \mathcal{R}$

(iii) f is Lipschitz w.r.t. y in $\mathcal{R}$:

$$|f(t,y_1) - f(t,y_2)| \le K|y_1 - y_2| \quad \text{in } \mathcal{R}$$

then the (IVP) has unique sol. for $|t-t_0| \le h := \min\{a, \frac{b}{M}\}$ and depends continuously on the data.

So, locally well-posed.

Note: $\frac{\partial f}{\partial y}$ bounded in $\mathcal{R}$ $\Rightarrow$ Lipschitz

Equivalent integral equ: (IVP) $\Leftrightarrow$ $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$

Proof of uniqueness: Suppose there are two solutions $y_1(t)$, $y_2(t)$ for $|t-t_0| \le h$. Let $v(t) = y_1(t) - y_2(t)$, want to show $v(t) \equiv 0$

Now $v(t) = y_1(t) - y_2(t) = \int_{t_0}^{t} [f(s, y_1(s)) - f(s, y_2(s))]\, ds$

$$\Rightarrow |v(t)| \le K \int_{t_0}^{t} |y_1(s) - y_2(s)|\, ds = K \int_{t_0}^{t} |v(s)|\, ds \qquad (1)$$

Let $V_h = \max_{|t-t_0| \le h} |v(t)|$. Then $|v(t)| \overset{(1)}{\le} K V_h (t-t_0) \le K V_h \cdot h \quad (2)$

Repeat: $|v(t)| \overset{(1)}{\le} K \int_{t_0}^{t} |v(s)|\, ds \le K \cdot K V_h \int_{t_0}^{t} (s-t_0)\, ds = V_h K^2 \frac{(t-t_0)^2}{2} \le V_h \frac{(Kh)^2}{2}$

$\Rightarrow |v(t)| \le K \int_{t_0}^{t} |v(s)| \cdot ds \le V_h \frac{(Kh)^3}{3!}$, repeat m times ...

$$|v(t)| \le V_h \frac{(Kh)^m}{m!} \longrightarrow 0 \text{ as } m \to \infty$$

$|t-t_0| \le h$

Therefore $|v(t)| \equiv 0$, so $y_1(t) \equiv y_2(t)$, so unique solution!