Crystals of single size ($N=1$)

Consider crystals of single size (and shape) evolving from an initial size x^* governed by

$$\text{(IVP)} \quad \frac{dx}{dt} = G(x) \quad , \quad x(0) = x^*$$

where $G(x) = k\left[c - c_{crit}\right] = k\left[\underbrace{c_1 - \mu x^3}_{c(t)} - \underbrace{c^* e^{\frac{\Gamma}{x}}}_{c_{crit}}\right]$

Equilibria are constant solutions (if any),
so, zeros of $G(x)$.

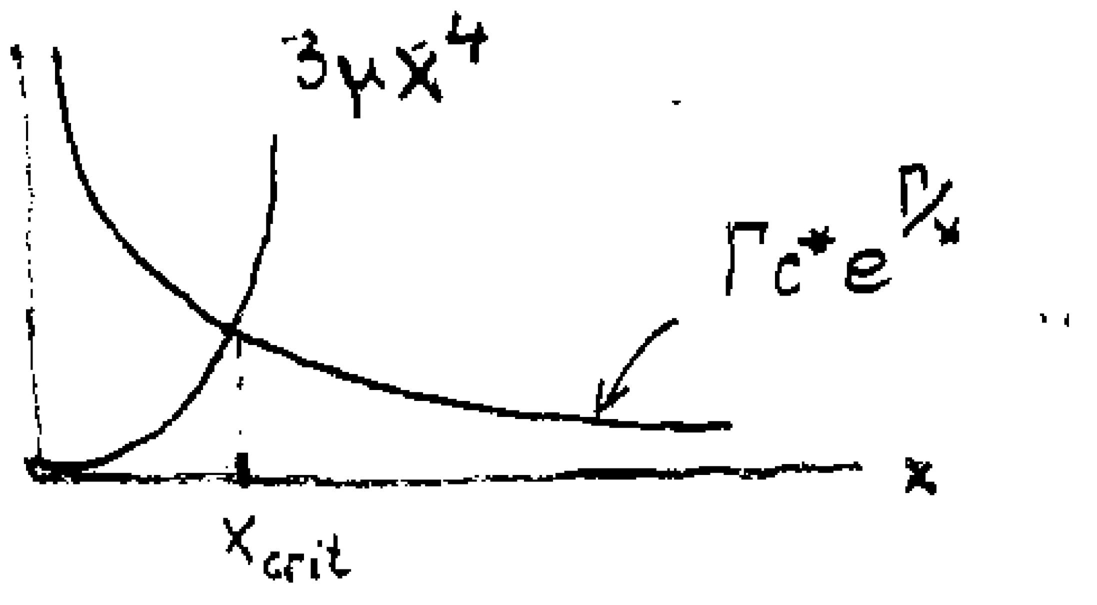
Theorem 1: There exist at most 2 positive equilibria $0<\xi_1<\xi_2$.
solutions of $c = c_{crit}$ i.e. of $\mu x^3 + c^* e^{\Gamma/x} = c_1$

Proof: They are values of x at which $y = f(x) := \mu x^3 + c^* e^{\frac{\Gamma}{x}}$ intersects $y = c_1$

Shape of $y = f(x)$: find max, min

$$f'(x) = 3\mu x^2 - \frac{\Gamma}{x^2} c^* e^{\Gamma/x} \overset{\text{want}}{=} 0$$

$$\Rightarrow \quad 3\mu x^4 = \Gamma c^* e^{\Gamma/x}$$



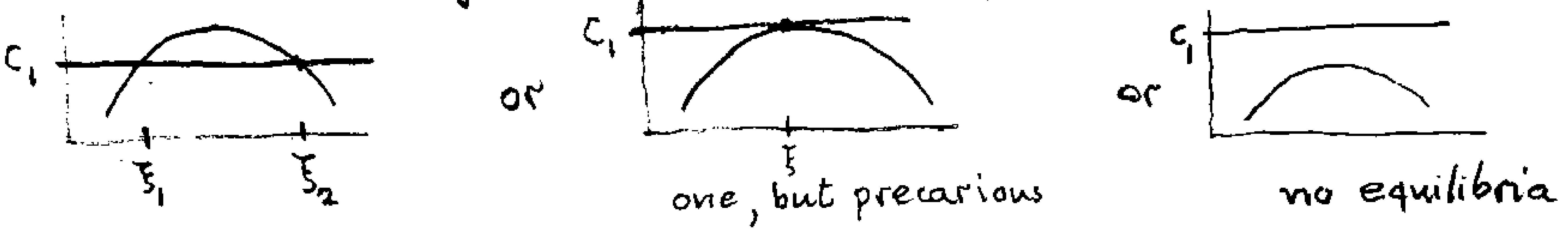
$f(x)$ has only one critical pt

$$f''(x) = \underbrace{6\mu x}_{>0} + \underbrace{\frac{2\Gamma}{x^3} c^* e^{\frac{\Gamma}{x}}}_{>0} + \underbrace{\frac{\Gamma^2}{x^4} c^* e^{\Gamma/x}}_{>0} > 0, \text{ so } x_{crit} \text{ is min}$$

$\therefore$ $f(x)$ has a single minimum,

$\therefore$ $G(x) = c_1 - f(x)$ has single max

Possibilities: depending on value of $c_1 = c_0 + \mu (x^*)^3$



two equilibria $0<\xi_1<\xi_2$ — one, but precarious — no equilibria

Single size ...

Assume c_0, x^* are chosen so as to have 2 distinct equilibria $0<\xi_1<\xi_2$

Theorem 2: If $x^* = \xi_1$ or $x^* = \xi_2$ then $x(t) \equiv x^*$ $\forall t \geq 0$.

Proof: ξ_1, ξ_2 are solutions of $G(x) \equiv k[c_1 - f(x)] = 0$
so then $G(x^*) = 0 \Rightarrow \frac{dx}{dt} = G(x^*) = 0,\ x(0) = x^* \Rightarrow x(t) \equiv x^*$ is solution

But, is it the only solution? can the (IVP) have more than one solution?
is the (IVP) well-posed?

Theorem 3: Our (IVP) is well-posed as long as $x \gneq 0$.

Proof: Need to verify the hypotheses of Well-Posedness Thm:
$G(x)$ continuous, bounded, Lip w.r.t. x: $G(x) = k[c_1 - f(x)]$
in some rectangle — obviously cont's.

$$\mathcal{R} = \{(t,x):\ \overset{0}{a} < t < b,\ A < x < B\}.$$

In $\mathcal{R}$: $|G| \leq k(|c_1| + |f(x)|) \leq k(c_1 + \mu B^3 + c^* e^{\Gamma/A}) =: M < \infty$ if $A \gneq 0$

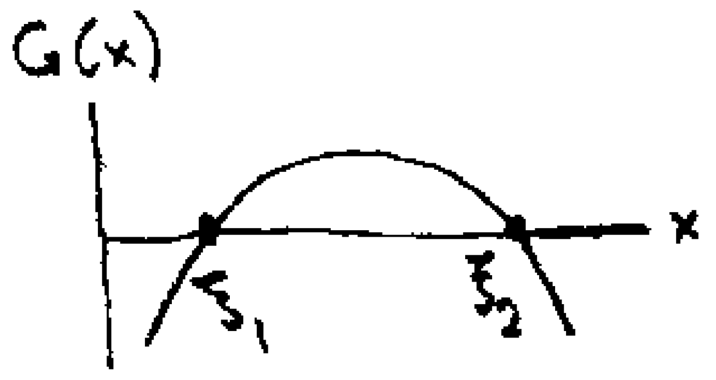
$$|G'| \leq 3\mu B^2 + \frac{c^*\Gamma}{A^2} e^{\frac{\Gamma}{A}} = K < \infty \quad \text{provided } A \gneq 0.$$

So, as long as $x \gneq 0$ we can choose $A > 0$, and we'll have well-posedness
$\therefore$ also uniqueness

Remark 1: Note usefulness of uniqueness property: to show something is **the** sol, all we need to show is that it is a solution!

Remark 2: If $x(t) = \xi_{1 \atop 2}$ at some time $t_{1 \atop 2}$ then $x(t) \equiv \xi_{1 \atop 2}$ ever after.

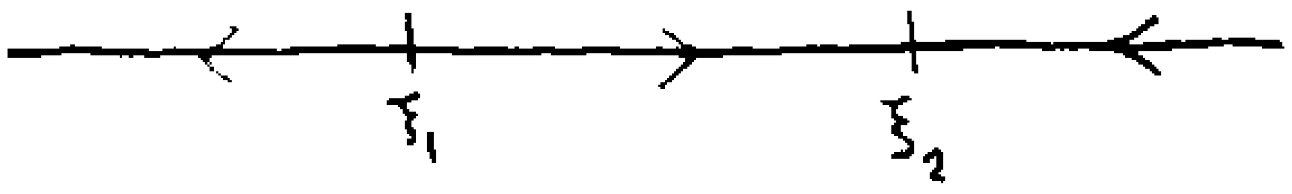
Single size ....



when $\xi_1 < x < \xi_2$, $G(x) > 0$ so $\frac{dx}{dt} = G(x) > 0 \Rightarrow x(t) \nearrow$

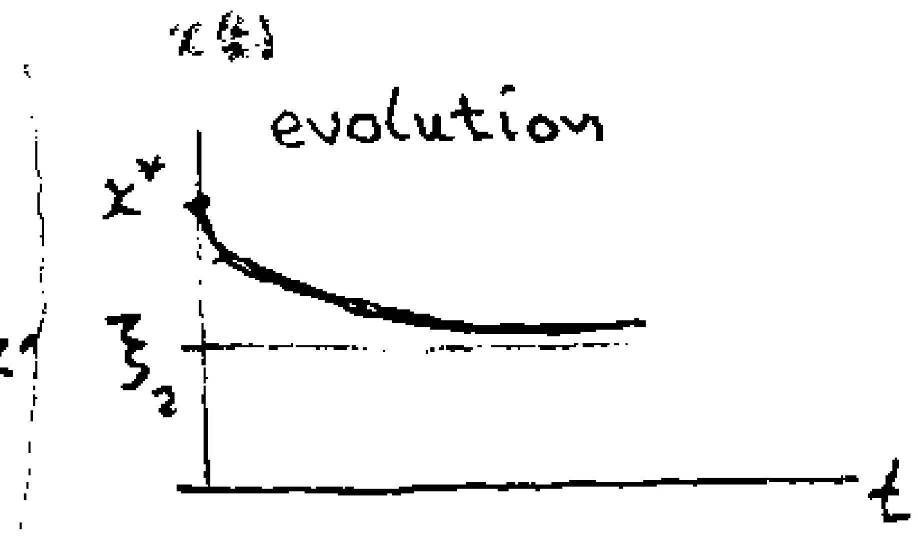
when $x < \xi_1$ or $x > \xi_2$, $G(x) < 0$ so $\qquad < 0 \Rightarrow x(t) \searrow$

Phase Line :



$\underline{\xi_1 \text{ is repelling}}$ equilibrium (unstable)

$\underline{\xi_2 \text{ is attracting}}$ equilibrium (stable)



<u>Theorem 4</u>: If $x^* > \xi_2$ then $x(t) \searrow$ and $\lim_{t\to\infty} x(t) = \xi_2$.

Proof: $x^* > \xi_2 \Rightarrow G(x^*) < 0 \Rightarrow x(t) \searrow$ as long as $x(t)$ remains $> \xi_2$

and $x(t)$ decreases slower and slower toward ξ_2 since $G(x) \to 0$.

Can $x(t)$ become $< \xi_2$? since $x(t)$ is continuous, it would have to first become $= \xi_2$ ($x(t_1) =$), and then $x(t) \equiv \xi_2$ ever after would be <u>the</u> sol by uniqueness.

If $x(t)$ ever became $< \xi_2$, then $\frac{dx}{dt} > 0$ so it would $\nearrow \xi_2$, and if a solution $x(t) \lneqq \xi_2$ existed we would have two sols, contradicting uniqueness, so $x(t) \geqslant \xi_2 \;\forall\, t > 0$.

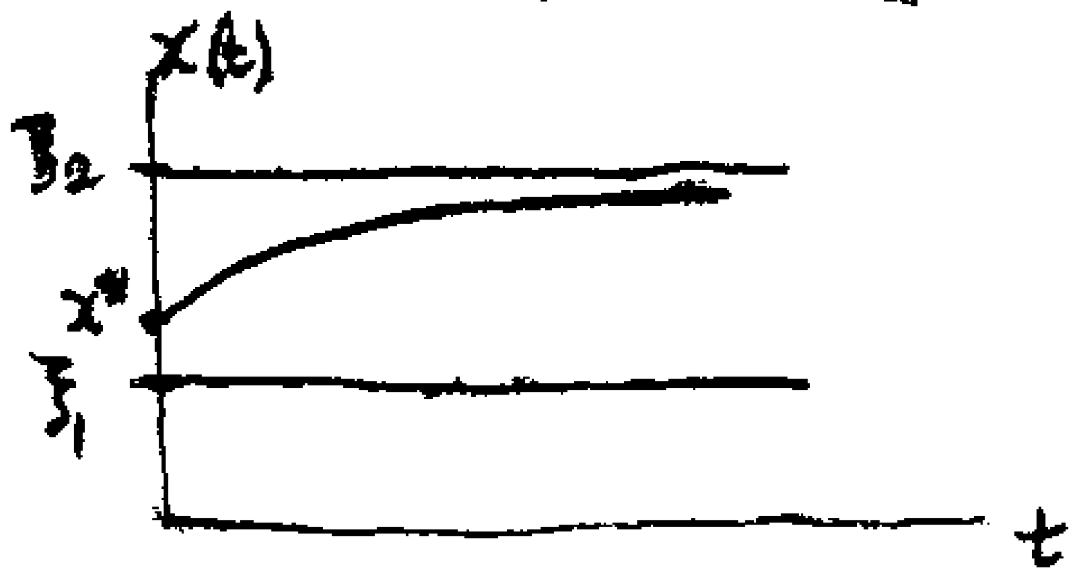
Can it get stuck strictly $> \xi_2$? suppose $\lim x(t) = \hat{\xi} > \xi_2$. Then $G(x) \lneqq 0$, so $x(t)$ must strictly decrease, can<u>not</u> flatten out as it $\to \hat{\xi}$ since $G(\hat{\xi}) < 0$.

It can flatten out only at ξ_2. So it must keep decreasing below $\hat{\xi}$ and $\to \xi_2$.

<u>Theorem 5</u>: If $\xi_1 < x^* < \xi_2$ then $x(t) \nearrow$ strictly and $\lim_{t\to\infty} x(t) = \xi_2$.

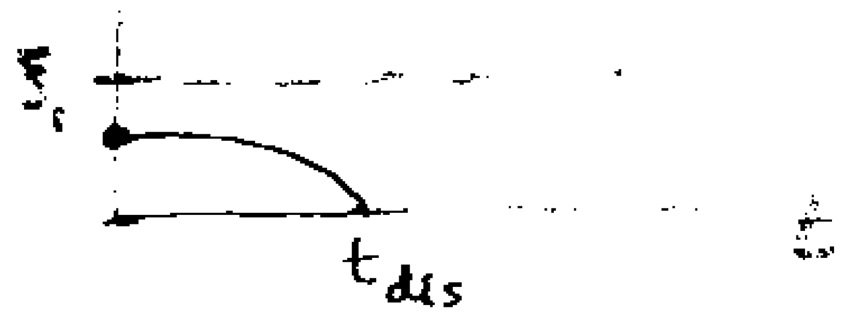
Proof: $G(x^*) > 0 \Rightarrow x(t) \nearrow$, $G(x) > 0 \Rightarrow x(t) \nearrow$ strictly $\forall\, t > 0$.

As in Thm 4, it cannot flatten out except at ξ_2, so must keep $\nearrow$ to ξ_2, but slower and slower!



<u>Theorem 6</u>: If $x^* < \xi_1$ then $x(t)$ strictly decreases to <u>zero</u>, faster and faster,

and dissolves at some <u>finite</u> time t_{dis}.



Proof: $0 < x^* < \xi_1 \Rightarrow G(x) < 0$ so $x(t) \searrow$,

faster and faster because $G(x)$ becomes more and more negative.

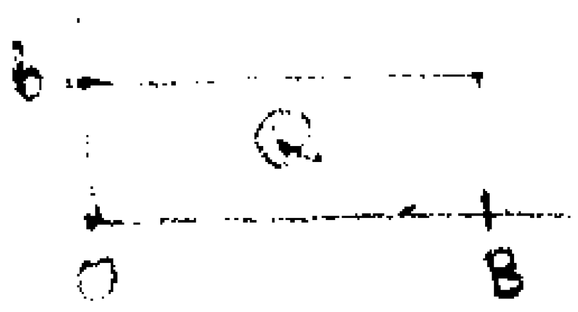
Can $\lim x(t) = \hat{\xi} > 0$? No! the smaller x becomes, the more negative $G(x)$, the faster $x(t)$ decreases, till it disolves ($x(t) \searrow 0$) and our model becomes inapplicable!

But this raises the question of **well-posedness**, which we proved for $0 \lneq A \le x \le B$.
Here we need $0 = A \le x \le B$, but then $|G|, |G'| \to \infty$, proof not valid!

<u>Theorem 7</u>: Our (IVP) is well-posed even for $x \geq 0$.

Proof: Consider the problem with x as indep. variable and $t(x)$ as unknown:

$$\frac{dt}{dx} = \frac{1}{G(x)} \quad , \quad t(x^*) = 0$$



Take as rectangle R: $0 = 0 \le A \le x \le B < \xi_1$,

$0 = a \le t \le b$.

Then $\frac{1}{G(x)}$ is continuous (since $G(x) \lneq 0$), $|G(x)| \geq k\left[c_1 - \mu \cdot 0 - c^* e^{\frac{r}{B}}\right] = M$

$$\Rightarrow \left|\frac{1}{G(x)}\right| \le \frac{1}{M} \quad , \text{ and } \quad \frac{\partial}{\partial t}\left(\frac{1}{G(x)}\right) \equiv 0 \, , \text{ so any } K > 0 \text{ will be Lip. constant.}$$

Hence, the RHS $\frac{1}{G(x)}$ is cont's, bounded, Lipschitz w.r.t. t
and we have **well-posedness**.

In Project 1 you will verify *numerically* these predictions
by solving the (IVP) numerically!

Need to discuss numerical methods for ODEs **now**.