Crystals of N sizes

Model for crystals of N sizes, starting with sizes $x_1^* < x_2^* < \dots < x_N^*$:

$$\begin{cases} \dfrac{dx_j}{dt} = G_j(x_1, \dots, x_N) := k\left[c(t) - c^* e^{\frac{\Gamma}{x_j}}\right], \\ x_j(0) = x_j^* \qquad , \quad j = 1:N \end{cases}$$

$$c(t) = \underbrace{c_0 + \sum_{i=1}^N \mu_i (x_i^*)^3}_{c_1} - \sum_{i=1}^N \mu_i x_i^3 = c_1 - \sum_{i=1}^N \mu_i x_i^3$$

Can be written neatly in vector form as

$$(IVP) \begin{cases} \dfrac{d\vec{X}(t)}{dt} = \vec{G}\left(\vec{X}(t)\right) \\ \vec{X}(0) = \vec{X}^* \end{cases} , \quad \vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_N \end{bmatrix}, \quad \vec{X}^* = \begin{bmatrix} x_1^* \\ \vdots \\ x_N^* \end{bmatrix}$$

Theorem 1: If $c_0 > c^*$ then $c^* < c(t) < c_1$ for all $t > 0$,

Therefore all $x_j(t)$ remain bounded (between 0 and some max size)

Proof: $c(0) = c_0 > c^*$, $c(t) = c_1 - \sum_{i=1}^N \mu_i x_i^3$

$$\Rightarrow \frac{dc(t)}{dt} = 0 - 3\sum \mu_i x_i^2 \frac{dx_i}{dt} = -3\sum \mu_i x_i^2 G_i(\vec{X})$$

Can $c(t)$ become $< c^*$? Being continuous, it would first $= c^*$.

Suppose at some first time t^*, $c(t^*) = c^*$. Then $G_j(\vec{X}(t^*)) = -kc^*\left[e^{\frac{\Gamma}{x_j}} - 1\right] < 0$

$$\Rightarrow \left.\frac{dc}{dt}\right|_{t=t^*} = -3\sum \mu_i x_i^2 G_i \Big|_{t=t^*} > 0 \rightarrow c(t) \nearrow$$

but it is coming <u>down</u> to c^*, contradiction!

Therefore, $c(t) > c^*$ $\forall t > 0$ $\Rightarrow c_1 - \sum \mu_i x_i^3 > c^* \Rightarrow \sum \mu_i x_i^3 < c_1 - c^*$ $\forall t$

$\Rightarrow x_j(t)$ bounded $\forall t, \forall j$

Theorem 2: $x_{j+1}(t) - x_j(t)$ strictly increases in time as long as $x_j(t) > 0 \;\; \forall j$

Therefore, ordering persists, and sizes grow further apart in time:

$$x_1(t) < \dots < x_j(t) < \dots < x_N(t) \quad \forall t > 0$$

Proof: Let $v_j(t) = x_{j+1}(t) - x_j(t)$. Initially $v_j(0) > 0$ and

$$\frac{dv_j}{dt} = G_{j+1} - G_j = k\left[c(t) - c^* e^{\frac{\Gamma}{x_{j+1}}} - c(t) + c^* e^{\Gamma/x_j}\right] = kc^*\left[e^{\frac{\Gamma}{x_j}} - e^{\frac{\Gamma}{x_{j+1}}}\right] > 0$$

as long as $x_j > 0$

$\Rightarrow$ $v_j(t)$ increases strictly, so $v_j(t) > 0 \;\; \forall t > 0$.

Note: $x_j < x_{j+1} \Rightarrow c^* e^{\frac{\Gamma}{x_{j+1}}} < c^* e^{\frac{\Gamma}{x_j}}$, so

if $c(t) < c^* e^{\frac{\Gamma}{x_{j+1}}} < c^* e^{\Gamma/x_j}$ then both decaying

if $c(t) > c^* e^{\Gamma/x_j} > c^* e^{\Gamma/x_{j+1}}$ then both growing

if $c^* e^{\Gamma/x_{j+1}} < c(t) < c^* e^{\Gamma/x_j}$ then x_{j+1} growing and x_j decaying

By Thm 2, smaller sizes must dissolve <u>before</u> larger sizes do.

Assume $k-1$ (smaller) sizes dissolve in finite time and k largest survive $\forall t > 0$,

so assume $x_k(t) < x_{k+1}(t) < \dots < x_N(t)$ survive $\forall t > 0$, for some $k < N$.

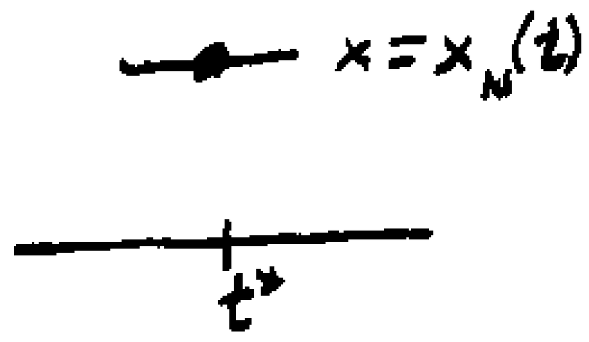
Note: $x_j(t)$ grows whenever $c(t) > c^* e^{\Gamma/x_j} \Leftrightarrow x_j(t) > \dfrac{\Gamma}{\ln \frac{c(t)}{c^*}} =: L^*(t)$

decays $<$ $<$

<u>Lemma:</u> The largest size $x = x_N(t)$ may intersect $x = L^*(t)$ at most once.

Proof: Let t^* be an intersection time, at which $x_N(t^*) = L^*(t^*)$

$$\Rightarrow \quad c(t^*) = c^* e^{\Gamma/x_N(t^*)} \Rightarrow G_N(\vec{X}(t^*)) = 0 \Rightarrow \left.\frac{dx_N}{dt}\right|_{t^*} = 0 \quad \text{so}$$

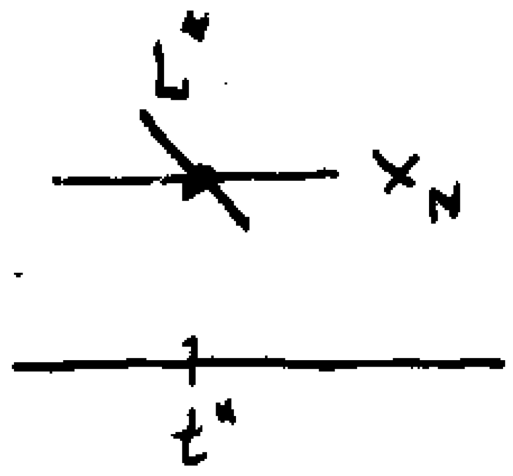


At $t = t^*$ we have $x_k(t^*) < \cdots < x_{N-1}(t^*) < x_N(t^*) = L^*(t^*)$

therefore they decay, $\therefore \ \dfrac{dx_j}{dt} < 0$ for $j = k, \ldots, N$

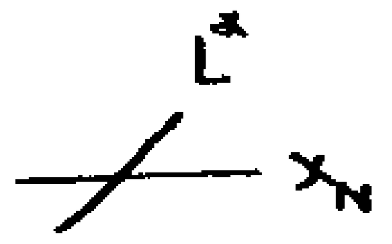
$$\Rightarrow \left.\frac{dc(t)}{dt}\right|_{t^*} = -3\sum_{j=k}^{N} \mu_j x_j^2 \left.\frac{dx_j}{dt}\right|_{t^*} > \sum_{j=k}^{N-1} 0 - 3\mu_N x_N^2 \left.\frac{dx_N}{dt}\right|_{t^*} = 0 - 0 = 0$$

$$\text{so } \left.\frac{dc}{dt}\right|_{t^*} \overset{>}{\neq} 0 \Rightarrow \left.\frac{dL^*}{dt}\right|_{t^*} = -\frac{\Gamma}{(\ln \frac{c}{c^*})^2} \cdot \left.\frac{\frac{dc}{dt}/c^*}{c/c^*}\right|_{t^*} < 0$$



$\Rightarrow \ x = L^*(t)$ is decreasing at t^*.

Therefore $x_N(t)$ and $L^*(t)$ <u>cannot</u> intersect again! would have to



impossible

Therefore, ever after such an intersection time t^*

either $x_N(t) > L^*(t) \Rightarrow \dfrac{dx_N}{dt} > 0 \quad \forall\, t > t^*$

or $x_N(t) < L^*(t) \Rightarrow \dfrac{dx_N}{dt} < 0 \quad \forall\, t > t^*$

$$\therefore \left.\begin{array}{l} x_N(t) \text{ monotone } \forall\, t > t^* \\ \text{and bounded (by Thm 1)} \end{array}\right\} \Rightarrow \lim_{t\to\infty} x_N(t) =: x_N(\infty) \ \underline{\text{exists}} < \infty$$

Corollary: Any surviving size tends to a finite Limit as $t \to \infty$,

Proof: We saw that $x_N(t)$ has finite Limit $x_N(\infty)$.

Then $x_N(t) - x_{N-1}(t)$ is increasing and bounded $\therefore$ $x_{N-1}(\infty)$ exists also,

and so on for $x_{N-2}, \ldots, x_k$, so all surviving sizes have limits:

$$x_k(\infty) \leq \ldots \leq x_N(\infty) \quad \text{exist and finite}$$

Therefore also $c(\infty) := c_1 - \sum_{j=k}^{N} \mu_j x_j(\infty)^3$ also exists and finite.

Theorem 3: All sizes except the Largest one will dissolve in finite time, and only the Largest size $x_N(t)$ can exist for all time (and $\to x_N(\infty)$)

Proof: Suppose some other x_k, $k \lneqq N$, also survives,

i.e. $x_k(\infty) \lneqq x_N(\infty) \Rightarrow e^{\Gamma/x_k(\infty)} \gneqq e^{\Gamma/x_N(\infty)}$

$$\Rightarrow G_k(\infty) = k\left[c(\infty) - c^* e^{\Gamma/x_k(\infty)}\right] \lneqq k\left[c(\infty) - c^* e^{\Gamma/x_N(\infty)}\right] = G_N(\infty).$$

So, at least one of the numbers $G_k(\infty) \lneqq G_N(\infty)$ must be $\neq 0$.

If $G_k(\infty) \neq 0$ then $\frac{dx_k}{dt} = G_k \to G_k(\infty) \neq 0$ so $x_k(t)$ could not have limit at ∞, contradiction.

The only way to avoid contradiction is $k = N$, i.e. only $x_N(t)$ survives!

Conclusion: The N-size model predicts that $x_1(t)$ will dissolve at some time t_1, then $x_2(t)$ at some $t_2, \ldots, x_{N-1}(t)$ at some t_{N-1}, and everafter only $x_N(t)$ remains, a single size!

For which we know its fate depends on its size (at t_{N-1}) relative to the two equilibria $\xi_1 < \xi_2$ of $\frac{dx_N}{dt} = G_N(x_N)$: either $x_N(t)$ will also dissolve at some t_N or $x_N(t)$ will tend to ξ_2!