Chemical Reactions via mass-action kinetics

velocity of reaction: V = rate of change of each species
+ for product, − for reactants

1. $A + B \xrightarrow{k} C$: A, B reactants, C product, k = rate constant

$$V = -\frac{dA}{dt} = -\frac{dB}{dt} = +\frac{dC}{dt} \Rightarrow A' = -C', \quad B' = -C'$$

$$\Rightarrow A + C = const. = A_0, \quad B + C = const = B_0$$

$$\Rightarrow A = A_0 - C, \quad B = B_0 - C$$

mass action: $\frac{dC}{dt} = k \cdot A \cdot B$

Set $y = C$: $\begin{cases} y' = k(A_0 - y)(B_0 - y) \\ y(0) = y_0 \end{cases}$ 1st order ODE, separable

2. $2A + 3B \xrightarrow{k} 4C$
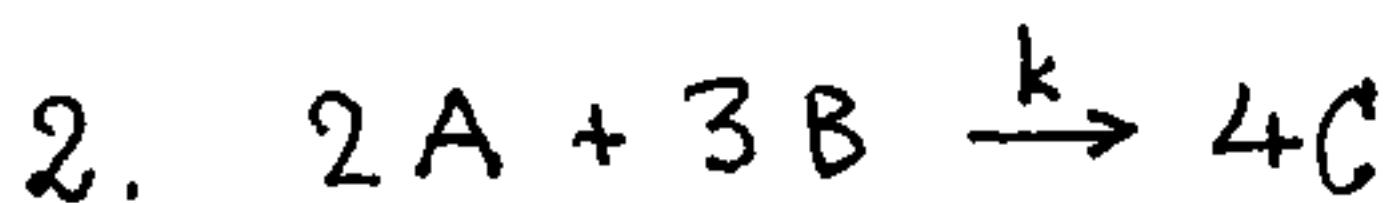
$$V = -\frac{1}{2}A' = -\frac{1}{3}B' = \frac{1}{4}C' \Rightarrow A' = -\frac{1}{2}C', \quad B' = -\frac{3}{4}C'$$

$$A = A_0 - \frac{1}{2}C, \quad B = B_0 - \frac{3}{4}C$$

$$\frac{dC}{dt} = k A^2 B^3$$

Set $y = C$: $\begin{cases} \frac{dy}{dt} = k\left(A_0 - \frac{1}{2}y\right)^2\left(B_0 - \frac{3}{4}y\right)^3 \\ y(0) = y_0 \end{cases}$

3. $A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C$ they are 2 reactions: $A + B \xrightarrow{k_1} C$ and $C \xrightarrow{k_2} A + B$
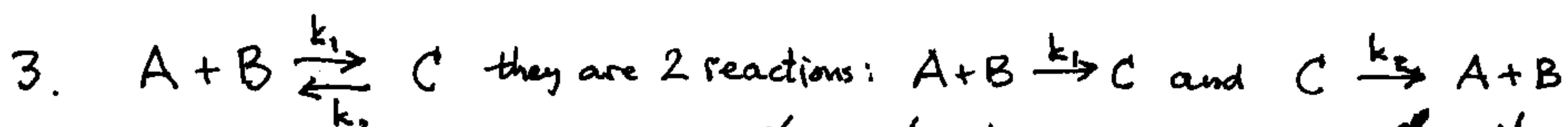
$$V_1 = -A' = -B' = C \qquad\qquad V_2 = -C' = A' = B'$$

$$A = A_0 - C, \quad B = B_0 - C$$

$$\frac{dC}{dt} = k_1 AB - k_2 C$$

$$\frac{dA}{dt} = -k_1 AB + k_2 C = -\frac{dC}{dt}$$

$$\frac{dB}{dt} = -k_1 AB + k_2 C = -\frac{dC}{dt}$$

Set $y = C$: $\frac{dy}{dt} = k_1(A_0 - y)(B_0 - y) - k_2 y$

$y(0) = y_0$

This type is fundamental in <u>receptor-ligand dynamics</u>, giving bound receptors (activated)

$$R + L \underset{k_{off}}{\overset{k_{on}}{\rightleftharpoons}} B \qquad v = -R' = -L' = B' \Rightarrow R + B = R_{tot}$$
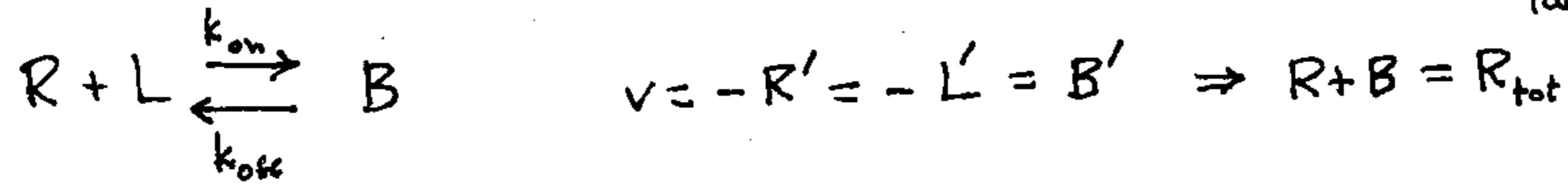
$$\Rightarrow \frac{dB}{dt} = k_{on} L (R_{tot} - B) - k_{off} B$$

$$\frac{dL}{dt} = -k_{on} L (R_{tot} - B) + k_{off} B \quad = -\frac{dB}{dt}$$

$$\left(\frac{dR}{dt} = -\frac{dB}{dt} \right)$$

4. <u>Michaelis-Menten kinetics</u> (enzymatic reactions)

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} ES \overset{k_2}{\rightarrow} E + P$$

E = enzyme : $E_0 = [E] + [ES]$
    (free) (bound)
S = substrate
P = product

(1) $\dfrac{d[ES]}{dt} = k_1 E \cdot S - k_{-1} [ES] - k_2 [ES]$
$\qquad \dfrac{dE}{dt} = -k_1 E \cdot S + k_{-1} [ES] + k_2 [ES] \quad = -\dfrac{d[ES]}{dt}$

(2) $\dfrac{dP}{dt} = k_2 [ES]$
$\qquad \dfrac{dS}{dt} = -k_1 E \cdot S$

Assume: $[ES]$ is quasi-stationary (changes much slower than $[S]$ and $[P]$) so $\dfrac{d[ES]}{dt} \approx 0 = -\dfrac{dE}{dt}$

$\qquad [E] + [ES] = const. =: E_0$

(1) $\Rightarrow \dfrac{d[ES]}{dt} = 0 \Rightarrow k_1 E \cdot S = (k_{-1} + k_2)[ES] \qquad \Rightarrow S \cdot E_0 = S \cdot [ES] + \underbrace{\left(\dfrac{k_{-1} + k_2}{k_1}\right)}_{= K_M} \cdot [ES]$

$\qquad\qquad\qquad \underset{\shortparallel}{} \qquad\qquad\qquad\qquad\qquad\qquad = (K_M + S)[ES]$
$\qquad\qquad k_1 (E_0 - [ES]) \cdot S$

$\qquad \Rightarrow [ES] = \dfrac{E_0 \cdot S}{K_M + S}$ of the form $C(x) = C_\infty \cdot \dfrac{x}{K_M + x}$
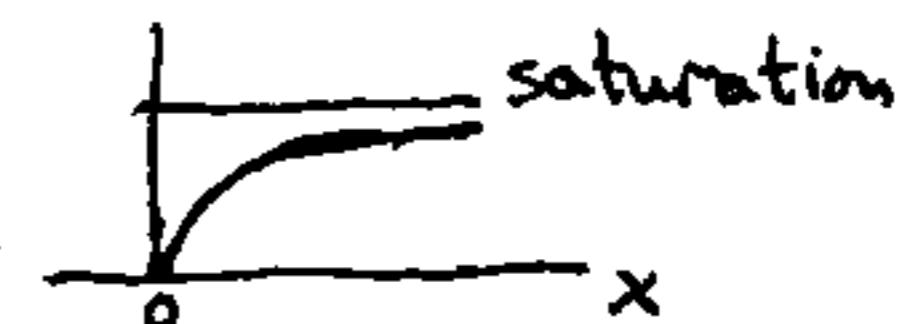
saturation

(2) $\Rightarrow \dfrac{dP}{dt} = k_2 E_0 \cdot \dfrac{S}{K_M + S} = \dfrac{V_{max} \cdot S}{K_M + S}$ = velocity of reaction = $v(S)$

$v(0) = 0$
$v(\infty) = V_{max}$
K_M = half max const

S (molar $M = \dfrac{mole}{liter}$)
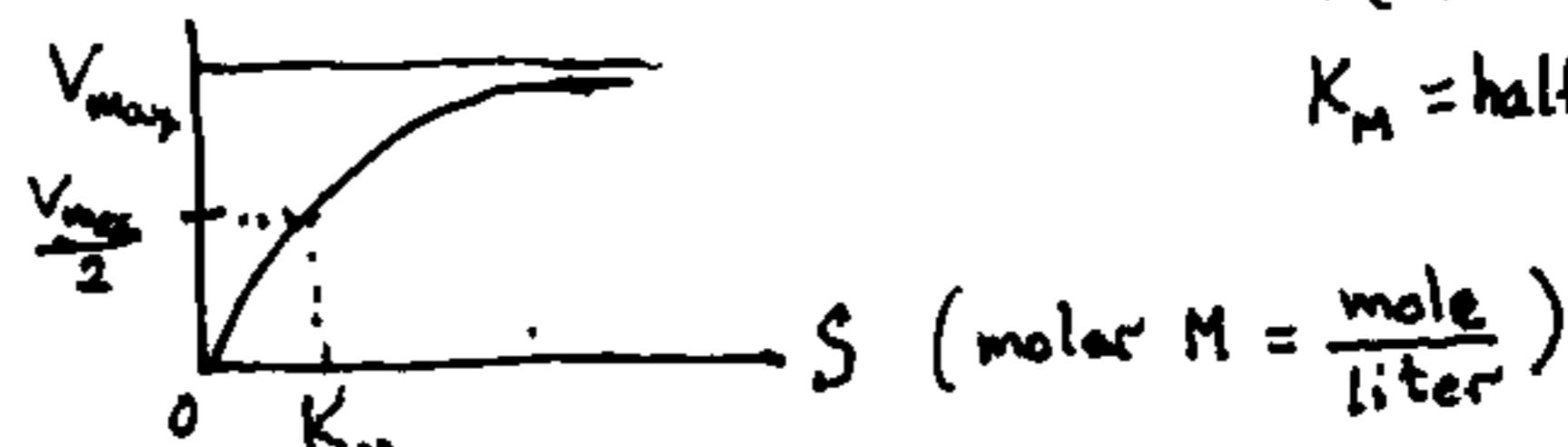
usually rates are hard to find (measure), but K_M, V_{max} can be found from experimental plot

Michaelis - Menten - Monod equation: $\qquad C(x) = C_\infty \dfrac{x}{K_M + x}$

C = fraction of enzyme bound to ligand

C_∞ = saturation value

x = free (unbound) ligand concentration

Hill equation: $\qquad C(x) = C_\infty \dfrac{x^n}{K_d + x^n}$

n = Hill coefficient

K_d = dissociation const. = $\dfrac{k_{-1}}{k_{+1}}$

K_M = half-saturation const.