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AVOIDING ANSCOMBE'S PARADOX

1. ANSCOMBE'S PARADOX

As noted by Anscombe (1976), the disposition of a set of proposals by majority rule does not preclude the existence of a majority of voters, each of whom disagrees with the outcomes in a majority of cases. It is intuitively clear, however, that when proposals are adopted or rejected, on average, by a sufficiently strong consensus, such a state of affairs cannot materialize. Indeed, we show for N voters, K proposals, and $0 < \alpha, \beta < 1$, that when the prevailing coalitions[1], across all proposals, comprise on average at least $(1-\alpha\beta)N$ voters, the set of voters who disagree with more than αK outcomes cannot exceed βN. Setting $\alpha = \beta = 1/2$, it follows that when prevailing coalitions comprise on average at least three-fourths of those voting, the set of voters disagreeing with a majority of outcomes cannot comprise a majority (see Wagner, 1983). Examples are provided to illustrate the "best possible" nature of these and related results.

2. THE RULE OF $1 - \alpha\beta$

Suppose that N individuals cast yes-or-no votes on K proposals. Given a decision procedure for deciding outcomes, there arises an $N \times K$ "A–D matrix", namely the matrix whose i–jth entry is A if voter i agrees with the outcome of voting on proposal j, as determined by the procedure, and D if he disagrees with that outcome. For example[2], the voting matrix

(2.1)

		Proposals	
	1	2	3
Voters 1	yes	yes	no
2	no	no	no
3	no	yes	yes
4	yes	no	yes
5	yes	no	yes

Theory and Decision **16** (1984) 233–238. 0040–5833/84/0163–0233$00.60.

gives rise to the A–D matrices

$$M_1 = \begin{bmatrix} A & D & D \\ D & A & D \\ D & D & A \\ A & A & A \\ A & A & A \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} D & D & A \\ A & A & A \\ A & D & D \\ D & A & D \\ D & A & D \end{bmatrix} \tag{2.2}$$

under the respective decision procedures D_1 (adopt a proposal iff more than half the voters approve) and D_2 (adopt a proposal iff more than two-thirds of the voters approve). Note that in both M_1 and M_2, a majority of voters disagree with two out of three outcomes, illustrating the possible state of affairs alluded to by Anscombe's paradox.

Given an $N \times K$ A–D matrix M, whatever the decision procedure by which it arises, if we denote by A_M the number of A's appearing in M, then A_M/K is the average size, across all proposals, of the prevailing coalitions, and A_M/NK the average fraction of voters comprising the prevailing coalitions.

THEOREM 2.1. *If M is an $N \times K$ A–D matrix, $0 < \alpha, \beta < 1$, and*

$$A_M > B(N, K) = NK - [\alpha K + 1][\beta N + 1], \tag{2.3}$$

(where $[x]$ denotes the greatest integer less than or equal to x) then no more than βN voters disagree with the outcomes on more than αK proposals.

Proof. Suppose, on the contrary, that more than βN voters each disagreed with more than αK outcomes. Then at least $[\beta N + 1]$ rows of M would each contain at least $[\alpha K + 1]$ D's. M would thus contain at least $[\alpha K + 1]$ $[\beta N + 1]$ D's, hence at most $NK - [\alpha K + 1][\beta N + 1]$ A's, contradicting (2.3).

The following corollary is an easily demonstrated consequence of Theorem 2.1:

COROLLARY 2.1 ("The Rule of $1 - \alpha\beta$"). For all N, $K \geqslant 2$, if *M is an $N \times K$ A–D matrix*, $0 < \alpha, \beta < 1$, and

$$A_M/NK \geqslant 1 - \alpha\beta, \tag{2.4}$$

then no more than βN voters disagree with the outcomes on more than αK proposals.

Proof. Since $B(N, K) = NK - [\alpha K + 1][\beta N + 1] < NK - \alpha\beta NK = (1-\alpha\beta)NK$, if $A_M/NK \geqslant 1-\alpha\beta$, then $A_M > B(N,K)$.

Setting $\alpha = \beta = 1/2$, it follows from Corollary 2.1 and the remarks preceding Theorem 2.1 that when prevailing coalitions comprise, on average, at least three-fourths of those voting, the set of voters disagreeing with a majority of outcomes cannot comprise a majority (see Wagner (1983, Theorem 2.2)). While Corollary 2.1 is a cruder result than Theorem 2.1, (2.3) is nearly equivalent to (2.4) for large N and K since, as may easily be shown, $B(N,K)/NK \to 1-\alpha\beta$ as $N, K \to \infty$.

The sharpness of inequality (2.3) as a condition sufficient to guarantee the conclusion of Theorem 2.1 may be illustrated by considering the case of simple majority rule. Any $N \times K$ A–D matrix M arising from a voting matrix by this decision procedure contains at least $[(N+1)/2]$ A's per column, hence at least $[(N+1)/2]K$ A's in all. Thus if $[(N+1)/2]K > NK - [\alpha K + 1][\beta N + 1] = B(N, K)$, no more than βN voters can disagree with more than αK outcomes, as determined by simple majority rule.

Suppose then that

$$(2.5) \qquad [(N+1)/2]K \leqslant NK - [\alpha K + 1][\beta N + 1] = B(N,K),$$

so that (2.3) is not automatically satisfied. In such cases, we can show that $B(N, K)$ is a best possible bound by exhibiting an $N \times K$ voting matrix V such that, for the A–D matrix M arising from V by simple majority rule, $A_M = B(N, K)$ and more than βN rows of M contain more than αK D's. Let $n = [\beta N + 1]$ and $k = [\alpha K + 1]$ and define $V = (v_{ij})$ as follows: $v_{ij} =$ yes iff

(2.6) $\quad 1^\circ \quad 1 \leqslant i \leqslant n,$

and

$\quad 2^\circ \quad \exists r \in R_i = \{(i-1)k + 1, (i-1)k + 2, \ldots, ik\}$ such that $j \equiv r \pmod K$.

We show first that the number of no votes in each column of V is at least $[(N+1)/2]$, so that all proposals are rejected. Consider the set of integers $I = R_1 \cup \ldots \cup R_n = \{1, 2, \ldots, nk\}$. If nk/K is an integer, I contains exactly nk/K complete residue systems (mod K), and so (2.6) implies that for each

$j \in \{1, \ldots, K\}$, v_{ij} = yes for nk/K values of $i \in \{1, \ldots, n\}$. Thus v_{ij} = no for $n - nk/K$ values of $i \in \{1, \ldots, n\}$. In addition, v_{ij} = no for $i > n$, so that v_{ij} = no for $N - nk/K$ values of $i \in \{1, \ldots, N\}$. It follows from (2.5) that $N - nk/K \geqslant [(N+1)/2]$.

If nk/K is not an integer, I contains $[nk/K]$ complete residue systems (mod K), plus a fragment of a complete residue system (mod K). Thus, by (2.6), for each $j \in \{1, \ldots, K\}$, v_{ij} = yes for at most $[nk/K] + 1$ values of $i \in \{1, \ldots, n\}$, and so v_{ij} = no for at least $n - [nk/K] - 1$ values of $i \in \{1, \ldots, n\}$. Since v_{ij} = no for $i > n$, v_{ij} = no for at least $N - [nk/K] - 1$ values of $i \in \{1, \ldots, N\}$. By (2.5), $N - nk/K \geqslant [(N+1)/2]$, and so $N - [nk/K] - 1 = [N - nk/K] \geqslant [(N+1)/2]$.

It follows from (2.6) that each of the first $n = [\beta N + 1]$ voters casts a yes vote on precisely $k = [\alpha K + 1]$ proposals. Since, as established above, all K proposals are rejected, it is the case that more than βN voters disagree with more than αK outcomes. Finally, we note that since the total number of D's in the A–D matrix corresponding to V is $nk = [\alpha K + 1][\beta N + 1]$, $A_M = NK - [\alpha K + 1][\beta N + 1] = B(N, K)$.

The preceding class of examples, along with the aforementioned observation that $B(N, K)/NK \to 1 - \alpha\beta$ as $N, K \to \infty$ show that, for A–D matrices arising from simple majority rule, if $1/2 < 1 - \alpha\beta$ (so that (2.4) is not automatically satisfied), the bound $1 - \alpha\beta$ of (2.4) cannot be replaced by any smaller *constant*. For suppose that $\delta < 1 - \alpha\beta$. Since $B(N, K)/NK \to 1 - \alpha\beta$ as $N, K \to \infty$, there exist integers N and K, with N even, such that $\max\{\delta, 1/2\} \leqslant B(N, K)/NK < 1 - \alpha\beta$. Since $B(N, K) \geqslant NK/2 = [(N+1)/2]K$, the class of examples constructed above yields an $N \times K$ A–D matrix M arising from simple majority rule for which $A_M/NK = B(N, K)/NK \geqslant \delta$ and yet more than βN voters disagree with more than αK outcomes.

3. REQUIRING THE ASSENT OF $1 - \alpha\beta$

Unless $1 - \alpha\beta = 1/2$, requiring the assent of at least $(1 - \alpha\beta)N$ voters in order to adopt a proposal is no guarantee that prevailing coalitions comprise, on average, at least $(1 - \alpha\beta)N$ voters. Hence requiring the assent of at least $(1 - \alpha\beta)N$ voters is no guarantee that no more than βN voters will disagree with more than αK outcomes. On the other hand, this decision rule does guarantee that no more than βN voters will disagree with a fraction greater than α of the subset of proposals *adopted* by this rule[3]. However, this conclusion fails in an infinite number of cases if $1-\alpha\beta$ is replaced by any smaller constant ϵ.

For given $\epsilon < 1-\alpha\beta$, choose positive integers a_1, a_2, b_1, and b_2 such that $a_1/a_2 \geqslant \alpha$, $b_1/b_2 \geqslant \beta$ and $\epsilon < 1-(a_1b_1/a_2b_2) \leqslant 1-\alpha\beta$. Since the increasing sequence $(a_2b_2 - a_1b_1)n/(a_2b_2n+1)$ approaches the limit $1-(a_1b_1/a_2b_2)$ as $n \to \infty$, there are an infinite number of integers n satisfying

$$\epsilon \leqslant (a_2b_2 - a_1b_1)n/(a_2b_2n+1) < 1-(a_1b_1/a_2b_2). \tag{3.1}$$

For each n satisfying (3.1), let $N = a_2b_2n+1$ and $K = a_2b_1n+1$. In defining the appropriate voting matrix $V=(v_{ij})$, it is convenient to label the N rows $i = 0, 1, \ldots, a_2b_2n$ and the K columns $j = 0, 1, \ldots, a_2b_1n$. We then set v_{ij} = yes iff $a_2b_1n+1 \leqslant i \leqslant a_2b_2n$, or $0 \leqslant i \leqslant a_2b_1n$ and $i+j \equiv r \pmod{a_2b_2n+1}$ for some $r \in \{0, 1, \ldots, (a_2-a_1)b_1n-1\}$. Each column of V then contains $(a_2b_2-a_1b_1)n$ yeses, hence by (3.1) at least $\epsilon(a_2b_2n+1) = \epsilon N$ yeses. So all K proposals are adopted. On the other hand, each of the first a_2b_1n+1 rows of V contains $(a_2-a_1)b_1n$ yeses and hence a_1b_1n+1 noes. Since $a_2b_1n+1 > (b_1/b_2)(a_2b_2n+1) \geqslant \beta N$ and $a_1b_1n+1 > (a_1/a_2)(a_2b_1n+1) \geqslant \alpha K$, it follows that more than βN voters disagree with more than αK proposals, although each of the K proposals is adopted by the assent of at least ϵN voters.

NOTES

[1] The prevailing coalition on a proposal is the set of voters agreeing with the outcome of voting on that proposal, as determined by whatever decision procedure is employed.

[2] This example is due to Gorman (1978).

[3] In particular, the rule requiring ratification of amendments to the U.S. Constitution by at least three-fourths of the States guarantees that the set of States whose legislatures have rejected a majority of the amendments thus adopted can never constitute a majority. See Wagner (1983, Section 2) for a fuller discussion of this example.

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