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A CHARACTERIZATION OF WEIGHTED ARITHMETIC MEANS*

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Abstract. We prove, among other things, that the set of weighted arithmetic means is identical with the set of functions $f: R^n \to R$ satisfying

(i) $\min\{x_j\} \leqq f(x_1, x_2, \cdots, x_n) \leqq \max\{x_j\}$

and

(ii) for $k = 2, 3$: $\sum_{i=1}^k x_{ij} = s$ $(j = 1, 2, \cdots, n) \Rightarrow \sum_{i=1}^k f(x_{i1}, x_{i2}, \cdots, x_{in}) = s$.

We call a function $f: R^n \to R$ an *averaging function* if

$$\min\{x_j\} \leqq f(x_1, x_2, \cdots x_n) \leqq \max\{x_j\}, \tag{1}$$

and a *weighted arithmetic mean* if $f(x_1, x_2, \cdots, x_n) = w_1x_1 + w_2x_2 + \cdots + w_nx_n$, where $0 \leqq w_j \leqq 1$ and $w_1 + w_2 + \cdots + w_n = 1$. It is easy to check that among the familiar averaging functions (weighted arithmetic, geometric, and harmonic means, weighted medians) weighted arithmetic means uniquely enjoy, for all $k \geqq 1$, what we shall call the *k-allocation property*:

For all $s \in R$, if (x_{ij}) is a $k \times n$ matrix with $x_{1j} + x_{2j} + \cdots + x_{kj} = s$ for $1 \leqq j \leqq n$, then $f(x_{11}, x_{12}, \cdots, x_{1n}) + f(x_{21}, x_{22}, \cdots, x_{2n}) + \cdots + f(x_{k1}, x_{k2}, \cdots, x_{kn}) = s$.

We prove in this note that the k-allocation property assumed only for $k = 2$ and $k = 3$, characterizes weighted arithmetic means in the set of all averaging functions. In fact, we obtain the following more general result:

THEOREM. *The function $f: R^n \to R$ satisfies the k-allocation property for $k = 2$ and $k = 3$ and is continuous at a point or bounded from one side on an (n-dimensional) interval or just on a set of positive measure if and only if there exist real numbers $w_1, w_2, \cdots, w_n$ with $w_1 + w_2 + \cdots + w_n = 1$ such that $f(x_1, x_2, \cdots, x_n) = w_1x_1 + w_2x_2 + \cdots + w_nx_n$.*

Proof. To postulate the k-allocation property for $k = 2, 3$ is equivalent to assuming that, for all $s \in R$,

$$f(x_1, x_2, \cdots, x_n) + f(s - x_1, s - x_2, \cdots, s - x_n) = s \tag{2}$$

and

$$f(x_1, x_2, \cdots, x_n) + f(y_1, y_2, \cdots, y_n) + f(s - x_1 - y_1, s - x_2 - y_2, \cdots, s - x_n - y_n) = s. \tag{3}$$

Setting $s = x_1$ in (2), and writing

$$-f(0, u_2, \cdots, u_n) = g(u_2, \cdots, u_n), \tag{4}$$

we have

$$f(x_1, x_2, \cdots x_n) = x_1 + g(x_1 - x_2, \cdots, x_1 - x_n). \tag{5}$$

Setting $s = x_1 + y_1$ in (3), and writing $u_j = x_1 - x_j$ and $v_j = y_1 - y_j$ $(2 \leqq j \leqq n)$, it

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follows from (4) and (5) that

$$g(u_2, \cdots, u_n)+g(v_1, \cdots, v_n)-g(u_2+v_2, \cdots, u_n+v_h)=0,$$

i.e.,

$$g(u_2+v_2, \cdots, u_n+v_n)=g(u_2, \cdots, u_n)+g(v_2, \cdots, v_n). \tag{6}$$

By [2] and [1, pp. 215–16 and p. 32], the general solution of (6), continuous at a point or bounded from one side on an interval or on a set of positive measure, is

$$g(u_2, \cdots, u_n)=a_2u_2+\cdots+a_nu_n. \tag{7}$$

Hence the general solution of (2) and (3), under these same weak regularity conditions, is, by (5),

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= (1+a_2+\cdots+a_n)x_1-a_2x_2-\cdots-a_nx_n \\ &= w_1x_1+w_2x_2+\cdots+w_nx_n, \end{aligned} \tag{8}$$

with $w_1+w_2+\cdots+w_n=1$, as asserted. Note that, in (8), one or more of the numbers w_j may be negative.

Now if f is assumed to be an averaging function, the aforementioned boundedness conditions are clearly satisfied, and setting $x_j=1$ and $x_k=0$ for $k\neq j$ yields $0\leqq w_j\leqq 1$. Thus we have as a corollary to the above theorem the following characterization of weighted arithmetic means:

COROLLARY. *Let $f: R^n \to R$. Then f is a weighted arithmetic mean if and only if f is an averaging function satisfying the k-allocation property for $k=2$ and $k=3$.*

Remark 1. In the statement of the above corollary the averaging function condition (1) may be replaced by a considerably weaker supposition. It is clearly sufficient, for example, that there exist some $c>0$ (no matter how small) such that $0\leqq x_j\leqq c$ $(j=1, 2, \cdots, n)$ implies $0\leqq f(x_1, x_2, \cdots, x_n)$.

Remark 2. The foregoing theorems arose in connection with a study of arithmetic averaging as a method of amalgamating a set of individual opinions as to the most appropriate values of some sequence of decision variables. (See [3].)

Suppose that there are k decision variables and n individuals and we denote by x_{ij} the opinion of individual j as to the most appropriate value of variable i. In many of these problems (such as the allocation of a fixed sum of money among k competing projects) the column sums of the matrix (x_{ij}) are required to have a common value s. If a group adopts as the consensual value of variable i the weighted arithmetic average $\bar{x}_i=w_1x_{i1}+w_2x_{i2}+\cdots+w_nx_{in}$ the consensual values have the highly desirable property $\bar{x}_1+\bar{x}_2+\cdots+\bar{x}_k=s$. The above results assert that weighted arithmetic means are the only averaging functions with this property.

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