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Generalized Finite Differences and Bayesian Conditioning of Choquet Capacities

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Rota's calculus of finite differences for a locally finite poset employs a difference operator based on the Möbius function. We investigate the behavior of this operator on products and quotients and employ the lattice-of-subsets case of our results to show that r-monotonicity is preserved under Bayesian conditioning of Choquet capacities. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let $(P, \leq)$ be a locally finite poset with least element and let μ be the Möbius function of P. As established by Rota [4], the operator ∇, defined for all $F\colon P \to \mathbf{R}$ by

$$\nabla F(j) = \sum_{i \leq j} \mu(i, j) F(i), \qquad \forall j \in P, \tag{1.1}$$

is the appropriate difference operator for functions on P, satisfying, as it does, the crucial inverse relation

$$F(j) = \sum_{i \leq j} \nabla F(i), \qquad \forall j \in P, \tag{1.2}$$

and reducing to the classical (backward) difference operator $\nabla F(j) = F(j) - F(j-1)$ for the poset $(\mathbf{N}, \leq)$ of nonnegative integers ordered in the usual way.

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For functions $F, G\colon \mathbf{N} \to [0,\infty)$ it is, of course, trivial to show that

$$\nabla F(j) \geq 0 \quad \text{and} \quad \nabla G(j) \geq 0, \quad \forall j > 0 \Rightarrow \nabla FG(j) \geq 0, \forall j > 0, \tag{1.3}$$

and

$$\nabla F(j) \leq 0 \quad \text{and} \quad \nabla G(j) \leq 0, \quad \forall j > 0 \Rightarrow \nabla FG(j) \leq 0, \forall j > 0, \tag{1.4}$$

where $FG(j) := F(j)G(j)$, and, if G is positive, that

$$\nabla G(j) \geq 0, \forall j > 0 \Rightarrow \nabla \frac{1}{G}(j) \leq 0, \forall j > 0, \tag{1.5}$$

and

$$\nabla G(j) \leq 0, \forall j > 0 \Rightarrow \nabla \frac{1}{G}(j) \geq 0, \forall j > 0, \tag{1.6}$$

where $(1/G)(j) := 1/G(j)$.

Interestingly, none of the implications (1.3)–(1.6) holds in general for functions defined on a poset, the implications (1.4) and (1.5) failing to hold in general even for finite lattices. On the other hand, as we prove below, (1.3) and (1.6) extend to all locally finite lattices with least element $\hat{0}$, from which it follows for such lattices L that if $F\colon L \to [0,\infty)$ and $G\colon L \to (0,\infty)$, then

$$\begin{aligned} \nabla F(j) \geq 0 \quad \text{and} \quad \nabla G(j) \leq 0, \quad & \forall j \in L - \{\hat{0}\} \\ \Rightarrow \nabla \frac{F}{G}(j) \geq 0, \quad & \forall j \in L - \{\hat{0}\}. \end{aligned} \tag{1.7}$$

This modest result has striking consequences for the (Bayesian) conditioning of a class of lower probabilities known as *r-monotone Choquet capacities*, i.e., functions $c\colon 2^X \to [0,1]$ such that $c(\phi) = 0, c(X) = 1$, and

$$c(A_1 \cup \cdots \cup A_r) \geq \sum_{\phi \neq I \subset \{1,\ldots,r\}} (-1)^{|I|-1} c\Bigl(\bigcap_{i \in I} A_i\Bigr), \tag{1.8}$$

for every sequence $A_1, \ldots, A_r$ of subsets of the finite set X. Specifically, if $c(E) > 0$ and one defines

$$c(A|E) = \inf\{p(A|E)\colon p \text{ is a probability measure on } 2^X \text{ and } p \geq c\}, \tag{1.9}$$

then, as we prove in Theorem 4.1, $c(\cdot|E)$ inherits r-monotonicity from c. Remarkably, preservation of r-monotonicity under Bayesian conditioning turns out to be a natural and transparent consequence of the lattice-of-subsets case of (1.7).

The remainder of this paper is organized as follows: In the next section we prove a preliminary lemma. In Section 3 we establish the aforementioned lattice generalizations of (1.3) and (1.6), adducing examples to show that our results are best possible. In Section 4 the lattice-of-subsets case of these results is applied to the analysis of Choquet capacities under Bayesian conditioning.

2. Preliminaries

In what follows the least and greatest element of a poset, should they exist, are denoted respectively by $\hat{0}$ and $\hat{1}$. For the requisite background on the Möbius function μ of a locally finite poset the reader is referred to Rota's classical paper [4] or Stanley's book [6, Chapt. 3]. Given elements l and m of the poset P such that $l \leq m$, it follows from the definition of μ that

$$\sum_{l \leq i \leq m} \mu(i, m) = \delta(l, m), \tag{2.1}$$

where δ denotes the Kronecker delta function. In particular, if μ is the Möbius function of the finite lattice $(L, \leq)$, then

$$\sum_{i \in L} \mu(i, \hat{1}) = \sum_{\hat{0} \leq i \leq \hat{1}} \mu(i, \hat{1}) = \delta(\hat{0}, \hat{1}) \geq 0. \tag{2.2}$$

The simplest case of the following lemma is identical with (2.2).

Lemma 2.1. *For every finite lattice $(L, \leq)$, every family $\{z_k \colon k \in L\}$ of nonnegative real numbers such that $\sum z_k < 1$, and every $p > 0$,*

$$\sum_{i \in L} \mu(i, \hat{1}) \left[1 - \sum_{k \leq i} z_k\right]^{-p} \geq 0. \tag{2.3}$$

Proof. We prove (2.3) by induction on $|L|$, the cases $|L| = 1, 2$ being obvious. Since, by (2.2), (2.3) holds when $z_k = 0$ for all $k \neq \hat{0}$, (2.3) will follow if we can show that for every $j \in L - \{\hat{0}\}$, the derivative with respect to z_j of the left-hand side of (2.3) is nonnegative, i.e., that

$$p \sum_{i \geq j} \mu(i, \hat{1}) \left[1 - \sum_{\substack{k \leq i \\ k \in L}} z_k \right]^{-p-1} \geq 0. \tag{2.4}$$

Consider the sublattice $U_j = \{k \in L\colon\ k \geq j\}$ of L, and the family $\{u_k\colon k \in U_j\}$ defined by

$$u_k = \sum_{\substack{h \in L: \\ h \vee j = k}} z_h. \tag{2.5}$$

Since the Möbius functions of L and U_j coincide on U_j [4, Proposition 4], (2.4) is equivalent to

$$\sum_{i \in U_j} \mu(i, \hat{1}) \left[1 - \sum_{k \leq i} u_k \right]^{-p-1} \geq 0, \tag{2.6}$$

which is true by the inductive hypothesis, since $|U_j| < |L|$. □

3. Generalized Finite Differences

THEOREM 3.1. *Let $(L, \leq)$ be a locally finite lattice with least element. If $F, G\colon L \to \mathbf{R}$ satisfy*

$$\nabla F(j) := \sum_{i \leq j} \mu(i, j) F(i) \geq 0, \qquad \forall j \in L, \tag{3.1}$$

and

$$\nabla G(j) := \sum_{i \leq j} \mu(i, j) G(i) \geq 0, \qquad \forall j \in L, \tag{3.2}$$

then

$$\nabla FG(j) := \sum_{i \leq j} \mu(i, j) F(i) G(i) \geq 0, \qquad \forall j \in L. \tag{3.3}$$

Proof. By (3.3), (1.2), (2.1), (3.1), and (3.2),

$$\begin{aligned} \nabla FG(j) &= \sum_{i \leq j} \mu(i, j) \sum_{k \leq i} \nabla F(k) \sum_{l \leq i} \nabla G(l) \\ &= \sum_{k \leq j} \nabla F(k) \sum_{l \leq j} \nabla G(l) \sum_{k \vee l \leq i \leq j} \mu(i, j) \\ &= \sum_{k \leq j} \nabla F(k) \sum_{l \leq j} \nabla G(l) \delta(k \vee l, j) \geq 0. \quad \square \end{aligned} \tag{3.4}$$

THEOREM 3.2. *Let $(L, \leq)$ be a locally finite lattice with least element. If $G\colon L \to (0,\infty)$ satisfies*

$$\nabla G(j) := \sum_{i \leq j} \mu(i,j) G(i) \leq 0, \qquad \forall j \in L - \{\hat{0}\}, \tag{3.5}$$

then

$$\nabla \frac{1}{G}(j) := \sum_{i \leq j} \mu(i,j) \frac{1}{G(i)} \geq 0, \qquad \forall j \in L - \{\hat{0}\}. \tag{3.6}$$

Proof. For each $j \in L - \{\hat{0}\}$ consider the finite lattice $L_j = \{k \in L\colon k \leq j\}$ and the family $\{z_k\colon k \in L_j\}$, where $z_{\hat{0}} = 0$, and $z_k = -\nabla G(k)/G(\hat{0})$ otherwise, recalling that the Möbius functions of L and L_j coincide on L_j. By (3.5), $z_k \geq 0$, $\forall k \in L_j$, and by (1.1) and the fact that $\nabla G(\hat{0}) = G(\hat{0})$,

$$\sum_{\substack{k \leq j \\ k \neq \hat{0}}} -\nabla G(k) = G(\hat{0}) - G(j) < G(\hat{0}), \tag{3.7}$$

whence $\sum_{k \in L_j} z_k < 1$. It follows from Lemma 2.1, with $p = 1$, that

$$\sum_{i \leq j} \mu(i,j) \left[1 + \sum_{\substack{k \leq i \\ k \neq \hat{0}}} \frac{\nabla G(k)}{G(\hat{0})} \right]^{-1} \geq 0, \tag{3.8}$$

and multiplying (3.8) by $1/G(\hat{0})$, and again using (1.1) yields

$$\sum_{i \leq j} \mu(i,j) \left[\sum_{k \leq i} \nabla G(k) \right]^{-1} = \sum_{i \leq j} \mu(i,j) \frac{1}{G(i)} \geq 0, \quad \square \tag{3.9}$$

Combining Theorems 3.1 and 3.2 yields the following obvious corollary:

COROLLARY 3.1. *Let $(L, \leq)$ be a locally finite lattice with least element. If $F\colon L \to [0,\infty)$ and $G\colon L \to (0,\infty)$ satisfy $\nabla F(j) \geq 0$ and $\nabla G(j) \leq 0$, $\forall j \in L - \{\hat{0}\}$, then $\nabla(F/G)(j) \geq 0$, $\forall j \in L - \{\hat{0}\}$.*

To see that Theorems 3.1 and 3.2 fail to hold in general for locally finite posets, consider the poset $(P, \leq)$ with Hasse diagram (Fig. 1). The Möbius

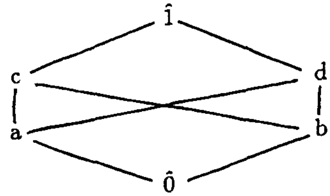

FIGURE 1

function μ of P is easily calculated from the defining conditions (1) $\mu(i,i) = 1$ and (2) $\mu(i,j) = -\sum_{i \le k < j}\mu(i,k)$ if $i < j$.

Consider first the function F: $P \to [0,\infty)$ defined by $F(\hat{0}) = \frac{1}{4}$, $F(a) = F(b) = \frac{3}{8}$, and $F(c) = F(d) = F(\hat{1}) = \frac{1}{2}$. It is easy to check that $\nabla F(j) \ge 0$, $\forall j \in P$. On the other hand, $\nabla F^2(\hat{1}) = \frac{1}{4} - 2(\frac{1}{4}) + 2(\frac{9}{64}) - \frac{1}{16} = \frac{-1}{32}$, so that Theorem 3.1 fails to hold here.

Consider next function G: $P \to (0,\infty)$ defined by $G(\hat{0}) = 1$, $G(a) = G(b) = \frac{3}{5}$, and $G(c) = G(d) = G(\hat{1}) = \frac{1}{5}$. One checks easily that $\nabla G(j) \le 0$, $\forall j \in P - \{\hat{0}\}$. On the other hand, $\nabla(1/G)(\hat{1}) = 5 - 2(5) + 2(\frac{5}{3}) - 1 = \frac{-8}{3}$, so that Theorem 3.2 fails to hold here.

We noted in the introduction that generalizations of (1.4) and (1.5), which are, essentially, Theorems 3.1 and 3.2 with all the inequalities reversed, fail to hold even in finite lattices. To see this, consider the lattice $(L, \le)$, where $L = \{\hat{0}, a, b, c\}$ and $\le$ is inherited from $(P, \le)$ above. The restriction of the above function G to L satisfies $\nabla G(j) \le 0$, $\forall j \in L - \{\hat{0}\}$. On the other hand, $\nabla G^2(c) = \frac{1}{25} - 2(\frac{9}{25}) + 1 = \frac{8}{25}$, so that a generalization of (1.4) fails to hold here. Finally, the function H: $L \to (0,\infty)$ defined by $H(\hat{0}) = 1$, $H(a) = H(b) = 2$, and $H(c) = 4$ satisfies $\nabla H(j) \ge 0$, $\forall j \in L - \{\hat{0}\}$. On the other hand, $\nabla \frac{1}{H}(c) = \frac{1}{4} - 2(\frac{1}{2}) + 1 = \frac{1}{4}$, so that a generalization of (1.5) fails to hold here.

4. Bayesian Conditioning of Choquet Capacities

For a fixed integer $r \ge 2$ and finite set X, a mapping c: $2^X \to [0,1]$ is called an *r-monotone capacity* [1] if $c(\emptyset) = 0$, $c(X) = 1$, and for every sequence $A_1, \dots, A_r$ of subsets of X,

$$c(A_1 \cup \cdots \cup A_r) \ge \sum_{\substack{I \subset [r] \\ I \ne \emptyset}} (-1)^{|I|-1} c\Big(\bigcap_{i \in I} A_i\Big), \tag{4.1}$$

where $[r] := \{1, \dots, r\}$. If c is r-monotone, then it is clearly s-monotone for $2 \le s \le r$. If c is 2-monotone, it is *superadditive* ($A \cap B = \phi \Rightarrow c(A \cup B) \ge c(A) + c(B)$), hence *monotone* ($A \subset B \Rightarrow c(A) \le c(B)$). If c is r-monotone for all $r \ge 2$, c is called an *infinitely monotone capacity*. As proved by Shapley [5], any 2-monotone capacity c admits a dominating probability measure, and so a natural way to condition c on a subset $E \subset X$ with $c(E) > 0$ is to set

$$c(A|E) := \inf_{p \in \mathscr{P}_c} \{p(A|E)\}, \tag{4.2}$$

where $\mathscr{P}_c$ is the set of all probability measures on 2^X such that $p(A) \ge c(A)$ for all $A \subset X$.

In establishing the aforementioned result, Shapely proved that for every nested pair $H_1 \subset H_2$ of subsets of X, there exists a $p \in \mathscr{P}_c$ such that $p(H_i) = c(H_i)$, $i = 1, 2$. It follows immediately that

$$c(A|E) = \frac{c(A \cap E)}{c(A \cap E) + 1 - c(A \cup \bar{E})}, \tag{4.3}$$

since, for all $p \in \mathscr{P}_c$,

$$\begin{aligned} p(A|E) &= \frac{p(A \cap E)}{p(A \cap E) + 1 - p(A \cup \bar{E})} \\ &\geq \frac{c(A \cap E)}{c(A \cap E) + 1 - p(A \cup \bar{E})} \\ &\geq \frac{c(A \cap E)}{c(A \cap E) + 1 - c(A \cup \bar{E})}, \end{aligned} \tag{4.4}$$

and there exists a $p \in \mathscr{P}_c$ such that $p(A \cap E) = c(A \cap E)$ and $p(A \cup \bar{E}) = c(A \cup \bar{E})$.

Walley [7] has shown, by an *ad hoc* argument, that if c is 2-monotone, then $c(\cdot|E)$ is also 2-monotone. Fagin and Halpern [2] and Jaffray [3] have shown that if c is infinitely monotone, then so is $c(\cdot|E)$, but their proofs are quite complicated and make use of special properties of infinitely monotone capacities, over and above their satisfying the inequalities (4.1). In fact, for every $r \geq 2$, r-monotonicity is preserved under Bayesian conditioning, as a natural and transparent consequence of the following lattice-of-subsets case of Corollary 3.1:

LEMMA 4.1. *If $r \geq 2$, $[r] := \{1, \ldots, r\}$, and F: $2^{[r]} \to [0, \infty)$ and G: $2^{[r]} \to (0, \infty)$ satisfy*

$$\nabla F(J) = \sum_{I \supset J} (-1)^{|I-J|} F(I) \geq 0, \qquad \forall J \subsetneqq [r] \tag{4.5}$$

and

$$\nabla G(J) = \sum_{I \supset J} (-1)^{|I-J|} G(I) \leq 0, \qquad \forall J \subsetneqq [r], \tag{4.6}$$

then $\nabla(F/G)(\phi) \geq 0$, *i.e.*,

$$\frac{F(\phi)}{G(\phi)} \geq \sum_{\phi \neq I \subset [r]} (-1)^{|I|-1} \frac{F(I)}{G(I)}. \tag{4.7}$$

Proof. Apply Corollary 3.1 to the lattice $(2^{[r]}, \supset)$ of subsets of $[r]$, noting that the Möbius function of this poset is given by $\mu(I, J) = (-1)^{|I-J|}$ [6, Ex. 3.8.3 and 4, Proposition 3]. □

THEOREM 4.1. *For every integer $r \geq 2$, if c is an r-monotone capacity on the finite set X, $E \subset X$, and $c(E) > 0$, then $c(\cdot\,|E)$, as defined by* (4.3), *is r-monotone.*

Proof. Since $c(A|E) = c(A \cap E|E)$ for all $A \subset X$, it suffices to show that for every sequence $A_1, \ldots, A_r$ of subsets of E,

$$\frac{c(A_\phi)}{c(A_\phi) + 1 - c\left(A_\phi \cup \bar{E}\right)} \geq \sum_{\phi \neq I \subset [r]} (-1)^{|I|-1} \frac{c(A_I)}{c(A_I) + 1 - c\left(A_I \cup \bar{E}\right)}, \tag{4.8}$$

where $A_\phi := A_1 \cup \cdots \cup A_r$ and $A_I := \bigcap_{i \in I} A_i$ for all nonempty $I \subset [r]$.

Define $F, G\colon 2^{[r]} \to \mathbf{R}$ by

$$F(\phi) := \sum_{\phi \neq I \subset [r]} (-1)^{|I|-1} c(A_I), \tag{4.9}$$

$$F(I) := c(A_I), \qquad \phi \neq I \subset [r], \tag{4.10}$$

$$G(\phi) := 1 - \sum_{\phi \neq I \subset [r]} (-1)^{|I|-1}\left(\left(c(A_I \cup \bar{E}\right) - c(A_I)\right), \tag{4.11}$$

and

$$G(I) := c(A_I) + 1 - c\left(A_I \cup \bar{E}\right), \qquad \phi \neq I \subset [r], \tag{4.12}$$

with

$$\nabla F(J) := \sum_{I \supset J} (-1)^{|I-J|} F(I), \qquad \forall J \subset [r] \tag{4.13}$$

and

$$\nabla G(J) := \sum_{I \supset J} (-1)^{|I-J|} G(I), \qquad \forall J \subset [r]. \tag{4.14}$$

Now $\nabla F([r]) = c(A_{[r]}) \geq 0$ and $\nabla F(J) \geq 0$ if $|J| = r - 1$ by monotonicity of c. If $0 \neq |J| \leq r - 2$, then by monotonicity and $r - |J|$-monotonicity of c,

$$\begin{aligned} c(A_J) &\geq c\Big(A_J \cap \bigcup_{i\in[r]-J} A_i\Big) = c\Big(\bigcup_{i\in[r]-J}(A_J \cap A_i)\Big) \\ &\geq \sum_{\phi\neq K\subset[r]-J} (-1)^{|K|-1} c\Big(\bigcap_{k\in K}(A_J\cap A_k)\Big) \\ &= \sum_{I \supsetneq J} (-1)^{|I-J|-1} c(A_I), \end{aligned} \tag{4.15}$$

and so $\nabla F(J) \geq 0$ for such J. Finally, it is easy to check that $\nabla F(\phi) = 0$, so that

$$\nabla F(J) \geq 0, \qquad \forall J \subset [r]. \tag{4.16}$$

Since by (1.2)

$$F(I) = \sum_{J\supset I} \nabla F(J), \qquad \forall I \subset [r], \tag{4.17}$$

it follows that

$$F(I) \geq 0, \qquad \forall I \subset [r]. \tag{4.18}$$

By (4.16), (4.17), and (4.18), if $F(\phi) = 0$, then $\nabla F(J) = 0$, $\forall J \subset [r]$, and so $F(I) = c(A_I) = 0$ if $\phi \neq I \subset [r]$, whence (4.8) holds trivially. In what follows we shall hence assume that $F(\phi) > 0$.

Now by (4.9), (4.11), and the r-monotonicity of c, applied to the sequence $A_1 \cup \overline{E}, \ldots, A_r \cup \overline{E}$,

$$\begin{aligned} G(\phi) &= F(\phi) + 1 - \sum_{\phi\neq I\subset[r]} (-1)^{|I|-1} c\big(A_I \cup \overline{E}\big) \\ &\geq F(\phi) + c\big(A_\phi \cup \overline{E}\big) - \sum_{\phi\neq I\subset[r]} (-1)^{|I|-1} c\big(A_I \cup \overline{E}\big) \\ &\geq F(\phi) > 0. \end{aligned} \tag{4.19}$$

Since, by 2-monotonicity of c, $c(A_I) + 1 - c(A_I \cup \overline{E}) \geq c(E) > 0$, it follows from (4.12) that $G(I) > 0$ if $\phi \neq I \subset [r]$, and so

$$G(I) > 0, \qquad \forall I \subset [r]. \tag{4.20}$$

Next we show that

$$\nabla G(J) \leq 0, \qquad \forall J \subsetneq [r]. \tag{4.21}$$

It is easy to check that $\nabla G(\phi) = 0$. If J is a nonempty proper subset of $[r]$, we may suppose with no loss of generality that $[r] - J = [s]$, for some positive integer $s < r$. Let $B_i := (A_J \cap A_i) \cup \bar{E}$ for all $i \in [s]$, $B_{s+1} := A_j \cap (A_1 \cup \cdots \cup A_s)$, $B_\phi := B_1 \cup \cdots \cup B_{s+1}$, and $B_K = \bigcap_{i \in K} B_i$ for all nonempty $K \subset [s+1]$. Since $s + 1 \leq r$, c is $s+1$-monotone, and so

$$\sum_{K \subset [s+1]} (-1)^{|K|} c(B_K) = c(B_\phi) - c(B_{\{s+1\}}) + \sum_{K \subset [s]} (-1)^{|K|} \big(c(B_K) - c(B_{K \cup \{s+1\}}) \big) \geq 0. \tag{4.22}$$

One checks easily that $B_\phi = [A_J \cap (A_1 \cup \cdots \cup A_s)] \cup \bar{E}$, $B_{\{s+1\}} = B_{s+1} = A_J \cap (A_1 \cup \cdots \cup A_s)$, and for all nonempty $K \subset [s]$, that $B_K = A_{J \cup K} \cup \bar{E}$ and $B_{K \cup \{s+1\}} = A_{J \cup K}$.

By 2-monotonicity of c, it follows from the above that $c(A_J \cup \bar{E}) - c(A_J) \geq c(B_\phi) - c(B_{\{s+1\}})$, and so (4.22) implies that

$$\begin{aligned} \sum_{K \subset [s]} (-1)^{|K|} \big(c(A_{J \cup K} \cup \bar{E}) - c(A_{J \cup K}) \big) &= \sum_{I \supset J} (-1)^{|I-J|} \big(c(A_I \cup \bar{E}) - c(A_I \cap E) \big) \\ &= -\nabla G(J) \geq 0, \end{aligned} \tag{4.23}$$

establishing (4.21) when $\phi \neq J \subsetneq [r]$.

By (4.18), (4.20), (4.16), and (4.21), it follows from Lemma 4.1, (4.10), and (4.12) that

$$\begin{aligned} \frac{F(\phi)}{G(\phi)} &\geq \sum_{\phi \neq I \subset [r]} (-1)^{|I|-1} \frac{F(I)}{G(I)} \\ &= \sum_{\phi \neq I \subset [r]} (-1)^{|I|-1} \frac{c(A_I)}{c(A_I) + 1 - c(A_I \cup \bar{E})}. \end{aligned} \tag{4.24}$$

But by r-monotonicity of c, $c(A_\phi) \geq F(\phi)$ and $c(A_\phi \cup \bar{E}) \geq 1 - G(\phi) + F(\phi)$, and so

$$\frac{c(A_\phi)}{c(A_\phi) + 1 - c(A_\phi \cup \bar{E})} \geq \frac{F(\phi)}{F(\phi) + 1 - c(A_\phi \cup \bar{E})} \geq \frac{F(\phi)}{G(\phi)}, \tag{4.25}$$

where the first inequality above follows from the fact that $x/(x+a)$ is a nondecreasing function of x if $x > 0$ and $a \geq 0$. Combining (4.25) and (4.24) yields (4.8), as desired. $\square$

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