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INTRANSITIVE INDIFFERENCE: THE SEMI-ORDER PROBLEM

1. THE PROBLEM OF INTRANSITIVITY

That an individual's preference or indifference relations may fail to be transitive is familiar. Such intransitivities are harmless, *per se*, and probably an inevitable part of the human condition. When such relations must function as principles of selection, however, intransitivity can lead to irrational choices. The task of devising rational methods for eliminating intransitivity would thus appear to be an important problem of decision theory. While there are usually many ways to eliminate intransitivity, our contention is that, in general, no one of these methods is rationally superior to any other. We argue for this position with respect to the well known problem, first considered by Luce [1956], of extending a semi-order to a weak order. In particular, we show that problems of sequential choice, which require a weak order, are equally well solved by any weak order extension of the original semi-order. We thus establish the rationality of a second-order indifference relation (see Frankfurt [1971] and Jeffrey [1974] for a discussion of higher order preference and indifference) among all possible ways of eliminating the intransitivity of (first-order) indifference.

2. THE CASE OF WINE

Let us consider the case of a buyer of wine. This individual, after extensive tasting, finds that he is equally attracted to wines A and B, and to wines B and C, yet prefers A to C. While he discerns no difference between A and B in a paired comparison and none between B and C in a similar comparison, he does discern a difference between A and C in such a comparison, and in favor of A. While we may not be able to account for such a scheme of preference and indifference, it is certainly not unusual. And it need not cause problems. It is perfectly adequate, for example, as a guide to a one-shot choice from the set $\{A, B, C\}$ *or any subset thereof*.

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But suppose that our wine buyer wishes to purchase a case of wine in the following situation: His wine merchant informs him that these wines will be available for purchase in a certain sequential fashion. Two of the wines, say x and y, will be available on Monday (and determined only then) and the remaining wine, z, may or may not be available on Tuesday. The merchant requires instructions from the buyer as to how to reserve for him a case of wine. The rules of this transaction require that the merchant make for the buyer a provisional choice of x or y, whatever they turn out to be. If z turns out not to be available on Tuesday, the buyer receives Monday's choice. If z is available, the merchant must choose z or Monday's choice. Suppose that the buyer informs the merchant of his preferences and indifferences, as described above, and instructs him to be guided by these, i.e., if the buyer is indifferent, the merchant may choose (provisionally or finally) either, and if he has a preference, the merchant must follow it. It turns out that wines A and B are available on Monday and wine C on Tuesday. As the buyer is indifferent between A and B, the merchant provisionally chooses B for him, and since the buyer is indifferent between B and C, the merchant makes for him the final choice of C. Thus, although A was available and preferred to C, the buyer receives C, in full compliance with his instructions.

Intransitivity of indifference is clearly the culprit in this exercise. The question is how to eliminate it. One suspects (correctly – see note 1 below) that such difficulties may be avoided if the buyer supplies a ranking (formally, a weak ordering) of the wines. But there are five rankings which preserve his original preference for A over C:

1	2	3	4	5
A	A-B	A	B	A
B	C	B-C	A	C
C			C	B

Which of these ways of interpolating additional preferences into the buyer's expressed scheme of preference and indifference is to be chosen, and on what considerations will the decision be based?

We might, of course, ask the buyer to reconsider his original report. Subtle differences in taste might manifest themselves upon re-tasting, enabling the buyer to refine his original scheme to a particular one of the aforementioned five rankings. But suppose that the buyer discerns no additional preferences, even after a thorough re-tasting of the wines.

The merchant, attempting to be helpful, might direct the buyer's attention to criteria other than taste, such as color or price. That such considerations may not solve the problem at hand will be clear to anyone familiar with the problem of integrating preferences based on several criteria. Indeed, even if the buyer can linearly order the wines based on an additional criterion and has a fairly well defined sense of the relative importance of this criterion, the problem may be far from solved.

Suppose, for example, that in a comparison of the colors of the wines, the buyer finds that he prefers the color of C to that of B, that of B to that of A, and, transitively, that of C to that of A. Moreover, he feels that color is a consideration secondary to that of taste and thus proposes to integrate his color preferences with his taste preferences lexicographically, with a color preference prevailing only in the face of indifference with respect to taste. As may easily be checked, this scheme would lead to the adoption of "grand" preferences for C over B, for B over A, and for A over C. Far from solving an intransitivity problem, this scheme compounds it! One might argue that these color preferences point to the superiority of rankings 4 and 5 since each of these at least incorporates two of the foregoing grand preferences, whereas rankings 1, 2 and 3 only incorporate one of these. Even if one were to accept this argument, however, one would still be left with the problem of choosing between rankings 4 and 5.

Assuming that reconsideration, based on taste alone or supplemented by the consideration of additional criteria, is indecisive, can one find "internal" evidence in the buyer's originally expressed scheme of preference and indifference that one of the five rankings deserves to be chosen over the others? Luce [1956] has suggested the adoption of the following formal rule for interpolating additional preferences: Suppose that we have a set S equipped with a semi-order (P, I), i.e., a pair of relations satisfying

S1. xIx

S2. $xPy \;\&\; yIz \;\&\; zPw \rightarrow xPw$

S3. $xPy \;\&\; yPz \;\&\; yIw \rightarrow \sim(xIw \;\&\; zIw)$

S4. Exactly one of the following: xPy, yPx, xIy.

(Note that our wine buyer's originally expressed scheme of preference and indifference satisfies these axioms.) If one defines relations $\bar{P}$ and $\bar{I}$

by

D1. $x\bar{P}y \Leftrightarrow xPy \vee [xIy \;\&\; \exists z(xIz \;\&\; zPy)] \vee [xIy \;\&\; \exists z(xPz \;\&\; zIy)]$

D2. $x\bar{I}y \Leftrightarrow \sim x\bar{P}y \;\&\; \sim y\bar{P}x$,

it follows that $(\bar{P}, \bar{I})$ constitutes a weak order on S in the sense of the following axioms:

W_1. $\bar{P}$ is transitive and asymmetric.
W_2. $\bar{I}$ is an equivalence relation (transitive, reflexive, & symmetric).
W_3. Exactly one of the following: $x\bar{P}y$, $y\bar{P}x$, $x\bar{I}y$.

It is easy to check that Luce's method leads to the adoption of the first of the five possible ways of extending the buyer's semi-order to a weak order. What is the justification for D1? As a principle for exposing originally undiscerned preferences based on the internal evidence of the original semi-order, it has an undeniable formal elegance. But we find it no more compelling than, say,

D1R. $x\bar{P}y \Leftrightarrow xPy \vee [xIy \;\&\; \exists z(xIz \;\&\; zPy)]$,

which, along with D2 always yields a weak order (in the case at hand, the second of the five rankings) or

D1L. $x\bar{P}y \Leftrightarrow xPy \vee [xIy \;\&\; \exists z(xPz \;\&\; zIy)]$,

which, along with D2, always yields a weak order (in the case at hand, the third of the five rankings).

It is true that D1 is in a certain sense more symmetrical than D1R or D1L, but that consideration seems indecisive. If, after all, the point is somehow to detect originally undiscerned preferences, based on nothing but the internal evidence of the original semi-order, what warrants an assumption about the structure of such preference? One can as easily imagine that the buyer's expressed indifference between A and B "goes down to the roots" and that the only undiscerned preference is for B over C (leading one to choose the second ranking) as that there are undiscerned preferences for A over B *and* for B over C. Indeed, it is perfectly consistent with the original semi-order that the undiscerned preferences might be for B over both A and C (leading one to choose the fourth ranking) or for both A and C over B (leading one to choose the 5th ranking). In the matter of discerning the indiscernible one has, in short, no grounds for choosing any of the five rankings over any other.

But does it matter? Let us consider the original sequential purchase problem. The buyer will, based on his original semi-order, be discontented only if he receives C when he could have had A. But the adoption of any one of the five rankings as a principle of choice precludes this result, as one may easily check.[1] In particular, once the wines available on Monday and Tuesday are fixed, the buyer will be indifferent, *qua* his original semi-order, between the wines purchased under any of the five rankings.[2] Thus there is no need to puzzle over the mysterious ways of undiscerned preferences or to argue over formal principles for detecting them. One can be truly and rationally indifferent between the five rankings. The happy buyer may tell his merchant "Extend my semi-order to a weak order in any of the five ways that you like and be guided by that weak order in making my purchase."

3. THE SCOTT-SUPPES THEOREM

Some readers will undoubtedly feel that we have been too hasty in rejecting Luce's solution to the problem at hand. Is the intransitivity of indifference in the buyer's semi-order, they may ask, not accounted for, as it is in the familiar cases of ordering by brightness, loudness, sweetness, etc., by a limited ability to discriminate stimuli? And in that case, is it not true that B *must*, on some deep level, lie between A and C, that there are undetected preferences for A over B and B over C that cumulate and manifest themselves finally in an articulated preference for A over C?

Some may see in the well known theorem of Scott and Suppes [1958] a formal justification for this view. This theorem asserts that if P is a relation on a finite set A, with $xIy \Leftrightarrow \sim xPy$ and $\sim yPx$, and δ is some positive number, then (A, P, I) is a semi-order if and only if there is a real valued function f on A such that $xPy \Leftrightarrow f(x) - f(y) > \delta$. It is tempting to view the number δ as some sort of threshold. According to this view the pair (f, δ) would constitute what might be termed a "semi-utility function". By simply deleting δ and treating f as an (ordinal) utility function (by specifying that $x\bar{P}y \Leftrightarrow f(x) > f(y)$) one might then intuit previously undetected preferences. In the wine buying example, it is easy to check that this approach always leads to Luce's solution, ranking 1. But even in such a simple case, there are problems with this approach. The problems derive from the general lack of empirical referents for f and δ, both for individual pairs (f, δ) and for the entire set of such pairs.

In more complicated problems the variety of possible semi-utility functions (f, δ) associated with a given semi-order may even, of course, yield different ways of extending that weak order to a semi-order. Suppose, for example, that an individual is indifferent between all pairs of alternatives from the set $\{A, B, C, D\}$ except for $\{A, D\}$ and he prefers A to D. Among other semi-utility functions "generating" this semi-order are: $f(A) = 3$, $f(B) = 2$, $f(C) = 2$, $f(D) = 1$ (with $\delta = 1$) and $g(A) = 4$, $g(B) = 3$, $g(C) = 2$, $g(D) = 1$ (with $\delta = 2$). Ignoring thresholds yields, in the first case, the weak order

$$\begin{array}{c} A \\ B - C \\ D \end{array}$$

and in the second, the weak order

$$\begin{array}{c} A \\ B \\ C \\ D \end{array}$$

So the method of employing semi-utility functions as if they were ordinal utility functions, aside from problems of empirical justification, does not always yield a unique solution to the problem of extending a semi-order to a weak order.

This method does, however, restrict the set of weak order extensions of a semi-order to those which involve a kind of centering strategy. We leave it to the reader to show that, in the above case, the only other weak order derivable by ignoring the threshold of a semi-utility function associated to the original semi-order is

$$\begin{array}{c} A \\ C \\ B \\ D \end{array}$$

So the alternative B and C are always "centered" between A and D.[3]

Is it rational to restrict the candidates for extending a semi-order to those weak orders derived by some sort of centering? Our answer is negative, for two reasons. First, whatever attractiveness centering may have depends on viewing intransitivity of indifference as the result of a lack of capacity to discriminate stimuli. But such intransitivity does not always arise in this way. A famous example of Armstrong [1939] illustrates the point. A boy is indifferent between receiving as a gift a bike or a pony, also indifferent between receiving a bike with a bell or a pony, but prefers receiving a bike with a bell to a bike. The boy is fully aware of what he is getting in each case. It is just that the bell is important (in a small, but significant way) only when his receiving the bike is already established. But a small thing like a bell just doesn't influence his choice between a pony and a bike with a bell.

Second, even if intransitivity of indifference is the result of a lack of discriminatory capacity, the case for centering is not established. To take the case of our wine buyer, for example, suppose that he will drink all the wine himself, and that he is not a wine snob, i.e., he is not interested in differences that he cannot taste. There is no reason for such a person to prefer centering even if it could somehow be established (on the authority of some expert, say) that there are differences between A and B and between B and C that our buyer cannot discriminate. For these are differences that do not make a difference to the buyer, and so even if his failure to discriminate such differences accounts in some sense for his intransitivity of indifference, this does not imply that centering is the right strategy. There is, so to speak, nothing irrational about not being a wine snob.

We conclude by reiterating the claim made at the beginning of this paper, namely, that there is no rationally mandated single way of eliminating intransitivity of indifference. In particular, it can be proved that an individual will be I-indifferent between the outcomes of sequential choice guided by any weak order extensions of his original semi-order (P, I). Hence, it is rational to have a (second-order) relation of indifference among all of the ways of extending a semi-order to a weak order.

NOTES

[1] This result holds for all problems of sequential choice. For let (P, I) be a semi-order on a finite set A and $(\bar{P}, \bar{I})$ a weak order on A such that $xPy \Rightarrow x\bar{P}y$. Define $\bar{R}$ by $x\bar{R}y \Leftrightarrow x\bar{P}y$ or $x\bar{I}y$. $\bar{R}$ may easily be shown to be transitive. Now let $A_1, \ldots, A_k$ be a sequence of nonempty, pairwise disjoint subsets of A and choose a sequence $x_1, \ldots, x_k$ as follows: x_1

is any element of A_1 such that $x_1\bar{R}y, \forall y \in A_1$, and for $i = 2, \ldots, k$, x_i is any element of $A_i \cup \{x_{i-1}\}$ such that $x_i\bar{R}y, \forall y \in A_i \cup \{x_{i-1}\}$. By transitivity of $\bar{R}$ each x_i satisfies $x_i\bar{R}y, \forall y \in A_1 \cup \ldots \cup A_i$. In particular x_k (the final choice resulting from this sequential procedure) satisfies $x_k\bar{R}y, \forall y \in A_1 \cup \ldots \cup A_k$. Thus sequential choice guided by a weak order never produces a final choice which is less preferred than some available alternative. This result is well known. But now since $\bar{P}$ extends P it also follows that there is no $y \in A_1 \cup \ldots \cup A_k$ such that yPx_k.

[2] This result also holds generally. For suppose in the above situation that $(\bar{P}', \bar{I}')$ is another weak order (with $\bar{R}'$ defined analogously to $\bar{R}$) such that $\bar{P}'$ extends P and choose a sequence $x'_1, \ldots, x'_k$ as above, guided by $(\bar{P}', \bar{I}')$. Then by the above argument, $x'_k\bar{R}'x_k$ and $x_k\bar{R}x'_k$, and so $\sim x_k\bar{P}'x'_k$ (hence $\sim x_kPx'_k$) and $\sim x'_k\bar{P}x_k$ (hence $\sim x'_kPx_k$). Thus $x_kIx'_k$.

[3] It is easy to show that taking any Scott-Suppes semi-utility function associated with a semi-order (P, I) and treating it as an ordinal utility function yields a weak order $(\bar{P}, \bar{I})$ which is "compatible" with the semi-order in the following sense: $\bar{P}$ extends P, and $a\bar{R}b\bar{R}c\bar{R}d$ & $aId \Rightarrow bIc$, where $\bar{R}$ is defined as in note 1 above. Moreover, all weak orders compatible with a given semi-order arise in this way. This result may be proved by modifying the proof of the Scott-Suppes Theorem which appears in Roberts [1979, pp. 260–262]. See Roberts [1979, Theorem 6.6] for an elaboration of the notion of compatibility.

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