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MINIMAL COVERS OF FINITE SETS

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Abstract. We enumerate the minimal covers of a finite set S, classifying such covers by their cardinality, and also by the number of elements in S which they cover uniquely.

1. Preliminaries

A *cover* of a set S is a collection of nonempty subsets of S, the union of which is S. Let $C(n)$ denote the number of covers of a set with finite cardinality n (abbreviated 'n-set' in what follows). Since any collection of nonempty subsets of an n-set S covers some subset of S, it follows that the numbers $C(n)$ are generated by the recurrence

$$C(n) = 2^{2^n - 1} - 1 - \sum_{n=1}^{n-1} \binom{n}{j} C(j).$$

A cover of S will be called *minimal* if none of its proper subsets covers S. Among the minimal covers of a set are all of its partitions. We recall that the number of partitions $P(n)$ of an n-set S is given by the formula

$$P(n) = \sum_{j=1}^{n} S(n, j),$$

where the $S(n, j)$, called Stirling numbers of the second kind, enumerate the partitions of S into j classes [1, p. 99].

2. Counting minimal covers

We shall find it useful to classify the minimal covers of a set S both by their cardinality and by the number of points in S which they cover uniquely. [A minimal cover of S consisting of k nonempty subsets of S will be called a *k-member minimal cover of S.* A point in S is said to be *covered uniquely* by a cover of S if it is an element of exactly one member of the cover, in which case that member is also said to cover the point uniquely.] Clearly, each member of a minimal cover of S covers at least one point of S uniquely. Hence, a minimal cover of an n-set may have no more than n members and a k-member minimal cover of S covers at least k points of S uniquely.

Let $M(n, k, j)$ denote the number of k-member minimal covers of an n-set S which cover j points of S uniquely. Each minimal cover of S of this specification induces a k-member partition of the j uniquely covered points in the obvious way. Thus, for $2 \leq k \leq j \leq n$, each minimal cover of this specification may be constructed by (i) choosing a j-subset of S, (ii) partitioning this subset into k classes, and (iii) adjoining the remaining $n-j$ elements of S to these k classes in such a way that each of these elements is adjoined to at least two of the classes. Hence, for $2 \leq k \leq j \leq n$,

$$(1) \qquad M(n, k, j) = \binom{n}{j} (2^k - k - 1)^{n-j} S(j, k) .$$

Clearly, $M(n, 1, j) = 0$ if $1 \leq j < n$ and $M(n, 1, n) = 1$ for all $n \geq 1$. Hence, with the convention $0^0 = 1$, formula (1) holds for $1 \leq k \leq j \leq n$.

Let $M(n, j)$ enumerate the minimal covers of an n-set which cover j points of that set uniquely and let $M^*(n, k)$ enumerate the k-member minimal covers of an n-set. It follows from (1) that

$$(2) \qquad M(n, j) = \binom{n}{j} \sum_{k=1}^{j} (2^k - k - 1)^{n-j} S(j, k)$$

and that

$$(3) \qquad M^*(n, k) = \sum_{j=k}^{n} \binom{n}{j} (2^k - k - 1)^{n-j} S(j, k) .$$

In particular, $M(n, 1) = 0$ if $n > 1$, $M(n, 2) = \binom{n}{2}$ and $M(n, n) = P(n)$. Also, $M^*(n, 1) = M^*(n, n) = 1$. In addition, it follows from elementary properties of the Stirling numbers [1, p. 43] that

$$M^*(n, n-1) = \tfrac{1}{2}n\,(2^n - n - 1)$$

and

$$M^*(n, 2) = \sum_{j=2}^{n} \binom{n}{j} S(j, 2) = S(n+1, 3)\,.$$

3. Tables and generating functions

Table 1
$M(n, j)$ for $n \le 7$

n \ j	1	2	3	4	5	6	7
1	1						
2	0	2					
3	0	3	5				
4	0	6	28	15			
5	0	10	190	210	52		
6	0	15	1,340	3,360	1,506	203	
7	0	21	9,065	60,270	48,321	10,871	877

Table 2
$M^*(n, k)$ for $n \le 7$

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	6	1				
4	1	25	22	1			
5	1	90	305	65	1		
6	1	301	3,410	2,540	171	1	
7	1	966	33,621	77,350	17,066	420	1

With regard to congruences for the numbers in Tables 1 and 2, we note that, for p prime, $M(p, j) \equiv 0 \pmod p$ for $1 \le j < p$, and $M^*(p, k) \equiv 0 \pmod p$ for $1 < k < p$. These assertions follow easily from (2), (3) and familiar congruences (mod p) for binomial coefficients and Stirling numbers.

For convenience in discussing generating functions for the rows of Tables 1 and 2, set $M(n, 0) = M^*(n, 0) = 0$, and let

$$(4) \qquad M_n(x) = \sum_{j=0}^{n} M(n, j) x^j$$

and

$$(5) \qquad M_n^*(x) = \sum_{k=0}^{n} M^*(n, k) x^k \ .$$

The Stirling numbers may be eliminated from (4) and (5) by writing $S(j, k) = \Delta^k 0^j / k!$ [1, p. 33]. Thus,

$$\begin{aligned} M_n(x) &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n} (2^k - k - 1)^{n-j} S(j, k) x^j \\ &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n} (2^k - k - 1)^{n-j} \frac{x^j}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} s^j \\ &= \sum_{k=0}^{n} \frac{1}{k!} \sum_{s=0}^{k} (-1)^{k-s} \sum_{j=0}^{n} \binom{n}{j} (2^k - k - 1)^{n-j} (sx)^j \\ &= \sum_{k=0}^{n} \frac{1}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} (2^k - k - 1 + sx)^n \ . \end{aligned}$$

Similarly,

$$M_n^*(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} (2^k - k - 1 + s)^n \ .$$

Note that $M_n(1) = M_n^*(1)$ = the total number of minimal covers of an n-set.

For p an odd prime,

$$M_p(x) = x^2 \sum_{j=2}^{p} M(p, j) x^{j-2} \ .$$

Since $M(p, p) = \Sigma_{k=1}^{p} S(p, k) \equiv 2 \pmod p$, $M(p, j) \equiv 0 \pmod p$ for

$2 \leq j < p$, and $M(p, 2) = \binom{p}{2} \not\equiv 0 \pmod{p^2}$, it follows that $M_p(x)$ is x^2 times an Eisenstein irreducible polynomial of degree $p-2$.

We conclude with an observation of L. Carlitz. Set

$$M(n, j, x) = \binom{n}{j} \sum_{k=0}^{j} (2^k - k - 1)^{n-j} S(j, k) x^k ,$$

so that $M(n, j, 1) = M(n, j)$. Then

$$M(n+j, j, x) = \binom{n+j}{j} \sum_{k=0}^{j} (2^k - k - 1)^n S(j, k) x^k$$

$$= \binom{n+j}{j} \sum_{k=0}^{j} S(j, k) x^k \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} 2^{ks} (k+1)^{n-s} .$$

Now set

$$P_j(x) = \sum_{k=0}^{j} S(j, k) x^k ,$$

so that

$$(\mathrm{D}x)^r P_j(x) = \sum_{k=0}^{j} (k+1)^r S(j, k) x^k \quad \left(\mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}x}\right) .$$

It follows that

$$M(n+j, j, x) = \binom{n+1}{j} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} (\mathrm{D}x)^{n-s} P_j(2^s x) .$$

In particular, $M(j, j, 1) = M(j, j) = P_j(1) = P(j)$, as noted in Section 2.

Reference

[1] J. Riordan, An introduction to combinatorial analysis (Wiley, New York, 1958).