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# Realistic Opinion Aggregation

## Lehrer-Wagner with a Finite Set of Opinion Values

### 1. Opinion Aggregation

Opinion aggregation problems arise when a number of individuals express different opinions on some set of variables (the 'agenda') and we wish to combine them into a single consistent 'collective' opinion on each variable. General methods for solving problems of this kind have been extensively studied in different domains - for instance in social choice theory, in statistics and in judgment aggregation theory – typically by identifying the class of methods satisfying one or more constraints. And although the kinds of opinions (votes, probability estimates, acceptances of propositions, etc.,) that that serve as inputs differ in these studies, as do the attendant notions of consistency of opinion, very similar constraints on aggregation are invoked in all of them.

Three of the most common of such constraints are especially relevant to our discussion. They are:

1. *Universal Domain*: The requirement that the method of opinion aggregation be applicable to any combination of individual opinion.
2. *Independence*: The requirement that the collective opinion on any particular variable depend only on the individuals' opinion on *that* variable and not on their opinions on any other variable.
3. *Unanimity Preservation*: The requirement that any opinion unanimously held by individuals be retained in the collective opinion.

Although the precise implication of these constraints on aggregation methods depends on the exact form of the opinion aggregation problem, in combination they severely constrain the class of admissible aggregation methods. Indeed in some well-known cases they suffice to restrict us to dictatorial methods i.e. to those methods which assign a collective value to each variable as a function of opinion of a single individual.

This paper studies the effect of these conditions in the context of a common type of opinion aggregation problem, termed an *allocation* problem, and in particular those in which the set of permissible opinions is finite. Allocation problems are opinion aggregation problems in which both individuals' opinions and the collective opinion are required to sum to a fixed value. A very simple and familiar kind of allocation problem is when individuals must vote for one, and only one, of a set of alternative proposals and one, and only one proposal, must be collectively accepted. Another is when a fixed amount of money is available to spend on a number of alternative goods or projects and individuals hold different opinions on how much should be spent on each. In this case the sum of the proposed amounts to be spent on each alternative, as well as the finally agreed amounts, must sum to the available budget. Finally, we face an allocation problem when individuals make probability judgments on a set of mutually exclusive and exhaustive propositions, for then the sum of these probabilities, as well as the sum of the aggregate ones, must equal one.

Any general method for forming a collective opinion in allocation problems will be termed an allocation aggregation method. In Lehrer and Wagner (1981) it was shown that in cases in which the set of possible opinion values is infinite (specifically an interval of real numbers) versions of the Universal Domain, Independence and Unanimity Preservation conditions suffice to constrain the class of allocation aggregation methods to those taking the form of linear averages with non-negative weights. In this note we extend their treatment to the case of 'realistic' allocation problems, namely ones in which the set of possible opinion values is finite. The main result of the paper is the following: *in realistic allocation aggregation problems the only aggregation methods consistent with only the three conditions are the dictatorial ones*.

We proceed as follows. In the next section, we review the results of Lehrer and Wagner (1981) characterizing linear averaging rules in the case infinite valuation domains before stating our ‘dictatorship’ result for finite domains. In subsequent sections, we consider the implications of the theorem and relation to the existing literature. All proofs are contained in an appendix.

**2. Allocation Aggregation Problems**

An $n \times m$ matrix $A = (a_{ij})$ is an s-*allocation matrix* if (1) each entry of A is a nonnegative real number and (2) the sums of the entries in each row of A are identically equal to some fixed positive real number s. Each such s-allocation matrix may be thought of as recording the opinions of n individuals regarding the most appropriate values of variables $x_1,\dots,x_m$, constrained to be nonnegative and to sum to s, with $a_{ij}$ denoting the value assigned by individual i to variable $x_j$. When n = 1, an s-allocation matrix is called an s-*allocation row vector.* In what follows, the $j^{th}$ column of a matrix A is denoted by $A_j$ , and the $j^{th}$ entry of the row vector a is denoted by $a_j$. The $n \times 1$ column vector with all entries equal to c is denoted by **c**. If $A = (a_{ij})$ and $B = (b_{ij})$ are any matrices with identical dimensions, we write $A \le B$ to indicate that $a_{ij} \le b_{ij}$ for all i and j.

Let V denote a subset of the interval [0,s] satisfying the closure conditions: (1) $0 \in V$, (2) if $x \in V$ then $s - x \in V$, and (3) if $x,y \in V$ and $x + y \le s$ then $x + y \in V$. Let $\mathcal{A}(n,m;s,V)$ denote the set of all $n \times m$ s-allocation matrices with entries limited to elements of V, and $\mathcal{A}(m;s,V)$ the set of all m-dimensional s-allocation row vectors with entries limited to elements of V. An *allocation aggregation method* (AAM) is any mapping F: $\mathcal{A}(n,m;s,V) \to \mathcal{A}(m;s,V)$, from profiles of individual allocations to a collective allocation. An AAM F is *dictatorial* if there exists an individual $d \in \{1,\dots,n\}$ such that for all $A \in \mathcal{A}(n,m;s,V)$, $F(A) = (a_{d1}, a_{d2},\dots,a_{dm})$.

Each AAM furnishes a method, applicable to every conceivable s-allocation matrix A, of reconciling, in the form of the group assignment $F(A) = a = (a_1,\dots,a_m)$, the (typically) different opinions recorded in A. By definition thus an AAM satisfies the Universal Domain requirement. We now state a version of the Independence requirement appropriate to allocation aggregation and a weak version of the Unanimity Preservation condition.

*Independence of Alternatives* (IA) For all $j \in \{1,\ldots,m\}$ and all $A,B \in \mathcal{A}(n,m;s,V)$:

$$A_j = B_j \Rightarrow F(A)_j = F(B)_j.$$

*Zero Preservation* (ZP) For all $j \in \{1,\ldots,m\}$ and all $A \in \mathcal{A}(n,m;s,V)$:

$$A_j = \mathbf{0} \Rightarrow F(A)_j = 0.$$

**Theorem 1.** (Lehrer and Wagner 1981) If $m \geq 3$ and $V = [0,s]$, then an AAM F satisfies IA and ZP if and only if there exists a *single* sequence $w_1,\ldots,w_n$ of weights, nonnegative and summing to one, such that for all $A \in \mathcal{A}(n,m;s,V)$ and all $j = 1,\ldots,m$, $F(A)_j = w_1a_{1j} + w_2a_{2j} + \ldots + w_na_{nj}$.

In Theorem 1, the *valuation domain* V, i.e., the set of values that may be assigned to the variables, is the infinite closed interval [0,s]. In real world allocation problems however valuation domains will necessarily be finite because of resource constraints. And, as we now show, if the domain of values is finite and satisfies the mild closure conditions listed above, then only dictatorial aggregation satisfies IA and ZP.

**Theorem 2.** If $m \geq 3$ and V is finite, then an AAM $F: \mathcal{A}(n,m;s,V) \rightarrow \mathcal{A}(m;s,V)$ satisfies IA and ZP if and only if F is *dictatorial*.

Theorem 2 is the main plank of our claim that realistic allocation aggregation must be dictatorial if it satisfies the trio of conditions: Universal Domain, Independence and Unanimity Preservation. In the final section, we examine the three conditions with a view to assessing the scope of the theorem. But before this we turn to a comparison of Theorem 2 with similar results to be found in the literature on opinion aggregation.

## 3. Dictatorship in Opinion Aggregation

The existing literature on opinion aggregation can be divided into two main groups. The first consists of work that assumes that every valuation domain is a continuum: prominent examples include the work on probability aggregation found mainly in statistics and the work on utility aggregation found in social choice theory. The second consists of work that assumes a binary valuation domain: prominent examples here include the work on ordinal preference aggregation in social choice theory and that in the new field of judgment aggregation.[1] Theorem 2 fills something of the space in between the two. In doing so it reveals an interesting connection between the characterizations of linear averaging found in the first group and the dictatorship results of the second. In short, conditions that allow arbitrary linear averaging when the valuation domain is a continuum force such averaging to take the extreme form of a dictatorship when the valuation domain is finite.

To add a bit more substance to this claim, let us consider how Theorem 2 sheds some light on the dictatorship results for proposition-wise independent judgment aggregation. In a typical judgment aggregation problem a set of individuals face a set of logically interconnected propositions, called the agenda, upon which they must reach a collective opinion. An agenda is assumed to be closed under negation, and individuals must either accept or reject each of its propositions. A judgment aggregation rule is a universal mapping from profiles of such consistent and complete individual judgments on the agenda to a consistent and complete collective judgment on it. The aggregation rule satisfies proposition-wise independence just in case the collective judgment on a proposition *p* depends only the individuals' judgments on *p*.

To form an allocation matrix from a judgment aggregation problem, let the column variables of the matrix be maximal consistent subsets of the agenda and the opinion values for each such set be 1 iff the individual accepts all propositions in the set (equivalently, accepts their conjunction), and 0 otherwise. The valuation domain is thus just the set {0,1}, which trivially satisfies the closure conditions of Theorem 3, and row opinion values must sum to 1. Since an opinion value of 1 on any particular maximal

[1] See List and Puppe (2009) for a survey of the field.

subset requires an opinion value of zero on the others, IA amounts to the requirement that the aggregate opinion value on each subset of the agenda depends only on which individuals accept the propositions in it. By Theorem 2 any judgment aggregation rule that preserves unanimous rejection of any maximal subset and satisfies this kind of independence is dictatorial.

This result is very close to Theorem 1 of Dietrich (2006) which proves that dictatorial rules are the only ones satisfying proposition-wise independence on what he calls 'states of the world' – essentially conjunctions of the propositions in the maximal subsets of the agenda – and a weak responsive condition that implies ZP in this context. But in general, although the allocation matrix representation of a judgment aggregation problem preserves all the information contained in the standard description of the problem, the application of IA to aggregation of the maximal subsets of the agenda has subtly different consequences from the requirement of proposition-wise independence of the agenda items. Nonetheless, the two conditions are close enough in spirit for Theorem 2 to shed some indirect light on the fact that proposition-wise independence implies dictatorial aggregation for a wide class of agendas. Specifically the contrast with the possibility of linear averaging in allocation problems involving probabilistic opinions suggests that the restriction to simply acceptance or rejection of propositions is playing an important role in the dictatorship results for proposition-wise independent judgment aggregation. This claim is considerably reinforced by Theorem 1 of Dietrich and List (2010), which shows that any proposition-wise aggregation rule that satisfies a condition that they call implication preservation (which requires the aggregate judgment set to be consistent with *p* implying *q* whenever all individuals' sets are), is a linear averaging rule when the judgments are probabilistic and a dictatorial rule when they are binary.

**4. Assessment**

In response to dictatorship results such as Theorem 2, it is natural to re-examine the conditions under which these results are derived. If further reflection suggests that they are unnecessarily stringent, then one might wish to seek out weaker conditions which allow for a broader class of aggregation methods while still ensuring a principled synthesis of individually differing allocations. Successful identification of such weaker conditions would show that the dictatorship results, while perhaps mathematically interesting, pose no real dilemma for group decision making.

How then might the conditions under which Theorem 2 was derived be modified? Condition ZP, which requires that aggregation respect the group's unanimity in assigning a variable the value zero, strikes us as eminently reasonable.[2] Finiteness of the valuation domain on the other hand is inevitable feature of any real world allocation problem. Only in idealized models is a continuum of possible values of the variables acceptable, with the upshot that arbitrary weighted linear averaging becomes admissible. To make the move to such models seems to us to give up on a realistic account of group decision making, the very point of the present paper. On the other hand, it is important to note that not every realistic opinion aggregation problem has the form of an allocation problem and that non-dictatorial opinion aggregation methods may be consistent with the usual conditions when the set of collective opinion values is finite, so long as this set is sufficiently larger than the set of permissible individual opinions. For instance if n individuals' opinions are draw from the set {1,2,…,n} but the collective opinion is drawn from the set {1,1/2,1/3,…,1/n,2,2/3,…,n} then some form of linear averaging may be possible.[3] Such cases lie outside the scope of this paper, however.

This leaves us with the independence condition IA. Conditions of this kind have come in for a fair amount of criticism, both in social choice theory and in the theory of probability aggregation. Yet they continue to feature (sometimes supplemented by normalization in order to satisfy an allocation constraint) in many treatments of group decision making.

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[2] Foregoing ZP might be reasonable if individuals in the group were only serving as advisors to an external decision maker with the ultimate power to choose an allocation. But, at least when V is a continuum, this does not significantly enlarge the set of acceptable AAMs. See Aczél, et al (1984). In the presence of IA, the condition ZP is equivalent to *s-Preservation* : For all $j \in \{1,\ldots,m\}$, and all $A \in A(n,m;s)$, $A_j = \mathbf{s} \Rightarrow F(A)_j = s$. So we may substitute s-Preservation for ZP in Theorems 2, 4, and 5. Indeed this might be preferable since, in isolation , s-Preservation is weaker than ZP, and, as Franz Dietrich has pointed out to us, s-Preservation corresponds nicely to the condition of *judgment-set-wise unanimity preservation* in judgment aggregation. See Dietrich (2006).
[3] We are grateful to the anonymous referee for this point.

The reason for this is to be found in the universal domain axioms that typically underlie, either explicitly or implicitly, axiomatic analyses of such decision making.[4] For if a method of aggregating differing opinions must be prepared to handle *every logically possible profile* of opinions, it is hard to imagine how to specify such a method without proceeding variable-by-variable (resp., proposition-by-proposition, state-by-state, or event-by-event). There are cases in which abandoning the Universal Domain condition allows for principled ways of resolving disagreement in a holistic manner not hobbled by IA.[5] But in the present one, it is hard to see how restricting the set of allocation matrices on which aggregation is intended to operate might open up the canon of acceptable AAMs.

In conclusion, we regard Theorem 2 as having genuine, though not devastating, limitative import. For allocation aggregation problems can always be resolved in practice by taking weighted arithmetic averages of the individual values assigned to each variable and, if those averages fail to lie in V, adjusting them by rounding up or down, while ensuring that the adjusted values continue to sum to s. While such adjustments may be small in magnitude, they will inevitably be ad hoc, with the upshot that the procedure will fall short of ideally rational aggregation.[6]

## 5. Appendix

In order to prove Theorem 2, we need to establish two preliminary lemmata. The first involves the following property of aggregation, which strengthens IA:

*Strong Label Neutrality* (SLN). For all $j,k \in \{1,\ldots,m\}$, and all $A, B \in \mathcal{A}(n,m;s,V)$,

$A_j = B_k \Rightarrow F(A)_j = F(B)_k$.

[4] In our setup, universal domain is implicit in our definition of an AAM as a mapping F with domain A(n,m;s,V).

[5] See, for example, Wagner (2010), where it is shown how one can avoid dictatorship results entailed by IA, ZP, and the demand for preservation of unanimous stochastic independence judgements by a natural restriction of the set of profiles of probability distributions to which an aggregation method must apply.

[6] We are indebted to Franz Dietrich, Christian List and an anonymous referee for their helpful comments on earlier drafts of the paper.

*Remark.* Whereas IA is equivalent to the existence of functions $f_j$: $V^n \rightarrow V$ such that, for all $A \in \mathcal{A}(n,m;s,V)$, $F(A)_j = f_j(A_j)$, SLN requires that the functions $f_j$ be identically equal to a single function f.

**Lemma 1.** Suppose that $m \geq 3$ and V is a subset of [0,s] satisfying the closure conditions (1), (2), and (3) introduced in Section 2 above. If an AAM
$F: \mathcal{A}(n,m;s,V) \rightarrow \mathcal{A}(m;s,V)$, satisfies IA and ZP, then it satisfies SLN.

*Proof.* See Lehrer and Wagner (1981, Theorem 6.2). Note that while their theorem is stated for the case V = [0,s], its proof only invokes conditions (1), (2), and (3). □

**Lemma 2.** If s > 0, a finite subset V of [0,s] with cardinality r + 1 satisfies
(1) $0 \in V$,
(2) $x \in V \Rightarrow s - x \in V$, and
(3) $x,y \in V$ and $x + y \leq s \Rightarrow x + y \in V$
if and only if $V = \{ks/r : k = 0,1,\dots,r\}$.

*Proof.* Sufficiency: obvious. Necessity: If r = 1, the result is obvious. Suppose then that $r \geq 2$, and let $\alpha$ be the smallest positive element of V. We shall show by a proof by contradiction that $\alpha$ = s/r. Suppose not. Then either (i) $\alpha$ < s/r or (ii) $\alpha$ > s/r.

(i) If $\alpha$ < s/r, then by repeated application of (3) it follows that
$k\alpha \in V$, k =0,1,…,r, and hence by (2) that $s - r\alpha \in V$. By assumption, $s - r\alpha > 0$. Moreover, $s - r\alpha < \alpha$, for otherwise $(r + 1)\alpha \leq s$, which would imply that $(r + 1)\alpha \in V$ and hence that |V| > (r + 1). But this contradicts the assumption that $\alpha$ is the smallest positive element of V.

(ii) If $\alpha$ > s/r, then $r\alpha > s$, and so $r\alpha \notin V$. Let m be the largest integer for which $m\alpha \leq s$, whence m < r, $m\alpha \in V$, and $s - m\alpha \in V$. Suppose that $m\alpha < s$. Then

$0 < s - m\alpha < \alpha$, again contradicting the assumption that $\alpha$ is the smallest positive element of V. So $\alpha = s/m$. Moreover, $V = \{0, s/m, \dots, (m-1)s/m, s\}$. Otherwise, there exists $\beta \in V$ such that $ks/m < \beta < (k+1)s/m$, where $1 \le k \le m-1$. Let $\beta^* := \beta + (m-1-k)s/m$. Then $(m-1)s/m < \beta^* < s$, and so $\beta^* \in V$ and $s - \beta^* \in V$. Furthermore, $0 < s - \beta^* < s/m$, contradicting the assumption that s/m is the smallest positive element of V. Hence $|V| = m + 1 < r + 1$, contradicting the assumption that $|V| = r + 1$.

By (i) and (ii), it follows that $\alpha = s/r$. Since $ks/r \in V$ for $k = 0,1,\dots,r$ and $|V| = r + 1$, it must be the case that $V = \{ks/r : k = 0,1,\dots,r\}$. □

**Theorem 2.** If $m \ge 3$ and V is finite, an AAM $F: \mathcal{A}(n,m;s,V) \to \mathcal{A}(m;s,V)$ satisfies IA and ZP if and only if F is *dictatorial*.

*Proof.* Sufficiency: obvious. Necessity: By Lemma 2, F satisfies SLN, and so there exists a function $f : V^n \to V$ such that, for all $A \in \mathcal{A}(n,m;s,V)$ and all $j \in \{1,\dots,m\}$, $F(A)_j = f(A_j)$. Moreover,

(4) $f(X + Y) = f(X) + f(Y)$ for all $X, Y \in V^n$ such that X, Y, and $X + Y \le \mathbf{s}$.

This follows from considering matrices A and B in $\mathcal{A}(n,m;s,V)$ defined (with vertical lines separating columns) by $A = (X \mid Y \mid \mathbf{s} - X - Y \mid \mathbf{0} \mid \dots \mid \mathbf{0})$ and $B = (X + Y \mid \mathbf{s} - X - Y \mid \mathbf{0} \mid \mathbf{0} \mid \dots \mid \mathbf{0})$, and noting that by Z $f(\mathbf{0}) = 0$. Summing the values of f over the columns of A and B then yields $f(X) + f(Y) + f(\mathbf{s} - X - Y) = s = f(X + Y) + f(\mathbf{s} - X - Y)$, and hence (4). Summing the values of f over the columns of $C = (\mathbf{s} \mid \mathbf{0} \mid \dots \mid \mathbf{0})$ shows that

(5) $f(\mathbf{s}) = s$.

By induction, the functional equation (4) can be extended to any finite number of summands X, Y, Z, …, so long as $X, Y, Z, \dots, X + Y + Z + \dots \le \mathbf{s}$. With (5), this yields

(6) $f(\mathbf{s}/r) = s/r$ and, more generally, $f(k\mathbf{s}/r) = ks/r$, $k = 0,\ldots,r$.

Next, associate with the function $f: V^n \rightarrow V$ functions $f^{\langle i\rangle} : V \rightarrow V$, $i = 1,\ldots,n$, defined for all $x \in V$ by $f^{\langle i\rangle}(x) = f(0,\ldots,0,x,0\ldots,0)$, where x occupies the $i^{th}$ position in the preceding vector. Clearly,

(7) $f(x_1,\ldots,x_n) = f^{\langle 1\rangle}(x_1) + f^{\langle 2\rangle}(x_2) + \ldots + f^{\langle n\rangle}(x_n)$ for all $(x_1,\ldots,x_n) \in V^n$,

and by (4),

(8) $f^{\langle i\rangle}(x + y) = f^{\langle i\rangle}(x) + f^{\langle i\rangle}(y)$ for all $x,y \in V$ such that x, y, and $x + y \leq s$.

Recall that $V = \{ks/r : k = 0,1,\ldots,r\}$. By (6) and (7),

(9) $f(\mathbf{s}/r) = f^{\langle 1\rangle}(s/r) + f^{\langle 2\rangle}(s/r) + \ldots + f^{\langle n\rangle}(s/r) = s/r$.

Since the values of f, and hence of the functions $f^{\langle i\rangle}$, are constrained to lie in V, this implies that there exists an individual $d \in \{1,\ldots,n\}$ such that

(10) $f^{\langle d\rangle}(s/r) = s/r$ and $f^{\langle i\rangle}(s/r) = 0$ for all $i \neq d$,

and repeated application of (8) to (10) then yields

(11) $f^{\langle d\rangle}(ks/r) = ks/r$ and $f^{\langle i\rangle}(ks/r) = 0$ for all $i \neq d$, $k = 0,\ldots,r$.

i.e., for all $x \in V$,

(12) $f^{\langle d\rangle}(x) = x$ and $f^{\langle i\rangle}(x) = 0$ for all $i \neq d$.

Hence, for all $A \in \mathscr{A}(n,m;s,V)$, $F(A) = (f(A_1), f(A_2),\ldots, f(A_n)) = (a_{d1}, a_{d2},\ldots,a_{dm})$. □

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