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Power hyper-sums enumerate quasi-monotone functions

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Abstract. We show that the sequences obtained by taking repeated partial sums of regular powers, falling factorials, and rising factorials enumerate certain classes of what we term quasi-monotone functions. In the latter two cases, a q-analogue is also provided.

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1 Introduction

In what follows, $\mathbb{P}$ denotes the set of positive integers, and $[n] = \{1, \dots, n\}$ for all $n \in \mathbb{P}$. If E is any finite set, then $|E|$ denotes the cardinality of E. If $\alpha : \mathbb{P} \to \mathbb{R}$, and $m \in \mathbb{P}$, the m-th degree hyper-sum $S_m^\alpha(n)$ is defined inductively by

$$S_1^\alpha(n) = \alpha(1) + \cdots + \alpha(n), \quad \text{and} \tag{1}$$

$$S_{m+1}^\alpha(n) = S_m^\alpha(n) + \cdots + S_m^\alpha(n) \quad \text{for all } m \in \mathbb{P}. \tag{2}$$

Since, for all $m \in \mathbb{P}$, the ordinary generating functions of the sequences $\{\alpha(n)\}_{n\geq 1}$ and $\{S_m^\alpha(n)\}_{n\geq 1}$ are clearly related by the equation

$$(1-x)^{-m} \sum_{n\geq 1} \alpha(n) x^n = \sum_{n\geq 1} S_m^\alpha(n) x^n, \tag{3}$$

it follows immediately that

$$S_m^\alpha(n) = \sum_{j=1}^{n} \alpha(j) \binom{n-j+m-1}{m-1}. \tag{4}$$

Let $r \in \mathbb{P}$. In what follows, we consider the special cases of the above given by (i) $\alpha(j) = j^r$, (ii) $\alpha(j) = j^{\underline{r}} := j(j-1)\cdots(j-r+1)$, and (iii) $\alpha(j) = j^{\overline{r}} := j(j+1)\cdots(j+r-1)$, denoting S_m^α in these three cases, respectively, by S_m^r, $S_m^{\underline{r}}$, and $S_m^{\overline{r}}$.

2 Quasi-monotone functions

If $r, m, n \in \mathbb{P}$, a function $f : [r+m] \to [n+m]$ is $(r, m, n)-$*quasi-monotone* if

$$f(i) < f(r+1) < f(r+2) < \cdots < f(r+m), \quad \text{for } i = 1, \ldots, r. \tag{5}$$

As shown below, the quantities $S^r_m(n)$ and $S^{\underline{r}}_m(n)$ each enumerate a certain class of $(r, m, n)-$quasi-monotone functions, and thus admit of simpler expressions than those furnished by formula (4). A slight variation on the notion of quasi-monotonicity facilitates a similar simplification of (4) in the case of $S^{\overline{r}}_m(n)$. Our analysis is based on three results from elementary combinatorics, namely, (i) $j^r = |\{f : [r] \to [j]\}|$, (ii) $j^{\underline{r}} = |\{f : [r] \to [j]$ such that f is injective$\}|$, and (iii) $j^{\overline{r}} =$ the number of distributions of balls labeled $1, \ldots, r$ among *contents-ordered* boxes labeled $1, \ldots, j$ [1, pp. 19–23].

THEOREM 2.1 *For all* $r, m, n \in \mathbb{P}$,

$$S^r_m(n) = \sum_{j=1}^{n} j^r \binom{n-j+m-1}{m-1} = \sum_{k=1}^{r} \sigma(r,k) \binom{n+m}{k+m}, \tag{6}$$

where $\sigma(r,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^r$ *is the number of surjective functions* $f : [r] \to [k]$.

Proof. The n-fold sum in (6), which follows from (4), enumerates the set of $(r, m, n)-$quasi-monotone functions $f : [r+m] \to [n+m]$, the j-th term of this sum enumerating those f for which $f(r+1) = j+1$. In the r-fold sum, the k-th term enumerates those f for which $|\text{range}(f)| = k+m$. □

When $m = 1$, (6) reduces to the well-known power sum formula

$$\sum_{j=1}^{n} j^r = \sum_{k=1}^{r} \sigma(r,k) \binom{n+1}{k+1}; \tag{7}$$

see, e.g., [4, 6]. Various q-analogues have been developed for power sums; see, e.g., [2]. We remark that (7) often appears in the variant form,

$$\sum_{j=1}^{n} j^r = \sum_{k=1}^{r} \frac{\left\{ {r \atop k} \right\}}{k+1} (n+1)^{\underline{k+1}}, \tag{8}$$

where $\left\{ {r \atop k} \right\} = \frac{\sigma(r,k)}{k!}$ is the Stirling number of the second kind.

REMARK 2.2 The n-fold sum in (6) may also be reduced to the r-fold sum by a more involved algebraic argument, using the fact that

$$j^r = \sum_{k=1}^{r} \sigma(r,k) \binom{j}{k}, \tag{9}$$

along with the binomial coefficient identity (see [3])

$$\sum_{j=1}^{n} \binom{n-j+m-1}{m-1} \binom{j}{k} = \binom{n+m}{k+m}. \tag{10}$$

We next consider the case when $\alpha(j) = j^{\underline{r}}$.

Theorem 2.3 *For all* $r, m, n \in \mathbb{P}$,

$$S_m^{\underline{r}}(n) = \sum_{j=1}^{n} j^{\underline{r}} \binom{n-j+m-1}{m-1} = \frac{(n+m)^{\underline{r+m}}}{(r+m)^{\underline{m}}}. \tag{11}$$

Proof. The n-fold sum in (11), which follows from (4), enumerates the set of *injective* $(r, m, n)-$quasi-monotone functions $f : [r+m] \to [n+m]$, where, as above, the j-th term in this sum counts those f for which $f(r+1) = j+1$. This sum may be simplified as indicated in (11) by showing that

$$(n+m)^{\underline{r+m}} = (r+m)^{\underline{m}} S_m^{\underline{r}}(n). \tag{12}$$

Let $F = \{f : [r+m] \to [n+m]$ such that f is injective$\}$ and $G = \{g : [r+m] \to [n+m]$ such that g is $(r, m, n) -$ quasi-monotone and injective$\}$. In what follows, we regard members of F as distributions of balls labeled $1, \ldots, r+m$ among boxes labeled $1, \ldots, n+m$, with at most one ball per box, and members of G as distributions of the aforementioned type such that (a) ball $r+i$ occupies a box with a smaller label than that of the box occupied by ball $r+i+1$, for $i = 1, \ldots, m-1$, and (b) each of the balls $1, \ldots, r$ occupies a box with smaller label than that of the box occupied by ball $r+1$. Now consider the map $\psi : F \to G$ defined as follows: Given a distribution $f \in F$, let $\psi(f) = g$, where (i) g has the same set $E \subseteq [n+m]$ of empty boxes as f, (ii) balls $1, \ldots, r$ are placed in the r boxes of $[n+m] - E$ with the smallest labels, and in the same order in which they appear in a left-to-right scan of the distribution f, and (iii) balls $r+1, \ldots, r+m$ are placed in the remaining boxes in their natural order.

$$f = \square_1 \boxed{5}_2 \square_3 \boxed{7}_4 \boxed{2}_5 \boxed{3}_6 \boxed{1}_7 \boxed{4}_8 \boxed{6}_9$$
$$\psi(f) = \square_1 \boxed{2}_2 \square_3 \boxed{3}_4 \boxed{1}_5 \boxed{4}_6 \boxed{5}_7 \boxed{6}_8 \boxed{7}_9$$

Figure 1: An illustration of the mapping ψ when $r = 3$, $m = 4$, and $n = 5$.

Clearly, each distribution in G has $(r+m)^{\underline{m}}$ pre-images in F under ψ. □

Theorem 2.4 *For all* $r, m, n \in \mathbb{P}$,

$$S_m^{\overline{r}}(n) = \sum_{j=1}^{n} j^{\overline{r}} \binom{n-j+m-1}{m-1} = \frac{(r+m+n-1)^{\underline{r+m}}}{(r+m)^{\underline{m}}} = \frac{n^{\overline{r+m}}}{(r+1)^{\overline{m}}}. \tag{13}$$

Proof. The n-fold sum in (13), which follows from (4), enumerates the distributions of balls labeled $1, \ldots, r+m$ among contents-ordered boxes labeled $1, \ldots, n+m$ such that (a) each of the balls $r+1, \ldots, r+m$ is the sole occupant of its box, (b) ball $r+i$ occupies a box with smaller label than that of the box occupied by ball $r+i+1$, for $i = 1, \ldots, m-1$, and (c) each of the balls $1, \ldots, r$ occupies a box with smaller label than that of the box occupied by ball $r+1$. The j-th term in this sum enumerates those distributions in which ball $r+1$ occupies box $j+1$. This sum may be simplified as indicated in (13) by the following argument.

Let Λ denote the set of distributions of balls labeled $1, \ldots, r+m$ among contents-ordered boxes labeled $1, \ldots, n+m$ in which boxes $n+1, \ldots, n+m$ remain empty. By an earlier observation, $|\Lambda| = n^{\overline{r+m}}$. Given $\lambda \in \Lambda$, let x be the *right-most* ball, in the sense that there are no balls in boxes with a greater label than that of the box occupied by x and, if there is more than one ball in the box containing x, then x occupies the right-most position in its box. We first move x to the right by m boxes (so, for example, if x occupied box n in the distribution λ, it would now occupy box $n+m$). We then move the second right-most ball y of λ to the right by $m-1$ boxes (so if y belonged to the same box as x, necessarily preceding x directly in that box, y would now occupy the box directly preceding the one now containing x). Continuing in this fashion, move the m right-most balls of λ such that the i-th right-most ball is moved to the right by $m-i+1$ boxes, for each $i \in [m]$.

Let λ^* denote the configuration (now allowing for any of the $n+m$ boxes to be occupied by balls) which arises after applying the above procedure to λ. It may be verified that the map $\lambda \mapsto \lambda^*$ is a bijection from Λ to Λ^*:= the set of distributions of balls labeled $1, \ldots, r+m$ among contents-ordered boxes labeled $1, \ldots, n+m$ in which the m right-most balls occupy distinct boxes. So also $|\Lambda^*| = n^{\overline{r+m}}$. But here we are interested only in those λ^* for which the m right-most balls are precisely $r+1, r+2, \ldots, r+m$, occurring in that order from left to right. Now the probability that a λ^* randomly chosen from Λ^* has this property is

$$\frac{r!}{(r+m)!} = \frac{1}{(r+m)(r+m-1)\cdots(r+1)} = \frac{1}{(r+1)^{\overline{m}}}.$$

This can be seen by fixing the number of elements that occupy each box, and then assigning the $r+m$ balls to the $r+m$ slots within the boxes to be occupied by at least one ball. It follows that

$$S_m^{\overline{r}}(n) = \frac{|\Lambda^*|}{(r+1)^{\overline{m}}} = \frac{n^{\overline{r+m}}}{(r+1)^{\overline{m}}}.$$

□

3 q-analogues

In this section, we consider q-analogues of the last two results. Given an indeterminate q, let $[j]_q = 1 + q + \cdots + q^{j-1}$ if $j \in \mathbb{P}$, with $[0]_q = 0$. Let $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ if $n \in \mathbb{P}$, with $[0]_q! = 1$, denote the *q-factorial* and let $\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$ denote the *q-binomial coefficient*, where $0 \leq m \leq n$. Given positive integers n and m, let $[n]_q^{\underline{m}} = [n]_q[n-1]_q \cdots [n-m+1]_q$ and $[n]_q^{\overline{m}} = [n]_q[n+1]_q \cdots [n+m-1]_q$, with $[n]_q^{\underline{0}} = [n]_q^{\overline{0}} = 1$.

Recall that the *number of inversions* in a word $w = w_1w_2 \cdots w_n$ over some alphabet of non-negative integers is the cardinality of the set $\{(i,j) : 1 \leq i < j \leq n \text{ with } w_i > w_j\}$, which is often denoted by $\text{inv}(w)$. We'll make use of the fact that the q-binomial coefficient $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is the generating function for the statistic that records the number of inversions in binary words of length n containing exactly m 1's (see [5, Prop. 1.3.17]).

We have the following q-generalization of the second identity in Theorem 2.3 above.

THEOREM 3.1 *For all* $r, m, n \in \mathbb{P}$,

$$\sum_{j=1}^{n} q^{m(j-r)} [j]_q^{\underline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q = \frac{[n+m]_q^{\underline{r+m}}}{[r+m]_q^{\underline{m}}}. \tag{14}$$

Proof. Note that the lower index of the sum on the left-hand side of (14) may be started from $j = r$ since $[j]_q^{\underline{r}} = 0$ if $j < r$. Let us assume further $n \geq r$, for otherwise both sides of (14) are zero. We provide a combinatorial proof of (14), rewritten in the form

$$[n+m]_q^{\underline{r+m}} = [r+m]_q^{\underline{m}} \sum_{j=r}^{n} q^{m(j-r)} [j]_q^{\underline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q. \tag{15}$$

First we extend $\mathbb{P}$ by adding the *infinity symbol* ∞, it being understood that $n < \infty$ for all $n \in \mathbb{P}$. Let $\mathcal{A}$ denote the set of words of length $n+m$ containing exactly $n-r$ infinity symbols and each member of $[r+m]$ once. Then $[n+m]_q^{\underline{r+m}}$ counts the members of $\mathcal{A}$ according to the number of inversions. To see this, first note that the $[n-r+1]_q$ factor accounts for the placement of the element $r+m$ amongst the $n-r$ infinity symbols, written in a row, since anywhere from 0 to $n-r$ inversions are created. Then $[n-r+2]_q$ accounts for the placement of the element $r+m-1$ once the position for $r+m$ has been determined, and, in general, $[n+m-i+1]_q$ accounts for the placement of the element i, $1 \leq i \leq r+m$, once the positions for all letters greater than i have been determined.

To show that the right-hand side of (15) also counts the members of $\mathcal{A}$ according to the number of inversions, we first describe a procedure for generating the members of $\mathcal{A}$. We start with a sequence ρ of length $n+m$ consisting of $n-r$ infinity symbols, $m-1$ zeros, and one occurrence of each element of $[r+1]$, where all the elements of $[r+1]$ occur to the left of all the zeros, the element $r+1$ occurs to the right of all the elements of $[r]$, and $r+1$ is in the $(j+1)$-st position for some $j \in [r, n] = \{r, r+1, \ldots, n\}$. We transform ρ into another sequence $\delta \in \mathcal{A}$ as follows: (i) Replace each letter in $[r+1]$ occurring in ρ with a zero, (ii) Replace m of the $r+m$ zeros in the word resulting from the first step with elements of $[r+1, r+m]$ so that each letter occurs once, and (iii) Replace the r remaining zeros with the elements of $[r]$ so that they occur in the same order in which they appear in a left-to-right scan of the word ρ. From this, we see that there are $(r+m)^{\underline{m}} \cdot j^{\underline{r}} \binom{n-j+m-1}{m-1}$ sequences $\delta \in \mathcal{A}$ in which the $(r+1)$-st left-most letter of δ that is not an infinity symbol occupies the $(j+1)$-st position, $r \leq j \leq n$.

Then the distribution of the inv statistic on the set consisting of such sequences $\delta \in \mathcal{A}$ is given by

$$[r+m]_q^{\underline{m}} \cdot q^{m(j-r)} [j]_q^{\underline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q,$$

whence (15) follows from summing over j. To see this, first note that the factor $[r+m]_q^{\underline{m}} = [r+m]_q [r+m-1]_q \cdots [r+1]_q$ accounts for both the choice of the positions for the members of $[r+1, r+m]$ relative to the positions of all the members of $[r+m]$ within δ and the inversions between two letters which aren't an ∞ in which at least one of the letters belongs to $[r+1, r+m]$. The factor $[j]_q^{\underline{r}} = [j]_q [j-1]_q \cdots [j-r+1]_q$ accounts for the choice of the positions for the left-most r members of $[r+m]$ within δ, the inversions between these members and infinity symbols, and inversions between two members of $[r]$ (note that the relative order of the members of $[r]$ did not change in the transformation from ρ to δ described above). The factor $q^{m(j-r)}$ accounts for the inversions between the left-most $j-r$ ∞'s and the right-most

m members of $[r+m]$ within δ. Finally, $\left[{n-j+m-1 \atop m-1}\right]_q$ accounts for the choice of the positions for the right-most $(n-r)-(j-r)=n-j$ ∞'s amongst the final $n+m-j-1$ positions of δ along with inversions involving these ∞'s. □

One may also generalize the second identity in Theorem 2.4 above.

THEOREM 3.2 *For all* $r, m, n \in \mathbb{P}$,

$$\sum_{j=1}^{n} q^{m(j-1)}[j]_q^{\overline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q = \frac{[n]_q^{\overline{r+m}}}{[r+1]_q^{\overline{m}}}. \tag{16}$$

Proof. A proof comparable to the one given for Theorem 3.1 above, the details of which we leave to the interested reader, may be given for (16), upon multiplying both sides by $[r+1]_q^{\overline{m}}$. Here, one would count sequences of length $r+m+n-1$ containing $n-1$ infinity symbols and each element of $[r+m]$ once according to the number of inversions. Note that in this case, if there are $j-1$ infinity symbols occurring to the left of the $(r+1)$-st left-most element of $[r+m]$ within such a sequence, then there are $m(j-1)$ inversions between these symbols and the m right-most elements of $[r+m]$ occurring in the sequence, whence the factor of $q^{m(j-1)}$. □

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