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Covering algebras and q-binomial generating functions

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Abstract

The theory of reduced incidence algebras of binomial posets furnishes a unified treatment of several types of generating functions that arise in enumerative combinatorics. Using this theory as a tool, we study 'reduced covering algebras' of binomial lattices and show that they are isomorphic to various algebras of q-binomial generating functions for certain modular binomial lattices.

1. Introduction

The polynomial identity

$$\binom{X}{r}\binom{X}{s}=\sum_{n}\binom{n}{r}\binom{r}{n-s}\binom{X}{n} \tag{1.1}$$

may be proved laboriously by finite difference methods or quickly by Vandermonde's theorem [7, p. 15]. In our view, however, the most satisfying proof of (1.1) is combinatorial, based on the observation that the product $\binom{n}{r}\binom{r}{n-s}$ enumerates the ordered pairs (A, B) of subsets of an n-set T for which $|A|=r$, $|B|=s$, and $A\cup B=T$. Thus, for every $X\in\mathbb{N}$ each side of (1.1) enumerates the ordered pairs (A, B) of subsets of an X-set S with $|A|=r$ and $|B|=s$, the right-hand side according to the cardinality n of $A\cup B$.

As we shall show, the identity (1.1) underlies the theory of formal binomial series, a type of generating function that has not, so far as we know, played much of a role in combinatorics. In fact, the connection with covers noted above suggests a theory of somewhat broader scope, the theory of 'covering algebras' of locally finite lattices,

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based on the product

$$f \diamond g(x,y) = \sum_{\substack{w,z\in[x,y]\\ w\vee z=y}} f(x,w)g(x,z), \quad \text{for } x \leqslant y. \tag{1.2}$$

We use the theory of reduced incidence algebras of binomial posets [3, Section 8; 9, Section 3.15], as a tool to study the 'reduced covering algebra' of a binomial lattice, constructing an isomorphic algebra of arithmetic functions that we call the *Newton algebra*. We conclude by showing that for modular binomial lattices in which intervals of length 2 have more than one atom, the Newton algebra may be construed as an algebra of formal q-binomial series.

In what follows we make extensive use of the theory of incidence algebras, a detailed account of which may be found in [8] or in [9]. It suffices here to recall that if P is a locally finite poset and $\mathrm{Int}(P)$ is the set of intervals of P, then the *incidence algebra of* P *over* $\mathbf{C}$ is the $\mathbf{C}$-algebra of all functions $f:\mathrm{Int}(P)\to\mathbf{C}$, with the usual $\mathbf{C}$-vector space structure and the multiplication $*$ defined by

$$f * g(x,y) = \sum_{z\in[x,y]} f(x,z)g(z,y). \tag{1.3}$$

The multiplicative identity of this algebra is the function δ, defined by $\delta(x,y)=\delta_{x,y}$, and f has a multiplicative inverse if and only if for all $x\in P$ the condition $f(x,x)\neq 0$ holds. In particular, the *zeta function* ζ, defined by $\zeta(x,y)=1$ for all $x\leqslant y$, is invertible. Its inverse, denoted by μ, is called the *Möbius function*, and embodies the principle of inclusion and exclusion in P. The *Möbius inversion principle* is simply the equivalence

$$g=f*\zeta \quad \text{if and only if } f=g*\mu. \tag{1.4}$$

What has been termed [3, p. 279] the *Schur product* $\square$, defined by $f\square g(x,y)=f(x,y)g(x,y)$, has heretofore played only a modest auxiliary role in the above theory (see, e.g. [3, Theorem 4.1]). In the present paper, however, the Schur product plays a central role, albeit in isomorphic disguise as the product $\diamond$ defined by (1.2).

In what follows we consider a variety of algebras of complex-valued functions. The sums (denoted $f+g$ or $F+G$) and scalar multiples (denoted $c.f$ or $c.F$) comprising the $\mathbf{C}$-vector space operations of such algebras are always taken pointwise. Quotients (denoted f/g or F/G) are also always taken pointwise.

2. Covering algebras of locally finite lattices

Given a locally finite lattice L and functions $f,g:\mathrm{Int}(L)\to\mathbf{C}$, define the product $f\diamond g$ by

$$f \diamond g(x,y) = \sum_{\substack{w,z\in[x,y]\\ w\vee z=y}} f(x,w)g(x,z). \tag{2.1}$$

The algebraic structure $(\mathbf{C}^{\mathrm{Int}(L)}, +, \diamond, .)$ is simply an isomorphic variant of the Schur algebra of L, a result that generalizes an old theorem of von Sterneck [10, p. 265].

Theorem 2.1. *The mapping* $f \mapsto f * \zeta$ *is an isomorphism from* $(\mathbf{C}^{\mathrm{Int}(L)}, +, \diamond, .)$ *to* $(\mathbf{C}^{\mathrm{Int}(L)}, +, \square, .)$.

Proof. Since this mapping is clearly a vector space isomorphism, we need only show that

$$(f \diamond g) * \zeta = (f * \zeta) \square (g * \zeta). \tag{2.2}$$

But for every interval $[x, y]$ in L,

$$\begin{aligned}[(f * \zeta) \square (g * \zeta)](x, y) &= \sum_{u \in [x, y]} f(x, u) \sum_{w \in [x, y]} g(x, w) \\ &= \sum_{z \in [x, y]} \sum_{\substack{u, w \in [x, z] \\ u \vee w = z}} f(x, u) g(x, w) \\ &= \sum_{z \in [x, y]} f \diamond g(x, z) \\ &= [(f \diamond g) * \zeta](x, y),\end{aligned}$$

which establishes (2.2). □

It follows from Theorem 2.1 that $(\mathbf{C}^{\mathrm{Int}(L)}, +, \diamond, .)$ is a commutative C-algebra with multiplicative identity δ. We call this algebra the *covering algebra of L*. Just as the kth powers $\zeta^{*k}(x, y)$ and $(\zeta - \delta)^{*k}(x, y)$ in the incidence algebra multiplication enumerate, respectively, the multichains and chains of length k from x to y in a locally finite poset [9, p. 115], the kth powers $\zeta^{\diamond k}$ and $(\zeta - \delta)^{\diamond k}$ enumerate an important class of sequences in a locally finite lattice L. If $[x, y] \in \mathrm{Int}(L)$, call a sequence $z_1, z_2, \ldots, z_k$ in $[x, y]$ (resp. $(x, y]$) an *ordered k-cover of y in* $[x, y]$*(resp.* $(x, y]$) if $z_1 \vee z_2 \vee \cdots \vee z_k = y$; denote the number of such sequences by $c_k(x, y)$ (resp. $c'_k(x, y)$). By an easy inductive argument,

$$c_k(x, y) = \zeta^{\diamond k}(x, y) \tag{2.3}$$

and

$$c'_k(x, y) = (\zeta - \delta)^{\diamond k}(x, y), \tag{2.4}$$

which explains our terminology 'covering algebra.' Since by Theorem 2.1,

$$\zeta^{\diamond k} = (\zeta^{\diamond k}) * \zeta * \mu = (\zeta * \zeta)^{\square k} * \mu, \tag{2.5}$$

eq. (2.3) yields

$$c_k(x, y) = \sum_{z \in [x, y]} |[x, z]|^k \mu(z, y). \tag{2.6}$$

Similarly,

$$c'_k(x,y)=\sum_{z\in[x,y]}|(x,z]|^k\mu(z,y). \tag{2.7}$$

Of course, one can also derive (2.6) and (2.7) directly by Möbius inversion without reference to the product $\diamond$.

Theorem 2.1 also implies that $f\in\mathbf{C}^{\mathrm{Int}(L)}$ is invertible with respect to $\diamond$ if and only if $f*\zeta$ is invertible with respect to $\square$, which is to say that for all intervals $[x,y]$ in L,

$$f*\zeta(x,y)=\sum_{z\in[x,y]}f(x,z)\neq 0. \tag{2.8}$$

In particular, ζ is invertible with respect to $\diamond$. We denote its inverse by ν and call it the *Newton function.* Using the isomorphism $f\mapsto f*\zeta$ from the covering algebra to the Schur algebra, one shows easily that

$$\nu=\left(\frac{\zeta}{\zeta*\zeta}\right)*\mu, \tag{2.9}$$

i.e.

$$\nu(x,y)=\sum_{z\in[x,y]}|[x,z]|^{-1}\mu(z,y). \tag{2.10}$$

We call the equivalence

$$g=f\diamond\zeta \quad \text{if and only if } f=g\diamond\nu \tag{2.11}$$

the *Newton inversion principle.* To render (2.11) more concretely, let us say of elements $z,w\in[x,y]$ that z is a *supplement of* w *in* $[x,y]$ if $w\vee z=y$, and denote by $S([x,y],w)$ the number of supplements of w in $[x,y]$. The Newton inversion principle then asserts the equivalence

$$g(x,y)=\sum_{w\in[x,y]}S([x,y],w)f(x,w) \quad \text{for all } x\leqslant y$$

if and only if

$$f(x,y)=\sum_{\substack{w,z\in[x,y]\\ w\vee z=y}}g(x,w)\nu(x,z) \quad \text{for all } x\leqslant y. \tag{2.12}$$

3. Reduced covering algebras of binomial lattices

A poset P is called a *binomial poset* if it satisfies the following three conditions:

(i) P is locally finite with $\hat{0}$ and contains an infinite chain.

(ii) In every interval $[x,y]$ of P all maximal chains have the same length. If this common length is n, write $l(x,y)=n$ and call $[x,y]$ an *n-interval.*

(iii) For all $n\in\mathbf{N}$ any two n-intervals contain the same number, $B(n)$, of maximal chains.

The theory of binomial posets originated in the work of Doubilet et al. [3], and a detailed exposition appears in [9]. For the sake of completeness, we outline below some basic results.

As a consequence of condition (ii) above, each n-interval $[x,y]$ of P has a rank function $\rho:[x,y]\to\{0,1,\dots,n\}$, where $\rho(z)=l(x,z)$ for all $z\in[x,y]$. Clearly, $z\in[x,y]\cap[x,y']$ has the same rank in $[x,y]$ as in $[x,y']$. Moreover, if $z\in[x,y]$ and $u\in[z,y]$, the rank of u in $[z,y]$ equals its rank in $[x,y]$, diminished by the rank of z in $[x,y]$.

As a consequence of conditions (ii) and (iii), any two n-intervals contain the same number $\left[{n \atop k}\right]$ of elements of rank k, and

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{B(n)}{B(k)B(n-k)}, \tag{3.1}$$

whence

$$B(n)=A(n)A(n-1)\cdots A(1), \tag{3.2}$$

with $A(r)=\left[{r \atop 1}\right]$, the number of atoms in an n-interval. (The numbers $\left[{n \atop k}\right]$ are the *incidence coefficients of P*.) Any two n-intervals in a binomial poset have the same cardinality G_n where

$$G_n=\begin{bmatrix} n \\ 0 \end{bmatrix}+\begin{bmatrix} n \\ 1 \end{bmatrix}+\cdots+\begin{bmatrix} n \\ n \end{bmatrix}. \tag{3.3}$$

For a binomial poset P, the set

$$\mathscr{R}(P):=\{f\in\mathbf{C}^{\mathrm{Int}(P)}\,|\,f(x,y)=f(x',y') \text{ whenever } l(x,y)=l(x',y')\} \tag{3.4}$$

is a subalgebra of the incidence algebra of P, called the *reduced incidence algebra of P*. The mapping $\psi:\mathscr{R}(P)\to\mathbf{C}^{\mathbf{N}}$ defined by

$$\psi f(n)=F(n):=f(x,y) \quad \text{for an } n\text{-interval } [x,y] \tag{3.5}$$

is an isomorphism from the algebra $(\mathscr{R}(P),+,*,.)$ to the algebra of arithmetic functions $(\mathbf{C}^{\mathbf{N}},+,\star,.)$, with product $\star$ defined by

$$F\star G(n)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}F(k)G(n-k), \tag{3.6}$$

i.e. to the algebra of generating functions of the form

$$\sum_{n\geqslant 0}F(n)\frac{X^n}{B(n)}. \tag{3.7}$$

Clearly, δ and ζ belong to $\mathscr{R}(P)$. Moreover, if $f, g\in\mathbf{C}^{\mathrm{Int}(P)}$ with $f*g=\delta$ and $f\in\mathscr{R}(P)$, it follows that $g\in\mathscr{R}(P)$ [9, p. 144]. Hence μ belongs to $\mathscr{R}(P)$. Let $D=\psi(\delta)$, $Z=\psi(\zeta)$, and $M=\psi(\mu)$. Under the isomorphism ψ, the Möbius inversion principle (1.4) becomes the equivalence, $G=F\star Z$ if and only if $F=G\star M$. That is,

$$G(n)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}F(k) \quad \text{for all } n\geqslant 0$$

if and only if

$$F(n)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}G(k)M(n-k) \quad \text{for all } n\geqslant 0. \tag{3.8}$$

It is trivial to show, but nevertheless worth noting, that $\mathscr{R}(P)$ is also closed under the Schur product $\square$ and that ψ is also an isomorphism $(\mathscr{R}(P), +, \square, .)$ to $(\mathbf{C}^{\mathbf{N}}, +, \blacksquare, .)$, where $F\blacksquare G(n):=F(n)G(n)$.

Suppose now that L is a *binomial lattice*, i.e. a lattice that is a binomial poset. By Theorem 2.1, for all $f, g\in\mathbf{C}^{\mathrm{Int}(L)}$,

$$f\diamond g=(f\diamond g)*\zeta*\mu=[(f*\zeta)\square(g*\zeta)]*\mu. \tag{3.9}$$

Thus, if $f, g\in\mathscr{R}(L)$, then $f\diamond g\in\mathscr{R}(L)$, since ζ and μ belong to $\mathscr{R}(L)$ which is closed under $*$ and $\square$. We call $(\mathscr{R}(L), +, \diamond, .)$ the *reduced covering algebra of* L. The next theorem shows that if $f\in\mathscr{R}(L)$ is invertible in $\mathbf{C}^{\mathrm{Int}(L)}$ with respect to $\diamond$, then its inverse belongs to $\mathscr{R}(L)$.

Theorem 3.1. *If $f\diamond g=\delta$ and $f\in\mathscr{R}(L)$, then $g\in\mathscr{R}(L)$.*

Proof. *If $f\diamond g=\delta$*, then by (2.2) we have $(f*\zeta)\square(g*\zeta)=(f\diamond g)*\zeta=\delta*\zeta=\zeta$. Since $f*\zeta$ is invertible with respect to $\square$ by Theorem 2.1, it follows that

$$g=\left(\frac{\zeta}{f*\zeta}\right)*\mu \tag{3.10}$$

and so g is clearly in $\mathscr{R}(L)$. $\square$

We now construct an algebra of arithmetic functions isomorphic to the reduced covering algebra of L. Given an interval $[x,y]$ in L and $r,s\in\mathbf{N}$, define

$$\left\{ \begin{matrix} [x,y] \\ r,s \end{matrix} \right\}=|\{(w,z)\,|\,w,z\in[x,y],\ w\vee z=y,\ \rho(w)=r, \text{ and } \rho(z)=s\}|. \tag{3.11}$$

Theorem 3.2. *If $[x,y]$ and $[x',y']$ are intervals in the binomial lattice L and $l(x,y)=l(x',y')$, then*

$$\left\{ \begin{matrix} [x,y] \\ r,s \end{matrix} \right\}=\left\{ \begin{matrix} [x',y'] \\ r,s \end{matrix} \right\}.$$

Proof. Define $f, g \in \mathbf{C}^{\mathrm{Int}(L)}$ by

$$f(x,y)=\begin{cases}1, & \text{if } l(x,y)=r,\\ 0, & \text{otherwise;}\end{cases} \tag{3.12}$$

$$g(x,y)=\begin{cases}1, & \text{if } l(x,y)=s,\\ 0, & \text{otherwise.}\end{cases}$$

The functions f and g are in $\mathscr{R}(L)$, so the product $f \diamond g$ is in $\mathscr{R}(L)$ as well. Thus $f \diamond g(x,y) = f \diamond g(x',y')$. But by (2.1), (3.11), and (3.12) it follows that

$$f \diamond g(x,y)=\left\{\begin{matrix}[x,y]\\ r,s\end{matrix}\right\} \text{ and } f \diamond g(x',y')=\left\{\begin{matrix}[x',y']\\ r,s\end{matrix}\right\}. \qquad \square$$

As a consequence of Theorem 3.2, the *covering coefficients of the binomial lattice L*,

$$\left\{\begin{matrix}n\\ r,s\end{matrix}\right\} := |\{(w,z) \mid w,z \in [x,y],\ l(x,y)=n,$$
$$w \vee z = y,\ \rho(w)=r, \text{ and } \rho(z)=s\}| \tag{3.13}$$

are well-defined for all $n, r, s \in \mathbf{N}$.

Consider now the mapping $\psi : \mathscr{R}(L) \to \mathbf{C}^{\mathbf{N}}$ defined by (3.5), with $\psi(f)=F$ and $\psi(g)=G$. For any n-interval $[x,y]$,

$$\begin{aligned}\psi(f \diamond g)(n) &= f \diamond g(x,y)\\ &= \sum_{\substack{w,z\in[x,y]\\ w\vee z=y}} f(x,w)g(x,z)\\ &= \sum_{0\leqslant r,s\leqslant n} F(r)G(s) \sum_{\substack{w,z\in[x,y]\\ w\vee z=y\\ \rho(w)=r,\ \rho(z)=s}} 1\\ &= \sum_{0\leqslant r,s\leqslant n} \left\{\begin{matrix}n\\ r,s\end{matrix}\right\} F(r)G(s).\end{aligned} \tag{3.14}$$

The following theorem is clearly a consequence of (3.14):

Theorem 3.3. *For $F, G \in \mathbf{C}^{\mathbf{N}}$, define $F \blacklozenge G$ by*

$$F \blacklozenge G(n) = \sum_{0\leqslant r,s\leqslant n} \left\{\begin{matrix}n\\ r,s\end{matrix}\right\} F(r)G(s), \tag{3.15}$$

where the $\left\{\begin{smallmatrix}n\\ r,s\end{smallmatrix}\right\}$ are the covering coefficients of the binomial lattice L. Then the mapping ψ defined by (3.5) *is an isomorphism from the reduced covering algebra of L to the algebraic structure* $(\mathbf{C}^{\mathbf{N}}, +, \blacklozenge, .)$.

Theorem 3.3 implies that the structure $(\mathbf{C}^{\mathbf{N}}, +, \blacklozenge, .)$ is a commutative $\mathbf{C}$-algebra with identity $D=\psi(\delta)$. We call this algebra the *Newton algebra of the binomial lattice*

L. Note that by Theorems 2.1 and 3.3 and the remarks following (3.5) and (3.8), the mapping

$$F \mapsto \psi(\psi^{-1}(F) * \zeta) = F \star Z \tag{3.16}$$

is an isomorphism from the Newton algebra to ($\mathbf{C}^{\mathbf{N}}$, $+$, $\blacksquare$, .). In Section 5 we show that for certain modular binomial lattices the Newton algebra may be viewed as an algebra of q-binomial generating functions.

The functions c_k and c_k' of Section 2 are clearly members of $\mathscr{R}(L)$. Writing $\psi(c_k) = C_k$ and $\psi(c_k') = C_k'$ we have from (2.3) and (2.5) that $C_k = Z^{\blacklozenge k} = (Z \star Z)^{\blacksquare k} \star M$. That is, by (3.6),

$$C_k(n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} G_j^k M(n-j), \tag{3.17}$$

where $G_j = [{j \atop 0}] + [{j \atop 1}] + \cdots + [{j \atop j}]$. Similarly,

$$C_k'(n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (G_j - 1)^k M(n-j). \tag{3.18}$$

Here $C_k(n)$ is the number of ordered k-covers of y in an arbitrary n-interval $[x, y]$ in the binomial lattice L, and $C_k'(n)$ is the number of such covers with no member equal to x.

By Theorem 3.1 the Newton function ν is also a member of $\mathscr{R}(L)$. With $N = \psi(\nu)$, it follows from (2.9) that $N = (Z/(Z \star Z)) \star M$. That is, by (3.6)

$$N(n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} G_j^{-1} M(n-j). \tag{3.19}$$

The Newton inversion principle (2.11) for the covering algebra reduces under the mapping ψ to the equivalence

$$G = F \blacklozenge Z \quad \text{if and only if } F = G \blacklozenge N. \tag{3.20}$$

To render (3.20) more concretely, define

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = \sum_{s=0}^{n} \begin{Bmatrix} n \\ r, s \end{Bmatrix} \tag{3.21}$$

so that $\{{n \atop r}\}$ is the number of ordered 2-covers of y in an n-interval $[x, y]$, the first member of which has rank r. By (3.15) and (3.21), equation (3.20) asserts the equivalence

$$G(n) = \sum_{r=0}^{n} \begin{Bmatrix} n \\ r \end{Bmatrix} F(r) \quad \text{for all } n \geqslant 0$$

if and only if

$$F(n) = \sum_{0 \leqslant r, s \leqslant n} \begin{Bmatrix} n \\ r, s \end{Bmatrix} G(r) N(s) \quad \text{for all } n \geqslant 0. \tag{3.22}$$

We conclude this section by deriving a formula for the covering coefficients of a binomial lattice. Note first that for fixed $r, s \in \mathbf{N}$,

$$\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}\begin{Bmatrix} k \\ r, s \end{Bmatrix} \quad \text{for all } n \geqslant 0 \tag{3.23}$$

since each side of (3.23) enumerates, for an arbitrary n-interval $[x, y]$, the set $\{(w, z) \mid w, z \in [x, y],\ \rho(w) = r, \text{ and } \rho(z) = s\}$, the right-hand side according to the rank k of $w \vee z$. By (3.8), equation (3.23) is equivalent to

$$\begin{Bmatrix} n \\ r, s \end{Bmatrix} = \sum_{k=0}^{n} M(n-k)\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ r \end{bmatrix}\begin{bmatrix} k \\ s \end{bmatrix} \quad \text{for all } n \geqslant 0. \tag{3.24}$$

As we shall see in Section 4, equations (3.23) and (3.24) are a generalization to binomial lattices of a classical 'inverse pair' of binomial coefficient identities [7, p. 15].

4. Modular binomial lattices

We now consider the special case of the preceding theory in which the binomial lattice L is modular, so that the rank function ρ of an interval $[x, y]$ in L satisfies [9, p. 104]

$$\rho(w \vee z) = \rho(w) + \rho(z) - \rho(w \wedge z) \quad \text{for all } w, z \in [x, y]. \tag{4.1}$$

Let $[x, y]$ be an n-interval in the modular binomial lattice L, with $A(n) = \left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]$ the number of atoms – i.e., elements of rank 1 – in $[x, y]$. By (4.1) the join of two distinct atoms of $[x, y]$ has rank 2. Thus, as Doubilet et al. [3, p. 310] note, the mapping $\{w, z\} \mapsto w \vee z$ is an

$$\binom{A(2)}{2}\text{-to-one surjection}$$

from the set of unordered pairs of distinct atoms of $[x, y]$ to the set of elements of rank 2 in $[x, y]$. Hence,

$$\binom{A(n)}{2} = \binom{A(2)}{2}\begin{bmatrix} n \\ 2 \end{bmatrix}, \tag{4.2}$$

which, with (3.1) and (3.2), implies that

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1\right) + 1. \tag{4.3}$$

We call the important number

$$q := \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \tag{4.4}$$

the *characteristic* of the modular binomial lattice L. The key combinatorial parameters of such a lattice are all functions of q. For example, from (4.3) and (4.4) an inductive argument shows that for a modular binomial lattice of characteristic q (henceforth, *q-lattice*), the number of atoms in any n-interval is given by

$$A(n)=\begin{bmatrix} n \\ 1 \end{bmatrix}=1+q+\cdots+q^{n-1}. \tag{4.5}$$

This result and (3.2) imply that the number of maximal chains $B(n)$ in an n-interval is given by

$$B(n)=(1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1+q)(1). \tag{4.6}$$

The following notation proves useful in the sequel:

$$n_q:=1+q+\cdots+q^{n-1}, \tag{4.7}$$

$$n!_q:=n_q(n-1)_q\ldots 1_q, \tag{4.8}$$

$$n_q^{\underline{k}}:=n_q(n_q-1_q)\cdots(n_q-(k-1)_q). \tag{4.9}$$

Note that

$$n_q^{\underline{n}}=q^{\binom{n}{2}}n!_q.$$

With this notation (3.1) and (4.6) imply that the incidence coefficient $[\begin{smallmatrix} n \\ k \end{smallmatrix}]$ of a q-lattice is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{n!_q}{k!_q(n-k)!_q} \qquad \left(=\frac{n_q^{\underline{k}}}{k_q^{\underline{k}}} \quad \text{if } q>0\right). \tag{4.10}$$

From (4.10) it is easy to establish the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix}=\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}+q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}, \tag{4.11}$$

for which there is also a combinatorial proof [2, p. 16].

Doubilet et al. [3, pp. 310, 311] were the first to note the importance of the class of modular binomial lattices. Among other results, the theory of such lattices provides, at long last, a satisfying account of the well-known parallels between binomial and Gaussian binomial coefficients: the lattice of finite subsets of an infinite set and the lattice of finite-dimensional subspaces of an infinite-dimensional vector space over a finite field of cardinality p^d are modular binomial lattices with respective characteristics $q=1$ and $q=p^d$. The incidence coefficients of these lattices are, respectively, binomial and Gaussian binomial coefficients.

In addition, the set of nonnegative integers $\mathbf{N}$ with the usual ordering is a modular binomial lattice with $q=0$. In this case the incidence coefficients $[\begin{smallmatrix} n \\ k \end{smallmatrix}]$ are equal to 1 if $0\leqslant k\leqslant n$ and equal to 0 otherwise. The results in this section all hold for the case $q=0$ as long as we define $0^0=1$.

Indeed, a slightly more general result of Doubilet et al. [3, Theorem 8.2] shows that every modular binomial lattice is isomorphic to $(\mathbf{N}, \leqslant)$, to the lattice of finite subsets of an infinite set, or to the lattice of finite-dimensional subspaces of an infinite dimensional vector space over a finite field. Consequently, one can establish general results for modular binomial lattices by offering separate proofs for these three special cases. (When a polynomial identity in q is at issue, one needs merely to argue for the cases $q = p^d$ since that provides sufficiently many instantiations of the identity.) If only for aesthetic reasons, however, it is satisfying to avoid case-by-case proofs. As the following three theorems show, one can furnish natural, comprehensive proofs of combinatorial results in the theory of modular binomial lattices with not much more effort than that required to establish such results in the special case of vector spaces.

We first derive an explicit formula for the covering coefficients of a q-lattice. This requires a preliminary result of independent interest. If $[x, y]$ is an interval in a locally finite lattice and $w \in [x, y]$, we call $z \in [x, y]$ a *complement of w in* $[x, y]$ if $w \vee z = y$ and $w \wedge z = x$.

Theorem 4.1. *Let* $[x, y]$ *be an n-interval in the q-lattice L. If* $z \in [x, y]$ *has rank k, then z has* $q^{k(n-k)}$ *complements in* $[x, y]$.

Proof. Let C denote the set of all complements of z in $[x, y]$. By (4.1) every element of C has rank $n-k$. For each $w \in C$, let D_w be the set of all maximal chains $x = a_0 < a_1 < \cdots < a_{n-k} = w$ in $[x, w]$, and let $D = \bigcup_{w \in C} D_w$. Clearly

$$|D| = B(n-k)|C| = (n-k)!_q |C|. \tag{4.12}$$

Note, however, that a chain $a_0 < a_1 < \cdots < a_{n-k}$ in $[x, y]$ belongs to D if and only if (i) $\rho(a_i) = i$, for $i = 0, 1, \ldots, n-k$ and (ii) $a_i \wedge z = x$, for $i = 0, 1, \ldots, n-k$. It is clear that condition (i) holds for chains in D, and condition (ii) holds for such chains since $x \leqslant a_i \wedge z \leqslant a_{n-k} \wedge z = x$, for $i = 0, 1, \ldots, n-k$. Conversely, a chain satisfying condition (i) is maximal in $[x, a_{n-k}]$, and by (4.1) and condition (ii) it follows that $\rho(a_{n-k} \vee z) = n$; so $a_{n-k} \vee z = y$ and $a_{n-k} \in C$. We enumerate D by counting chains satisying conditions (i) and (ii).

Such a chain must have $a_0 = x$. Supposing that $a_0, a_1, \ldots, a_i$ satisfy conditions (i) and (ii), how many choices are there for a_{i+1}? Clearly, a_{i+1} must be an atom of the $(n-i)$-interval $[a_i, y]$, and there are $(n-i)_q$ such atoms a, some of which, however, fail to satisfy $a \wedge z = x$. But for any atom a of $[a_i, y]$, equation (4.1) yields

$$\begin{aligned} \rho(a \wedge z) &= \rho(a) + \rho(z) - \rho(a \vee z) \\ &\leqslant \rho(a) + \rho(z) - \rho(a_i \vee z) \\ &= \rho(a) - \rho(a_i) + \rho(x) \\ &= 1. \end{aligned} \tag{4.13}$$

So the number of choices for a_{i+1} is equal to $(n-i)_q - |A|$, where $A := \{a \mid a$ is an atom of $[a_i, y]$ and $\rho(a \wedge z) = 1\}$.

To determine $|A|$, let B denote the set of atoms in $[x, z]$. If a and a' are distinct elements of A, then $(a \wedge z) \wedge (a' \wedge z) = (a \wedge a') \wedge z = a_i \wedge z = x$, and so $a \wedge z \neq a' \wedge z$. Thus, the mapping $a \mapsto a \wedge z$ from A into B is injective.

If $b \in B$, then $x \leqslant a_i \wedge b \leqslant a_i \wedge z = x$, and so by (4.1) we have $\rho(a_i \vee b) = \rho(a_i) + \rho(b) - \rho(x) = i+1$. Hence, $\rho((a_i \vee b) \wedge z) = \rho(a_i \vee b) + \rho(z) - \rho((a_i \vee b) \vee z) = (i+1) + k - \rho(a_i \vee z) = (i+1) + k - (i+k) = 1$. Thus, the mapping $b \mapsto a_i \vee b$ takes B into A. Moreover, if b and b' are distinct elements of B, then $\rho((a_i \vee b) \vee (a_i \vee b')) = \rho(a_i \vee (b \vee b')) = \rho(a_i) + \rho(b \vee b') - \rho(a_i \wedge (b \vee b')) = i + 2 - \rho(x) = i+2$ (since $x \leqslant a_i \wedge (b \vee b') \leqslant a_i \wedge z = x$) and so $a_i \vee b \neq a_i \vee b'$. Therefore, this mapping is injective.

Thus $|A| = |B| = k_q$ and the number of choices for a_{i+1} is $(n-i)_q - k_q$. It follows that

$$|D| = \prod_{i=0}^{n-k-1} ((n-i)_q - k_q), \tag{4.14}$$

which, with (4.12), implies that $|C| = q^{k(n-k)}$. □

Theorem 4.2. *If L is a q-lattice and $n, r, s \in \mathbb{N}$, then the covering coefficient $\left\{ {n \atop r,s} \right\}$ of L, defined by* (3.13), *is given by the formula*

$$\left\{ {n \atop r,s} \right\} = \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r \\ n-s \end{bmatrix} q^{(n-r)(n-s)}. \tag{4.15}$$

Proof. Let $[x, y]$ be an n-interval in L, and let $A = \{(w, z) \mid w, z \in [x, y],\ w \vee z = y,\ \rho(w) = r$, and $\rho(z) = s\}$ and $B = \{(w, v) \mid w \in [x, y],\ v \in [x, w],\ \rho(w) = r$, and $\rho(v) = r + s - n\}$. Then

$$|A| = \left\{ {n \atop r,s} \right\} \text{ and } |B| = \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r \\ r+s-n \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r \\ n-s \end{bmatrix}.$$

Formula (4.15) follows by noting that the mapping $(w, z) \mapsto (w, w \wedge z)$ is a $q^{(n-r)(n-s)}$-to-one surjection from A to B. For by (4.1) it follows that this mapping is into B and that the inverse image of any $(w, v) \in B$ consists of precisely those pairs (w, z) such that z is a complement of w in $[v, y]$. Since the rank of w in $[v, y]$ is $r - (r + s - n) = n - s$ and $l(v, y) = n - (r + s - n) = 2n - r - s$, Theorem 4.1 implies that the number of such complements is $q^{(n-s)(n-r)}$. □

As has long been known, based on the proofs of the separate cases $q = 0$, $q = 1$, and $q = p^d$ [8, pp. 345, 347, 352], the Möbius function of a q-lattice takes a particularly nice form.

Theorem 4.3. *The Möbius function μ of a q-lattice L is given by*

$$\mu(x, y) = (-1)^{l(x,y)} q^{\binom{l(x,y)}{2}}. \tag{4.16}$$

Proof. Since L is a binomial lattice, we know that $\mu \in \mathscr{R}(L)$. It suffices to show that

$$M(n)=(-1)^n q^{\binom{n}{2}}, \tag{4.17}$$

where, as previously, $M=\psi(\mu)$. Since the equation $M \star Z = D$ uniquely determines M, equation (4.17) follows if we can prove that

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} = \begin{cases} 1, & \text{if } n=0, \\ 0, & \text{if } n>0. \end{cases} \tag{4.18}$$

This identity is easily verified using (4.11). □

By Theorems 4.2 and 4.3 many of the results of Section 3 particularize to q-lattices. By (3.17), for example, the number of ordered k-covers of y in any n-interval $[x, y]$ of a q-lattice is given by

$$C_k(n)=\sum_{j=0}^{n} (-1)^{n-j} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} G_j^k. \tag{4.19}$$

If $q=1$ then $G_j=2^j$, and (4.19) yields the familiar formula $C_k(n)=(2^k-1)^n$ for the number of ordered covers of an n-set with k blocks. If $q=0$, then $G_j=j+1$ and (4.19) yields $C_k(n)=(n+1)^k-n^k$, as a simple counting argument also shows.

Similarly, (3.19) becomes

$$N(n)=\sum_{j=0}^{n} (-1)^{n-j} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} G_j^{-1}. \tag{4.20}$$

If $q=1$ then $N(n)=(-1/2)^n$, and if $q=0$, then $N(0)=1$ and $N(n)=-1/(n^2+n)$ for $n \geqslant 1$.

Hence, for $q=1$ the q-lattice version of the Newton inversion principle (3.22) yields

$$G(n)=\sum_{r=0}^{n} \binom{n}{r} 2^r F(r) \quad \text{for all } n \geqslant 0$$

if and only if

$$F(n)=2^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} G(r) \quad \text{for all } n \geqslant 0. \tag{4.21}$$

Of course, this equivalence may also be proved by ordinary binomial inversion. When $q=0$, with $F(0)=G(0)=0$, eqs (3.22) and (4.20) yield the curious equivalence

$$G(n)=F(1)+\cdots+F(n-1)+(n+1)F(n) \quad \text{for all } n \geqslant 1$$

if and only if

$$F(n)=(n+1)^{-1}(G(n)-n^{-1}(G(1)+\cdots+G(n-1))) \quad \text{for all } n \geqslant 1. \tag{4.22}$$

Finally, for a q-lattice the inverse pair (3.23) and (3.24) becomes the equivalence

$$\begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} r \\ k-s \end{bmatrix} q^{(k-r)(k-s)} \quad \text{for all } n \geqslant 0$$

if and only if

$$\begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r \\ n-s \end{bmatrix} q^{(n-r)(n-s)} = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} k \\ s \end{bmatrix} \quad \text{for all } n \geqslant 0. \tag{4.23}$$

When $q=1$, equation (4.23) reduces to a classical inverse pair of binomial coefficient identities [7, p. 15].

5. Formal q-binomial series

In this section we show that the Newton algebra ($\mathbf{C}^{\mathbf{N}}$, $+$, $\blacklozenge$, $.$) of a q-lattice may be viewed as an algebra of formal q-binomial series whenever $q>0$. Capital Roman letters denote elements of $\mathbf{C}^{\mathbf{N}}$ with the indeterminate $X := (0,1,0,\dots)$ as usual. If $F, G \in \mathbf{C}^{\mathbf{N}}$, the simple juxtaposition FG denotes their Cauchy product – i.e. $FG(n) := \sum_{j=0}^{n} F(j)G(n-j)$. Where no confusion arises, we denote the arithmetic function $(c.D) = (c, 0, 0, \dots)$ simply by c.

An arithmetic function $P \in \mathbf{C}^{\mathbf{N}}$ is, as usual, a *polynomial* if P is finitely nonzero. The symbol $\mathscr{P}$ denotes the set of all polynomials. We begin by introducing the sequence of *Newton polynomials*

$$\left(\begin{bmatrix} X \\ n \end{bmatrix} \right)_{n \geqslant 0}, \quad \text{where} \begin{bmatrix} X \\ 0 \end{bmatrix} = 1$$

and for all $n \geqslant 1$

$$\begin{bmatrix} X \\ n \end{bmatrix} = \frac{1}{n_q^{\underline{n}}} X(X-1_q) \cdots (X-(n-1)_q). \tag{5.1}$$

It follows from (4.10) that for all $m \geqslant 1$,

$$\begin{bmatrix} m_q \\ n \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}, \tag{5.2}$$

which leads to a quick proof of the following theorem.

Theorem 5.1. *The Newton polynomials of a modular binomial lattice L of characteristic $q>0$ satisfy*

$$\begin{bmatrix} X \\ r \end{bmatrix} \begin{bmatrix} X \\ s \end{bmatrix} = \sum_{n} \begin{Bmatrix} n \\ r,s \end{Bmatrix} \begin{bmatrix} X \\ n \end{bmatrix} \tag{5.3}$$

for all $r, s \geqslant 0$, where the $\{{}_{r,s}^{\,n}\}$ are covering coefficients of L.

Proof. Setting $n=m$ and $k=n$ in (3.23) and using (5.2), we have

$$\begin{bmatrix} m_q \\ r \end{bmatrix}\begin{bmatrix} m_q \\ s \end{bmatrix}=\sum_n \begin{Bmatrix} n \\ r,s \end{Bmatrix}\begin{bmatrix} m_q \\ n \end{bmatrix} \quad \text{for all } m\geqslant 0, \tag{5.4}$$

which establishes the polynomial identity (5.3). □

Since the Newton algebra of a q-lattice is isomorphic to $(\mathbf{C}^{\mathbf{N}}, +, \blacksquare, .)$, it clearly contains zero divisors. Hence $(\mathbf{C}^{\mathbf{N}}, +, \blacklozenge, .)$ is not isomorphic to the algebra of formal power series over **C**. Its subalgebra $(\mathscr{P}, +, \blacklozenge, .)$ is, however, isomorphic to the algebra of polynomials over **C**.

Theorem 5.2. *The mapping $\phi : \mathscr{P} \to \mathscr{P}$ defined by*

$$\phi(P)=\sum_n P(n)\begin{bmatrix} X \\ n \end{bmatrix} \tag{5.5}$$

is an isomorphism from $(\mathscr{P}, +, \blacklozenge, .)$ to the algebra of polynomials over **C**.

Proof. Clearly, ϕ is an automorphism of the **C**-vector space $\mathscr{P}$. Also, for $P, Q \in \mathscr{P}$, by (5.5), (3.15), and (5.3),

$$\begin{aligned}
\phi(P \blacklozenge Q) &= \sum_n P \blacklozenge Q(n)\begin{bmatrix} X \\ n \end{bmatrix} \\
&= \sum_n \sum_{r,s} \begin{Bmatrix} n \\ r,s \end{Bmatrix} P(r)Q(s)\begin{bmatrix} X \\ n \end{bmatrix} \\
&= \sum_{r,s} P(r)Q(s)\sum_n \begin{Bmatrix} n \\ r,s \end{Bmatrix}\begin{bmatrix} X \\ n \end{bmatrix} \\
&= \sum_{r,s} P(r)Q(s)\begin{bmatrix} X \\ r \end{bmatrix}\begin{bmatrix} X \\ s \end{bmatrix} \\
&= \sum_r P(r)\begin{bmatrix} X \\ r \end{bmatrix}\sum_s Q(s)\begin{bmatrix} X \\ s \end{bmatrix} \\
&= \phi(P)\phi(Q). \quad \square
\end{aligned} \tag{5.6}$$

Note that, unless the sequence (a_n) is finitely nonzero, a sum of the form

$$\sum_n a_n \begin{bmatrix} X \\ n \end{bmatrix}$$

does not converge in the usual power-series metric (see [6] or [9, p. 6]) since the

$$\text{'power-series degree' of } \left[\!\left[\begin{matrix} X \\ n \end{matrix} \right]\!\right] \not\to \infty \quad \text{as } n \to \infty.$$

Suppose, however, that we define a sequence of polynomials

$$\left(\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle \right)_{n \geqslant 0} \text{ by } \left\langle \begin{matrix} X \\ 0 \end{matrix} \right\rangle = 1$$

and, for all $n \geqslant 1$,

$$\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle = \frac{1}{n_q^{\underline{n}}} X \blacklozenge (X - 1_q) \blacklozenge \cdots \blacklozenge (X - (n-1)_q). \tag{5.7}$$

By (5.5) we have

$$\phi(X^n) = \left[\!\left[\begin{matrix} X \\ n \end{matrix} \right]\!\right].$$

But by (5.6), (5.7), and (5.1) it follows that

$$\phi\left(\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle \right) = \left[\!\left[\begin{matrix} X \\ n \end{matrix} \right]\!\right].$$

Since ϕ is injective, it follows that

$$\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle = X^n.$$

Hence, arbitrary sums of the form

$$\sum_n a_n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle$$

are well-defined, and for any $F \in \mathbf{C}^{\mathbf{N}}$ we have

$$F = \sum_n F(n) \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle. \tag{5.8}$$

Trivially, by (5.8),

$$\left[\sum_r F(r) \left\langle \begin{matrix} X \\ r \end{matrix} \right\rangle \right] \blacklozenge \left[\sum_s G(s) \left\langle \begin{matrix} X \\ s \end{matrix} \right\rangle \right] = F \blacklozenge G = \sum_n F \blacklozenge G(n) \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle. \tag{5.9}$$

That this is not sterile formalism is shown by the following longer derivation of (5.9), the first, second, and fourth equalities of which are established by straightforward convergence arguments in $\mathbf{C}^{\mathbf{N}}$ equipped with the usual power-series metric:

$$\left[\sum_n F(r)\left\langle {X \atop r} \right\rangle\right] \blacklozenge \left[\sum_s G(s)\left\langle {X \atop s} \right\rangle\right] = \sum_r \left[F(r)\left\langle {X \atop r} \right\rangle \blacklozenge \sum_s G(s)\left\langle {X \atop s} \right\rangle\right]$$

$$= \sum_r \left[\sum_s F(r)G(s)\left\langle {X \atop r} \right\rangle \blacklozenge \left\langle {X \atop s} \right\rangle\right]$$

$$= \sum_r \left[\sum_s F(r)G(s)\left(\sum_n \left\{ {n \atop r,s} \right\}\left\langle {X \atop n} \right\rangle\right)\right]$$

$$= \sum_n \left(\sum_{r,s} \left\{ {n \atop r,s} \right\} F(r)G(s)\right)\left\langle {X \atop n} \right\rangle$$

$$= \sum_n F \blacklozenge G(n)\left\langle {X \atop n} \right\rangle. \tag{5.10}$$

Hence, the Newton algebra $(\mathbf{C}^{\mathbf{N}}, +, \blacklozenge, .)$ of a q-lattice for which $q>0$ may be construed as the algebra of formal q-binomial series. Thus by Theorem 3.3, the reduced covering algebra of such a q-lattice is isomorphic to the algebra of formal q-binomial series.

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