ADVANCES IN APPLIED MATHEMATICS 11, 49–62 (1990)

Random Variables Related to a Class of Ordered Structures

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We describe a combinatorial model which encompasses the enumeration of many types of ordered structures and determine the behavior of three random variables which record certain numerical parameters of such structures. Examples to which our results are applicable include chains in binomial posets, direct sum decompositions of finite vector spaces, binary words, and Fishburn's generalized weak orders.

1. INTRODUCTION

Let $(c_n)_{n \geq 0}$ be a nonnegative sequence with $c_0 = 0$ and let $(g_n)_{n \geq 0}$ be a positive sequence with $g_0 = 1$. Define a formal generating function

$$F(z) = \sum_{n \geq 0} c_n z^n / g_n, \tag{1.1}$$

and a set of generalized k-nomial coefficients

$$\begin{bmatrix} n \\ n_1, \ldots, n_k \end{bmatrix} = g_n / g_{n_1} \cdots g_{n_k} \tag{1.2}$$

for $n \geq 0$, $k \geq 2$ and nonnegative integers n_i summing to n, abbreviating $\begin{bmatrix} n \\ j, n-j \end{bmatrix}$ by $\begin{bmatrix} n \\ j \end{bmatrix}$. It is then easy to show that the triangular array $(d(n, k))_{0 \leq k \leq n < \infty}$ is defined equivalently by the generating functions

$$F^k(z) = \sum_{n \geq k} d(n, k) z^n / g_n, \qquad k \geq 0, \tag{1.3}$$

0196-8858/90 $7.50

the recurrence relation

$$d(n,k) = \sum_{j=1}^{n-k+1} \begin{bmatrix} n \\ j \end{bmatrix} c_j d(n-j, k-1), \tag{1.4}$$

with $d(n,0) = \delta_{n,0}$, or the explicit formulas

$$d(n,k) = \sum_{n_i > 0} \begin{bmatrix} n \\ n_1 \dots, n_k \end{bmatrix} c_{n_1} \cdots c_{n_k}, \tag{1.5}$$

for $k \geq 2$, with $d(n,0) = \delta_{n,0}$ and $d(n,1) = c_n$. In the above context it is also straightforward to show that the sequence $(d(n))_{n \geq 0}$ is equivalently defined by the sum

$$d(n) = \sum_{k=0}^{n} d(n,k), \tag{1.6}$$

the generating function

$$\sum_{n \geq 0} d(n) z^n / g_n = (1 - F(z))^{-1}, \tag{1.7}$$

or the recurrence relation

$$d(n) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} c_k d(n-k), \tag{1.8}$$

with $d(0) = 1$.

The foregoing combinatorial model arises in the enumeration of many types of ordered structures. In perhaps the most familiar example, $c_n = 1$ for all $n \geq 1$, $g_n = n!$, $F(z) = e^z - 1$, and $d(n,k)$ enumerates the ordered partitions of an n-set into k blocks. Numerous additional examples are described in Section 6. With (c_n) only stipulated to be nonnegative, it is possible that $d(n) = 0$ for all $n \geq 1$. The following lemma provides a useful condition necessary and sufficient to ensure that $d(n) > 0$ for sufficiently large n:

LEMMA 1.1. *Let $S = \{n: c_n > 0\}$ and let (S) be the ideal generated by S. There exists an n^* such that $d(n) > 0$ for all $n \geq n^*$ if and only if $(S) = \mathbf{Z}$.*

Proof. Sufficiency. If $1 \in S$, then $c_1 > 0$ and so $d(n,n) > 0$ by (1.5) and hence $d(n) > 0$ by (1.6), for all $n \geq 0$. If $1 \notin S$ and $(S) = \mathbf{Z}$, there is an $r \geq 2$ and a subset $\{n_1, \dots, n_r\}$ of S such that g.c.d. $(n_1, \dots, n_r) = 1$. By elementary number theory [3, p. 38, Ex. 19,20], if g.c.d. $(n_1, n_2) = d$ then sufficiently large multiples of d are linear combinations, with nonnegative integral coefficients, of n_1 and n_2, and this result may be extended by induction to the case g.c.d. $(n_1, \ldots, n_r) = d$ using g.c.d. $(n_1, \ldots, n_r) =$ g.c.d. (n_1, md^*), where g.c.d. $(n_1, m) = 1$ and $d^* =$ g.c.d. $(n_2, \ldots, n_r)$. When $d = 1$, as above, it follows that there is an n^* such that for all $n \geq n^*$ there exist nonnegative integers $a_1, \ldots, a_r$ such that $a_1 n_1 + \cdots + a_r n_r = n$. By (1.5), $d(n, a_1 + \cdots + a_r) > 0$ and so $d(n) > 0$ by (1.6).

Necessity. If $(S) \neq \mathbf{Z}$, then $(S) = m\mathbf{Z}$ for some $m \geq 2$. It follows from (1.5) and (1.6) that $d(n) = 0$ for all $n \not\equiv 0 \pmod{m}$.

Our aim in this paper is to describe the behavior of three random variables, I_n, L_n, and A_n, defined for all n such that $d(n) > 0$ by

$$\Pr(I_n = k) = \begin{bmatrix} n \\ k \end{bmatrix} c_k d(n-k)/d(n), \qquad 1 \leq k \leq n, \tag{1.9}$$

$$\Pr(L_n = k) = d(n,k)/d(n), \qquad 1 \leq k \leq n, \tag{1.10}$$

and

$$A_n = n/L_n. \tag{1.11}$$

By (1.8) and (1.6), (1.9) and (1.10) are indeed density functions whenever $d(n) > 0$. When $d(n)$ is the number of ordered partitions of an n-set S, and $d(n,k)$ the number of such partitions with k blocks, the random variables I_n, L_n, and A_n record, respectively, the *initial* block size, the *length* (i.e., the number of blocks), and the *average* block size of a randomly chosen ordered partition of S. Similar natural interpretations of these random variables exist for the examples in Section 6, which motivate our interest in a general analysis of their behavior.

We show, under the restrictions on (c_n) postulated in Lemma 1.1, that if (1.1) has positive radius of convergence, then the sequence (I_n) converges in distribution to a random variable I with density $\Pr(I = k) = c_k p^k / g_k$, where p is the unique positive real solution of $F(z) = 1$, and the sequence (L_n) is, with a single exception, asymptotically normal. We also show that $E(A_n) = E(I_n)$.

2. Preliminaries

The following results will often be used in the remainder of this paper:

LEMMA 2.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have a pole of order $m+1$ at $p \neq 0$ and be analytic elsewhere in $\{z : |z| \leq rp\}$ for some $r > 1$. If $s(z) = \sum_{k=0}^{m} b_k/(z-p)^{k+1}$ is the singular part of f at p, then*

$$a_n p^{n+1} = Q_m(n) + o(r^{-n}), \tag{2.1a}$$

where

$$Q_m(n) = \sum_{k=0}^{m} (-1)^{k+1}\binom{n+k}{n} b_k/p^k. \tag{2.1b}$$

Proof. Expand $s(z)$ as a power series about zero. Since the power series of $f(z) - s(z)$ converges at $z = rp$, we obtain

$$\lim_{n\to\infty}\left[a_n p^n - \sum_{k=0}^{m}(-1)^{k+1}\binom{n+k}{n} b_k/p^{k+1}\right] r^n = 0,$$

which, multiplied by p, yields (2.1). We remark that in the case of a pole p of order $1(m = 0)$, (2.1) asserts that $a_n p^{n+1} = -b_0 + o(r^{-n})$, where $b_0 \neq 0$.

LEMMA 2.2. *If, for some $\mu \neq 0$, $r > 1$, and $k \geq 0$, the sequences (u_n), (v_n), and (w_n) satisfy* (i) $u_n = \mu + O(r^{-n})$, (ii) $u_n v_n = w_n + O(r^{-n})$, *and* (iii) $|w_n| = O(n^k)$, *then*

$$v_n = w_n \mu^{-1} + O(n^k r^{-n}).$$

Proof. By (ii) $\exists M_1$ such that $|u_n v_n - w_n| \leq M_1 r^{-n}$ for sufficiently large n. By (i) $\exists M_2$ such that $|u_n - \mu| \leq M_2 r^{-n}$ for sufficiently large n. Since $\mu \neq 0$, by (i) and (ii) $\exists M_3$ such that $|v_n| = |u_n^{-1} u_n v_n| \leq M_3 |w_n|$ for sufficiently large n. Hence

$$\begin{aligned} |v_n - w_n\mu^{-1}| &= |\mu|^{-1}|\mu v_n - w_n| \\ &\leq |\mu|^{-1}(|u_n v_n - w_n| + |u_n - \mu|\,|v_n|) \\ &\leq |\mu|^{-1}(M_1 r^{-n} + M_2 M_3 |w_n| r^{-n}) \end{aligned}$$

for sufficiently large n. Combining this result with (iii) it follows that $\exists M$ such that $|v_n - w_n\mu^{-1}| \leq M(n^k r^{-n})$ for sufficiently large n.

LEMMA 2.3. *Let $(g_n)_{n\geq 0}$ be a positive sequence with $g_0 = 1$. Let $(c_n)_{n \geq 0}$ be a nonnegative sequence such that $c_0 = 0$ and the ideal generated by $S = \{n\colon c_n \neq 0\}$ is $\mathbf{Z}$. If*

$$F(z) = \sum_{n=0}^{\infty} c_n z^n / g_n \tag{2.2}$$

has positive radius of convergence, then there exists a unique positive p such that $F(p) = 1$ and there exists a $\rho > p$ such that $(1 - F(z))^{-1}$ is analytic for $|z| < \rho$, except for a simple pole at p.

Proof. Let R be the radius of convergence of (2.2). Since the coefficients c_n are nonnegative and there exists at least one $n > 0$ for which $c_n > 0$, F

is strictly increasing on $(0, R)$ and $|F(z)| \le F(|z|)$ whenever $|z| < R$. Thus if $R < \infty$, $\lim_{x \to R-} F(x) = +\infty$. If $R = +\infty$, F is a nonconstant entire function, hence unbounded, and we also have $\lim_{x \to R-} F(x) = +\infty$. Since $F(0) = 0$, there is a unique positive p satisfying $F(p) = 1$. Moreover, $F'(p) \neq 0$, and since

$$\lim_{z \to p} \frac{z - p}{1 - F(z)} = \frac{-1}{F'(p)},$$

the pole of $(1 - F(z))^{-1}$ at p is simple.

Now let $\rho = \min\{|z| : F(z) = 1, z \neq p\}$, setting $\rho = R$, if this set is empty. Since $|F(z)| \le F(|z|)$, $\rho \ge p$. If $\rho = p$, there is a $w \in C$ with $|w| = p$ and $1 = F(w) \le |F(w)| \le F(|w|) = F(p) = 1$. Let $n_0 = \min S = \min\{n : c_n \neq 0\}$. We have $\arg w^{n_0} = \arg w^n$ for all $n \in S$. Moreover, $\arg w^{n_0} = \arg F(w) = 0$, so that $w^n > 0$ for all $n \in S$. Since the ideal generated by S is $\mathbf{Z}$, $w = p$, which is a contradiction. Hence $\rho > p$, and $(1 - F(z))^{-1}$ is analytic for $|z| < \rho$, except for the simple pole at p, as asserted.

3. The Limiting Distribution of (I_n)

Under the hypotheses of Lemma 2.3, with p the unique positive solution of $F(z) = 1$, we may define a random variable I by

$$\Pr(I = k) = c_k p^k / g_k, \qquad k \ge 1, \tag{3.1}$$

since $\sum_k c_k p^k / g_k = F(p) = 1$. The limiting behavior of the sequence of random variables (I_n) is then specified by the following theorem:

THEOREM 3.1. *Let $(g_n)_{n \ge 0}$ be a positive sequence with $g_0 = 1$, and let $(c_n)_{n \ge 0}$ be a nonnegative sequence such that $c_0 = 0$ and the ideal generated by $S = \{n : c_n > 0\}$ is $\mathbf{Z}$. If*

$$F(z) = \sum_{n=0}^{\infty} c_n z^n / g_n \tag{3.2}$$

has positive radius of convergence, and I_n and I are defined, respectively, by (1.9) *and* (3.1), *then (I_n) converges to I in distribution and the moments of I_n converge to the moments of I.*

Proof. Let ϕ be the characteristic function of I. By (3.1) and (3.2),

$$\phi(t) = F(pe^{it}). \tag{3.3}$$

Let ϕ_n be the characteristic function of I_n, which is defined for all n such that $d(n) > 0$. For such n, by (1.9),

$$d(n)\phi_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} c_k d(n-k) e^{ikt} \tag{3.4}$$

and we stipulate (3.4) as the definition of the otherwise meaningless expression $d(n)\phi_n(t)$ when $d(n) = 0$. With this stipulation, let

$$G(t, z) = \sum_{n=0}^{\infty} d(n)\phi_n(t) z^n / g_n. \tag{3.5}$$

Substituting (3.4) into (3.5) and interchanging summation yields

$$\begin{aligned} G(t, z) &= \sum_{k=0}^{\infty} c_k e^{ikt} \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} d(n-k) z^n / g_n \\ &= \sum_{k=0}^{\infty} c_k e^{ikt} z^k / g_k \sum_{n=k}^{\infty} d(n-k) z^{n-k} / g_{n-k} = \frac{F(ze^{it})}{1 - F(z)}, \end{aligned} \tag{3.6}$$

by (3.2) and (1.7).

By (3.6) and Lemma 2.3, for a fixed t, $G(t, z)$ has a simple pole at $z = p$. Hence by (3.3),

$$G(t, z) = \frac{-\phi(t)}{F'(p)(z - p)} + H(t, z), \tag{3.7}$$

where $H(t, z)$ is analytic for z near p. Applying Lemma 2.1, with $m = 0$, to (3.5) and (3.7) yields

$$p^{n+1} d(n)\phi_n(t)/g_n = \frac{\phi(t)}{F'(p)} + o(r^{-n}) \tag{3.8}$$

for some $r > 1$. Since, for all $n \geq 1$, $d(n)\phi_n(0) = d(n)$ by (3.4) and (1.8), setting $t = 0$ in (3.8) yields

$$p^{n+1} d(n)/g_n = \frac{1}{F'(p)} + o(r^{-n}). \tag{3.9}$$

Dividing (3.8) by (3.9) for n sufficiently large to ensure that $d(n) > 0$, it follows that ϕ_n converges pointwise to ϕ and hence that (I_n) converges in distribution to I. Note that (3.9) also yields the nice asymptotic formula,

$$d(n) \sim \frac{g_n}{p^{n+1}F'(p)}. \tag{3.10}$$

To show that the moments of I_n converge to the moments of I, let j be a fixed positive integer and calculate $(-i)^j(\partial^j/\partial t^j)G(0, z)$, both from (3.5) and from (3.7). Setting the results of these two calculations equal to each other yields

$$\sum_{n=1}^{\infty} d(n)E(I_n^j)z^n/g_n = \frac{-E(I^j)}{F'(p)(z-p)} + L(z), \tag{3.11}$$

where $L(z)$ is analytic near p. As in the case of $d(n)\phi_n(t)$ above, the identity

$$d(n)E(I_n^j) = \sum_{k=1}^{n} k^j \begin{bmatrix} n \\ k \end{bmatrix} c_k d(n-k), \tag{3.12}$$

which holds for $d(n) > 0$, is stipulated as a definition of the otherwise meaningless expression $d(n)E(I_n^j)$ when $d(n) = 0$. Applying Lemma 2.1 to (3.11) yields

$$\frac{p^{n+1}d(n)E(I_n^j)}{g_n} = \frac{E(I^j)}{F'(p)} + o(r^{-n}) \tag{3.13}$$

for some $r > 1$. Dividing (3.13) by (3.9) then shows that $E(I_n^j) \to E(I^j)$.

4. The First Moments of A_n and I_n

In light of the example of ordered partitions of a set, where A_n and I_n record initial and average block cardinalities, one might at least expect some sort of asymptotic relation to hold between $E(A_n)$ and $E(I_n)$ in the general case. Perhaps a bit surprisingly, the following identity may be proved:

THEOREM 4.1. *If*

$$\alpha_n = \sum_{k=1}^{n} \frac{n}{k} d(n, k) \tag{4.1}$$

and

$$\beta_n = \sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} c_k d(n-k), \tag{4.2}$$

then, for all $n \geq 1$, $\alpha_n = \beta_n$, and so, for all n such that $d(n) > 0$, $E(A_n) = \alpha_n/d(n) = \beta_n/d(n) = E(I_n)$.

Proof. It suffices to prove the formal generating function identity

$$\sum_{n\geq 1} \alpha_n z^n/g_n = \sum_{n\geq 1} \beta_n z^n/g_n = zF'(z)(1-F(z))^{-1}, \tag{4.3}$$

where F is given by (1.1), and then use (1.9) and (1.11). By (1.3) we have

$$\begin{aligned} -\log(1-F(z)) &= \sum_{k\geq 1} \frac{F^k(z)}{k} = \sum_{k\geq 1} \frac{1}{k} \sum_{n\geq k} d(n,k) z^n/g_n \\ &= \sum_{n\geq 1} \sum_{k=1}^{n} \frac{d(n,k)}{k} z^n/g_n. \end{aligned} \tag{4.4}$$

Applying zD_z to (4.4), and using (4.1), we obtain

$$zF'(z)(1-F(z))^{-1} = \sum_{n\geq 1} \alpha_n z^n/g_n. \tag{4.5}$$

Formulas (4.4) and (4.5) are to be interpreted in terms of formal logarithms and derivatives applied to formal power series. See [5, pp. 878–879] for further details.

The other half of (4.3) follows by multiplication of the series $zF'(z) = \sum_{n\geq 0} nc_n z^n/g_n$ and $(1-F(z))^{-1} = \sum_{n\geq 0} d(n) z^n/g_n$.

We remark in conclusion that when $F(z) = z + z^2$, $E(A_3^2) \neq E(I_3^2)$, so that higher moments of A_n and I_n do not necessarily coincide. An elaboration of this example appears in Section 6.

5. The Asymptotic Normality of (L_n)

Let $(g_n)_{n\geq 0}$ be a positive sequence with $g_0 = 1$ and let $(c_n)_{n\geq 0}$ be a nonnegative sequence with $c_0 = 0$. Recall from Section 1 the formal power series identities:

$$F(z) = \sum_{n=0}^{\infty} c_n z^n/g_n, \tag{5.1}$$

$$F^k(z) = \sum_{n=k}^{\infty} d(n,k) z^n/g_n, \qquad k \geq 0, \tag{5.2}$$

and

$$(1-F(z))^{-1} = \sum_{n=0}^{\infty} d(n) z^n/g_n, \tag{5.3}$$

and the random variables L_n, defined for all n such that $d(n) > 0$ by

$$\Pr(L_n = k) = d(n, k)/d(n), \qquad 1 \leq k \leq n. \tag{5.4}$$

Suppose that the ideal generated by $S = \{n : c_n > 0\}$ is $\mathbf{Z}$ and that F has positive radius of convergence, so that $d(n)$ is positive for n sufficiently large (Lemma 1.1) and there is a unique real positive p such that $F(p) = 1$ and an $r > 1$ such that $(1 - F(z))^{-1}$ is analytic for $|z| \leq rp$, except for a simple pole at p (Lemma 2.3). We may then prove the following theorem:

THEOREM 5.1. *The sequence of random variables* (L_n) *is asymptotically normal unless* $c_k = 0$ *for all* $k \neq 1$, *in which case* L_n *is a point mass at* n.

Proof. Let $\phi_n(t)$ denote the characteristic function of L_n and $E(L_n^m)$ the mth moment of L_n, defined for all n such that $d(n) > 0$. For such n, we have by (5.4)

$$d(n)\phi_n(t) = \sum_{k=0}^{n} d(n, k)e^{ikt} \tag{5.5}$$

and

$$d(n)E(L_n^m) = \sum_{k=0}^{n} k^m d(n, k), \tag{5.6}$$

and we stipulate these formulas as definitions of the otherwise meaningless expressions $d(n)\phi_n(t)$ and $d(n)E(L_n^m)$ when $d(n) = 0$. Let

$$G(t, z) = \sum_{n=0}^{\infty} d(n)\phi_n(t)z^n/g_n. \tag{5.7}$$

Substituting (5.5) into (5.7), interchanging summation, and using (5.2) then yields

$$G(t, z) = \sum_{k=0}^{\infty} F^k(z)e^{ikt} = \left[1 - e^{it}F(z)\right]^{-1}, \tag{5.8}$$

valid for $|F(z)| < 1$, i.e., for $|z| < p$, where p is the unique real positive solution of $F(z) = 1$. By (5.7),

$$(-i)^m \frac{\partial^m}{\partial t^m} G(0, z) = \sum_{n=0}^{\infty} d(n)E(L_n^m)z^n/g_n, \tag{5.9}$$

which by (5.8) is a rational function of $F(z)$ with denominator $(1 - F(z))^{m+1}$, and hence has a pole of order $m + 1$ at $z = p$. Now let

$$q_m(n) = F'(p)Q_m(n), \tag{5.10}$$

where $Q_n(m)$ is given by (2.1b) for the function $f(z) = (-i)^m \partial^m / \partial t^m G(0, z)$. It follows from Lemma 2.1 that

$$d(n) E(L_n^m) p^{n+1} / g_n = q_m(n) / F'(p) + o(r^{-n}), \tag{5.11}$$

for some $r > 1$. Letting $m = 0$ in (5.11) and noting that $q_0(n) = 1$ yields

$$d(n) p^{n+1} / g_n = 1/F'(p) + o(r^{-n}), \tag{5.12}$$

and dividing (5.11) by (5.12) yields

$$E(L_n^m) = q_m(n) + O(n^m / r^n). \tag{5.13}$$

By the Continuity Theorem, (L_m) is asymptotically normal if, for all $t \in \mathbf{R}$,

$$\lim_{n \to \infty} e^{-i\mu_n t/\sigma_n} \phi_n(t/\sigma_n) = e^{-t^2/2}, \tag{5.14}$$

where $\mu_n = E(L_n)$ and $\sigma_n^2 = \mathrm{Var}(L_n)$. By (5.13), $q_1(n)$ and $(q_2(n) - q_1^2(n))^{1/2}$ are asymptotic expressions for μ_n and σ_n, respectively. We will show that σ_n^2 is either bounded for all n or grows like a linear polynomial in n. In the former case, we show that each L_n is a point mass at n. In the latter case we show that (5.14) is equivalent to

$$\lim_{n \to \infty} e^{-iq_1(n)t/\sigma_n} \phi_n(t/\sigma_n) = e^{-t^2/2}, \tag{5.15}$$

and prove (5.15), thus establishing the asymptotic normality of (L_n). To establish these results we shall need convenient expressions for $q_1(n)$ and σ_n, as well as a uniform estimate of $\phi_n(t)$ for $|t|$ small, which we derive by examining the poles of $G(t, z)$ for fixed small t. First consider the case $t = 0$. By (5.8), $G(0, z) = (1 - F(z))^{-1}$ and so by Lemma 2.3 there is an $r > 1$ such that $G(0, z)$ is analytic for $|z| \le rp$, except for a simple pole at $z = p$. Let such an r be chosen and fixed. We claim that there exists a $\delta > 0$ such that for every t satisfying $|t| \le \delta$, $G(t, z)$ is analytic for $|z| \le rp$, except for a simple pole at a point we shall denote by $z = p(t)$.

To prove this claim, note first that by (5.8) $G(t, z)$ has a pole whenever $F(z) = e^{-it}$. Since $F'(p) \ne 0$, F is one-to-one on an open set Ω containing p. Thus we can define an analytic function $p(t)$, for $|t|$ sufficiently small, by

$$p(t) = F^{-1}(e^{-it}). \tag{5.16}$$

Note that $p(0) = p$. Clearly $p(t)$ is a simple pole of $G(t, z)$ and the only pole of $G(t, z)$ in Ω. We must prove the existence of some $\delta > 0$ such that for all t satisfying $|t| \le \delta$, $p(t)$ is the only pole of $G(t, z)$ in $\{z : |z| \le rp\}$.

Suppose that no such δ exists. Then there are sequences (t_n) and (w_n) satisfying $\lim_{n \to \infty} t_n = 0$, $F(w_n) = e^{-it_n}$, $|w_n| \le rp$, and $w_n \notin \Omega$. Since (w_n)

lies in a compact set there is no loss in generality in assuming that (w_n) converges, say to the limit w, whence $F(w) = \lim_{n\to\infty} F(w_n) = \lim_{n\to\infty} e^{-it_n} = 1$. Since $|w| \le rp$, we must have $w = p$, and hence $w_n \in \Omega$ for sufficiently large n, which is a contradiction.

Now the residue of $G(t, z)$ at $z = p(t)$ is

$$\lim_{z\to p(t)} \frac{z - p(t)}{1 - e^{it}F(z)} = \frac{-1}{e^{it}F'(p(t))}. \tag{5.17}$$

Since $F(p(t)) = e^{-it}$, we have $F'(p(t)) = -ie^{-it}/p'(t)$, and so

$$G(t, z) = \frac{-ip'(t)}{z - p(t)} + H(t, z), \tag{5.18}$$

where $H(t, z)$ is an analytic function of z for $|t| \le \delta$ and $|z| \le rp$. Applying Lemma 2.1 to (5.7) and (5.18) yields

$$d(n)\phi_n(t)p^{n+1}(t)/g_n = ip'(t) + o(r^{-n}) \tag{5.19}$$

for $|t| \le \delta$. In particular, when $t = 0$ we have

$$d(n)p^{n+1}/g_n = ip'(0) + o(r^{-n}). \tag{5.20}$$

Dividing (5.19) by (5.20) yields

$$\phi_n(t)\left(\frac{p(t)}{p}\right)^{n+1} = \frac{p'(t)}{p'(0)} + O(r^{-n}). \tag{5.21}$$

Since r does not depend on t, we have

$$\lim_{n\to\infty} \phi_n(t)\left(\frac{p(t)}{p}\right)^{n+1} = \frac{p'(t)}{p'(0)}, \tag{5.22}$$

uniformly for $|t| \le \delta$.

Equation (5.18) also yields formulas for $q_1(n)$ and $q_2(n) - q_1^2(n)$, to be used as asymptotic estimates of the mean and variance of L_n, in terms of the derivatives of $p(t)$ at zero. Let $p_k = p^{(k)}(0)$, where $p(t)$ is given by (5.16), noting that $p_0 = p$. Using (2.1b), (5.10), (5.16), and (5.18), we may derive the formulas

$$q_1(n) = i\left(\frac{(n+1)p_1}{p} - \frac{p_2}{p_1}\right) \tag{5.23}$$

and

$$\sigma_n^2 = (n+1)\left(\frac{p_2}{p} - \frac{p_1^2}{p^2}\right) + \left(\frac{p_2^2}{p_1^2} - \frac{p_3}{p_1}\right) + o(nr^{-n}). \tag{5.24}$$

By (5.24), either (σ_n) is bounded, or $\lim_{n\to\infty}\sigma_n = \infty$. The former possibility is equivalent to the condition $pp_2 = p_1^2$, which is equivalent, by (5.16) and the definition $p_k = p^{(k)}(0)$, to

$$p^2F''(p) - p^2(F'(p))^2 + pF'(p) = 0. \tag{5.25}$$

Now the left-hand side of (5.25) is the variance of the random variable I defined by $\Pr(I = k) = c_k p^k/g_k$, $k \geq 1$. Hence I is a point mass at 1 and so $F(z) = c_1 z/g_1$, i.e., $c_k = 0$ whenever $k \neq 1$. It follows from (5.2), (5.3), and (5.4) that L_n is a point mass at n.

Now assume that $\lim_{n\to\infty}\sigma_n = \infty$. Then $\lim_{n\to\infty}|\mu_n - q_1(n)|/\sigma_n = 0$ and thus (5.14) is equivalent to (5.15). In (5.15), moreover, the constant term in $q_1(n)$ can be deleted, since $\lim_{n\to\infty}\sigma_n = \infty$ and the exponential factor has constant modulus one. Thus (5.15) is equivalent to

$$\lim_{n\to\infty} e^{(n+1)p_1 t/p\sigma_n}\phi_n(t/\sigma_n) = e^{-t^2/2}. \tag{5.26}$$

For fixed $t \in \mathbf{R}$, $\lim_{n\to\infty} t/\sigma_n = 0$ and so for n sufficiently large $|t/\sigma_n| \leq \delta$. Hence by (5.22),

$$\lim_{n\to\infty} \phi_n(t/\sigma_n)\left(p(t/\sigma_n)/p\right)^{n+1} = 1. \tag{5.27}$$

By (5.27), (5.26) then holds if

$$\lim_{n\to\infty} \left(e^{-p_1 t/p\sigma_n}p(t/\sigma_n)/p\right)^{-(n+1)} = e^{-t^2/2},$$

which is implied by

$$e^{-p_1 t/p\sigma_n}p(t/\sigma_n)/p = 1 + t^2/2(n+1) + O(n^{-3/2}). \tag{5.28}$$

Now

$$e^{-p_1 t/p}p(t)/p = 1 + \left(\frac{p_2}{p} - \frac{p_1^2}{p^2}\right)\frac{t^2}{2} + O(t^3). \tag{5.29}$$

Substituting t/σ_n for t in (5.29) and using (5.24) then yields (5.28), which completes the proof.

6. Applications

The combinatorial model specified by formulas (1.1)–(1.11) arises in the study of many types of ordered structures. The following examples, with $c_0 = 0$, $g_0 = 1$, and c_n and g_n defined as indicated for $n \geq 1$, illustrate the generality of our results.

As noted in Section 1, the prototypical example of this model is the case in which $c_n \equiv 1$, $g_n = n!$, and $F(z) = e^z - 1$, where $d(n, k)$ enumerates the *ordered partitions of an n-set into k blocks*. It follows in this case from Theorem 5.1 that (L_n) is asymptotically normal and from Theorem 3.1 that $\Pr(I_n = k) \to (\log 2)^k/k!$, the latter result having been established by a different argument in [4].

When $c_n \equiv g_n \equiv 1$ and $F(z) = (1 - z)^{-1} - 1$, it is easy to see that $d(n, k)$ enumerates the *ordered partitions of the integer n into k positive summands*. In this case one may confirm, without using Theorems 3.1 and 5.1, that (I_n) converges in distribution to a geometric random variable with $p = \frac{1}{2}$, and, using the central limit theorem for binomial distributions, that (L_n) is asymptotically normal.

Note that there is a one-to-one correspondence between ordered partitions of an n-set S with k blocks and *chains* $\phi \subset S_1 \subset \cdots \subset S_k = S$ *of subsets of* S, and between ordered partitions of the integer n having k positive parts and *chains* $0 < n_1 < \cdots < n_k = n$ *of positive integers*. Hence the aforementioned results may be rephrased in terms of such chains. More generally, if P is any binomial poset, $c_n \equiv 1$, and $g_n =$ the number of maximal chains in an n-interval, then $d(n, k)$ enumerates the *chains* $x = x_0 < x_1 < \cdots < x_k = y$ *of length k, where* $[x, y]$ *is an n-interval* [7, pp. 142–146]. Theorems 3.1 and 5.1 hold in all such cases, with the obvious interpretations of I_n and L_n. A particularly interesting example of the foregoing is the case in which $g_n = (1 + q)\dots(1 + q + \cdots + q^{n-1})$, where $d(n, k)$ enuemrates *chains of subspaces* $\{0\} \subset V_1 \subset \cdots \subset V_k = V$ *of an n-dimensional* $GF(q)$*-vector space* V. A variant of the latter example, to which Theorems 3.1 and 5.1 also apply, is the case in which $c_n \equiv 1$ and $g_n = (q^n - 1)(q^n - q)\dots(q^n - q^{n-1})$, where $d(n, k)$ enumerates the *ordered direct sum decompositions of V into k subspaces* [1].

The class of generalized weak orders [2] furnishes examples of our model for which (c_n) is not constant. If $c_n = 2^{n(n+1)/2}$ and $g_n = n!$, then $d(n, k)$ enumerates *the generalized weak orders (GWOs) on an n-set S partitioning S into k generalized indifference classes* [8]. Here I_n records the number of "optimal elements" and L_n the number of generalized indifference classes of a randomly chosen GWO on S. In this case $F(z)$ converges only at zero, so Theorems 3.1 and 5.1 do not apply. In [4] it is shown in this case that $(I_n - n)$ converges to zero with probability one. Special cases of the above example, where $F(z)$ has positive radius of convergence, are (1) $c_n =$

$B(n+1)$, the $n+1$th Bell number, and $g_n = n!$, for which $d(n, k)$ enumerates the *transitive GWOs on an n-set S which partition S into k generalized indifference classes* and (2) $c_n \equiv 2$ and $g_n = n!$, where $d(n, k)$ enumerates the *transitive, negatively transitive GWOs partitioning S into k generalized indifference classes*. See [8, 4] for further details.

We conclude with two examples where (c_n) is finitely nonzero. First, let $c_1 = c_2 = 1$, $c_n = 0$ for $n \geq 3$, and $g_n \equiv 1$. It is then easy to show that $d(n,k) = \binom{k}{n-k}$ and, for all $n \geq 0$, $d(n) = F_n$, the nth Fibonacci number. The hypotheses of Theorems 3.1 and 5.1 are clearly satisfied in this case. Recalling that F_n enumerates the binary words of length $n-1$ with no two zeroes adjacent, we leave the interpretation of I_n, L_n, and A_n in this case as an exercise.

Finally, for fixed $s \geq 2$, let $c_1 = w_1$, $c_2 = w_2, \ldots, c_s = w_s$, where $0 \leq w_i \leq 1$ and $\Sigma w_i = 1$, and $c_n = 0$ for $n > s$, and let $g_n \equiv 1$, so that $F(z) = w_1 z + w_2 z^2 + \cdots + w_s z^s$. For a die with sides numbered 1 through s and weighted so that side i appears with probability w_i, $d(n, k)$ is the probability that after k rolls of this die the sum of the numbers which have appeared is exactly n, and $d(n)$ is the probability that the sum n is attained at some point in a sequence of rolls. Suppose that g.c.d. $\{i : w_i > 0\} = 1$. By Lemma 1.1, $d(n)$ is then positive for n sufficiently large. The unique positive solution p of $F(z) = 1$ is $p = 1$, and $F'(1) =$ the expected value of the number appearing on a roll of the die. By (3.10), $d(n)$ converges to the reciprocal of this expected value, or $\frac{2}{7}$ in the case of an ordinary die (cf. [6]). As for the random variables I_n and L_n, we have $\Pr(I_n = k) = w_k d(n-k)/d(n)$ and $\Pr(L_n = k) = d(n, k)/d(n)$. Hence, for the set of sequences of rolls for which the sum n is attained at some point, I_n records the number appearing on the initial roll and L_n the number of the roll on which the sum n is attained. By Theorem 3.1, $\Pr(I_n = k) \to w_k$, as one would expect, and by Theorem 5.1, (L_n) is asymptotically normal.

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