Recursive Formulae for Power Sums 1 April 2013

Carl Wagner

If r and n are nonnegative integers, the r^{th} *power sum* $S_r(n)$ is defined by

(1) $$S_r(n) := \sum_{k=0}^{n} k^r, \text{ where } 0^0 := 1.$$

In particular,

(2) $$S_0(n) = 1 + n.$$

Along with (2), the following recurrence provides a recursive formula for $S_r(n)$:

(3) $$S_r(n) = \frac{1}{r+1}\{(n+1)^{r+1} - \sum_{i=0}^{r-1}\binom{r+1}{i} S_i(n)\}, \text{ for all } r \geq 1.$$

Proof of (3): By the binomial theorem,

(4) $$(k+1)^{r+1} - k^{r+1} = \sum_{i=0}^{r}\binom{r+1}{i} k^i,$$

and so

(5) $$\sum_{k=0}^{n}[(k+1)^{r+1} - k^{r+1}] = \sum_{k=0}^{n}\sum_{i=0}^{r}\binom{r+1}{i} k^i = \sum_{i=0}^{r}\binom{r+1}{i}\sum_{k=0}^{n} k^i = \sum_{i=0}^{r}\binom{r+1}{i} S_i(n).$$

Since the left-most sum above telescopes to $(n+1)^{r+1}$, this yields the implicit recurrence

(6) $$\sum_{i=0}^{r}\binom{r+1}{i} S_i(n) = (n+1)^{r+1},$$

which yields (3) upon solving for $S_r(n)$. See Wagner [2] for a combinatorial proof of (6). □

From (2) and (3) it follows by induction on r that $S_r(n)$ *is a polynomial in* n *of degree* $r+1$. Accordingly, let us write

(7) $$S_r(n) = \sum_{j=0}^{r+1} a(r, j) n^j.$$

From (2) and the recurrence (3), we may compute

(8) $S_1(n) = \frac{1}{2}n + \frac{1}{2}n^2$,

(9) $S_2(n) = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$,

(10) $S_3(n) = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$, and

(11) $S_4(n) = -\frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5$, etc.

Tabulating the coefficients of these polynomials clarifies the connection between the coefficients of $S_r(n)$ and those of $S_{r-1}(n)$.

Table 1. $a(r, j)$ for $0 \le r \le 4$ and $0 \le j \le 5$

	$j=0$	$j=1$	$j=2$	j=3	$j=4$	$j=5$
$r=0$	1	1				
$r=1$	0	½	½			
$r=2$	0	1/6	½	1/3		
$r=3$	0	0	¼	½	¼	
$r=4$	0	-1/30	0	1/3	½	1/5

A careful examination of the above table leads us to conjecture the following theorem:

Theorem. For all $r \ge 1$, $a(r,0) = 0$ and

(12) $\sum_{j=0}^{r+1} a(r, j) = \sum_{j=1}^{r+1} a(r, j) = 1$, whence,

(13) $a(r,1) = 1 - \sum_{j=2}^{r+1} a(r, j)$.

For all $r \ge 1$ and $j \ge 2$,

(14) $a(r, j) = \frac{r}{j}\, a(r-1, j-1)$.

Proof (Owens [1]). Setting $n=1$ in (7) yields $\sum_{j=0}^{r+1} a(r,j) = S_r(1) = 0^r + 1^r = 1.$ Setting $n=0$ in (7) for $r \geq 1$ yields $a(r,0) = S_r(0) = 0^r = 0.$ Now consider the polynomial

(15) $$S_r(x) := \sum_{j=0}^{r+1} a(r,j)x^j.$$

In what follows we use the easily established fact that, applied to a polynomial $p(x)$, the finite difference operator Δ and the differentiation operator D commute, i.e.,

(16) $$\Delta(Dp(x)) = D(\Delta p(x)).$$

Now by (1), it is the case that for fixed $r \geq 0$ and all $n \geq 0$, $S_r(n+1) - S_r(n) = (n+1)^r$, which implies the polynomial identity

(17) $$\Delta\, S_r(x) = S_r(x+1) - S_r(x) = (x+1)^r.$$

From (16) and (17) it follows that

(18) $$\Delta(DS_r(x)) = D(\Delta\, S_r(x)) = D(x+1)^r = r(x+1)^{r-1} = r\Delta\, S_{r-1}(x) = \Delta r S_{r-1}(x).$$

Hence,

(19) $$\Delta(D\, S_r(x) - rS_{r-1}(x) = 0,$$

from which it follows that the polynomial $DS_r(x) - rS_{r-1}(x)$ is equal to a constant[1], namely, the constant $c = DS_r(0) - rS_{r-1}(0) = a(r,1) - r\delta_{r,1}.$ Hence,

(20) $DS_r(x) = rS_{r-1}(x) + c$, which implies that

(21) $$\sum_{j=1}^{r+1} ja(r,j)x^{j-1} = \sum_{j=0}^{r} (j+1)a(r,j+1)x^j = \sum_{j=0}^{r} ra(r-1,j)x^j + c.$$

Comparing coefficients of x^j for $j \geq 1$ yields

(22) $(j+1)a(r,j+1) = ra(r-1,j)$ for all $r, j \geq 1,$ i.e.,

(23) $ja(r,j) = ra(r-1,j-1)$ for all $r \geq 1$ and all $j \geq 2.$ □

Remark. The numbers B_r defined by $B_0 = a(0,1)$, $B_1 = -a(1,1)$, and $B_r = a(r,1)$ for all $r \geq 2$, are called *Bernoulli numbers.* We have

$B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, B_9 = 0, B_{10} = 5/66,$ $B_{11} = 0$, $B_{12} = -691/2730$, etc. It may be proved that $B_r = 0$ for all odd $r \geq 3$, and that B_r alternates in sign for r even. If we define a function $\beta : \mathbb{R} \to \mathbb{R}$ by

(24) $$\beta(x) = \frac{x}{e^x - 1} \text{ if } x \neq 0 \text{ and } \beta(0) = 1 \ (= \lim_{x \to 0} \frac{x}{e^x - 1}) \ ,$$

then

(25) $$\sum_{r=0}^{\infty} B_r \frac{x^r}{r!} = \beta(x) \quad (\text{and so} \quad \sum_{r=0}^{\infty} a(r,1) \frac{x^r}{r!} = \beta(x) + x\,) \quad \text{for } |x| < 2\pi \ .$$

Notes

[1] If $p(x) = \sum_{j=0}^{m} c_j \binom{x}{j}$, and $\Delta p(x) = \sum_{j=1}^{m} c_j \binom{x}{j-1} = 0$, then $c_1 = \cdots = c_m = 0$, since the set of polynomials $\{ \binom{x}{j-1}, \ j = 1, ..., m\}$ is a basis for the vector space of polynomials of degree $\leq m-1$. So $p(x) = c_0$.

References

1. R. Owens, Sums of powers of integers, *Mathematics Magazine* **65** (1992), 38-40.

2. C. Wagner, Combinatorial proofs of formulas for power sums, *Archiv der Mathematik* **68** (1997), 464-467.