Surjections, Differences, and Binomial Lattices

By Carl G. Wagner

The elementary problem of counting surjections from an n-set to a k-set is generalized to that of enumerating solutions of $a_1 \vee \cdots \vee a_n = y$, with each a_i an atom of the k-interval $[x,y]$ in a binomial lattice L. When L is modular, the number of such solutions is representable as a q-difference and satisfies a simple recurrence.

1. Introduction

Let $[0] = \phi$ and $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{P}$. If $n, k \in \mathbb{N}$, the surjections from $[n]$ to $[k]$, the ordered partitions of $[n]$ with k blocks, and the chains $\phi = S_0 \subset S_1 \subset \cdots \subset S_k = [n]$ all have the same cardinality, which we denote by $\sigma(n,k)$. On any of these interpretations, it is obvious that $\sigma(n,0) = \delta_{n,0}$ and that $\sigma(n,k) = 0$ if $n < k$.

Apart from these boundary values, there are many different representations of $\sigma(n,k)$. Perhaps the most basic is the formula

$$\sigma(n,k) = \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \in \mathbb{P}}} \frac{n!}{n_1! \cdots n_k!}, \qquad 1 \le k \le n, \tag{1.1}$$

which immediately yields the generating functions

$$\sum_{n \ge 0} \sigma(n,k) \frac{z^n}{n!} = (e^z - 1)^k, \qquad k \in \mathbb{N}, \tag{1.2}$$

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and the recurrence

$$\sigma(n,k) = \sum_{j\geq 1}\binom{n}{j}\sigma(n-j,k-1), \qquad 1 \leq k \leq n. \tag{1.3}$$

In addition, there is the formula

$$\sigma(n,k) = \sum_{j\geq 0}(-1)^j\binom{k}{j}(k-j)^n, \tag{1.4}$$

derived by the sieve method. This may also be written

$$\sigma(n,k) = \Delta^k 0^n := \Delta^k x^n|_{x=0}, \tag{1.5}$$

and derived in the latter form from the easily established polynomial identity

$$x^n = \sum_{k\geq 0}\sigma(n,k)\frac{x^{\underline{k}}}{k!}, \tag{1.6}$$

where $x^{\underline{0}}=1$ and $x^{\underline{k}}=x(x-1)\cdots(x-k+1)$ for $k\in\mathbb{P}$. Finally $\sigma(n,k)$ satisfies the simple recurrence

$$\sigma(n,k) = k\sigma(n-1,k-1)+k\sigma(n-1,k), \qquad 1 \leq k \leq n. \tag{1.7}$$

In the work of Doubilet, Rota and Stanley [1], the numbers $\sigma(n,k)$ appear simply as a special case of the solution to the general problem of determining CHAIN(n,k), the number of chains of length k between the endpoints of an n-interval in a binomial poset P. Formula (1.1) generalizes to

$$\text{CHAIN}(n,k) = \sum_{\substack{n_1+\cdots+n_k=n\\ n_i\in\mathbb{P}}}\frac{B(n)}{B(n_1)\cdots B(n_k)}, \tag{1.8}$$

where $B(n)$ is the number of maximal chains in each n-interval of P. By a general formal rule [2, pp. 49–50], (1.8) immediately yields generalizations of (1.2) and (1.3). As we show in the next section, the numbers CHAIN(n,k) also satisfy a generalization of (1.4). But nothing resembling (1.5)–(1.7) holds for these numbers.

In this paper, we describe a generalization, SURJ(n,k), of $\sigma(n,k)$ for which versions of (1.5)–(1.7) do survive. The numbers SURJ(n,k) count the sequences $(a_1,\ldots,a_n)$ of atoms in a k-interval $[x,y]$ of a binomial lattice L such that $a_1 \vee \cdots \vee a_n = y$. These numbers satisfy a version of (1.4) and, when L is modular, of (1.5)–(1.7), generalizing $\sigma(n,k)$ in a manner nearly disjoint from that of CHAIN(n,k). It thus appears that the multitude of representations of $\sigma(n,k)$, comprising almost an *embarras du choix*, arises from a happy coincidence in lattices of sets of two otherwise distinct combinatorial constructs.

2. Chains in binomial posets

The standard references on incidence algebras and binomial posets are, respectively, Rota's classical paper [3] and Stanley's book [4, Chap. 3]. The following is a brief review of pertinent results.

Let P be a locally finite poset, with Int(P) the set of intervals of P. The incidence algebra of P is the $\mathbb{C}$-algebra $(\mathbb{C}^{\mathrm{Int}(P)},+,*,.)$ with pointwise addition and scalar multiplication and product $*$ defined for f,g: Int$(P)\to\mathbb{C}$ by

$$f*g(x,y) = \sum_{z\in[x,y]} f(x,z)g(z,y). \tag{2.1}$$

The multiplicative identity of this algebra is the function δ, defined by $\delta(x,y)=\delta_{x,y}$. The function f has a multiplicative inverse iff, for all $x\in P, f(x,x)\neq 0$. In particular the zeta function ζ, defined by $\zeta(x,y)=1$ for all $x\leq y$, is invertible. Its inverse, denoted by μ, is called the Möbius function of P. The Möbius inversion principle is simply the equivalence

$$g = f*\zeta \Leftrightarrow f = g*\mu. \tag{2.2}$$

If $[x,y]$ is an interval in the locally finite poset P and $k\in\mathbb{N}$, let multichain $([x,y],k)$ denote the number of sequences $(z_0,z_1,\dots,z_k)$ in $[x,y]$ such that $x=z_0\leq z_1\leq\cdots\leq z_k=y$, and let chain $([x,y],k)$ denote the number of such sequences satisfying $z=z_0<z_1<\cdots<z_k=y$. By (2.1), it is clear that

$$\text{multichain}([x,y],k) = \zeta^{*k}(x,y) \tag{2.3}$$

and

$$\text{chain}([x,y],k) = (\zeta-\delta)^{*k}(x,y). \tag{2.4}$$

A poset P is called a binomial poset if it satisfies the following three conditions:

(I) P is locally finite with $\hat{0}$ and contains an infinite chain.

(II) Every interval $[x,y]$ of P is graded, i.e., all maximal chains in $[x,y]$ have the same length. If this common length is n, write $l(x,y)=n$ and call $[x,y]$ an n-interval.

(III) For all $n \in \mathbb{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains.

As a consequence of (II), each n-interval $[x,y]$ of a binomial poset admits a rank function $\rho_{x,y}\colon [x,y] \to \{0,1,\ldots,n\}$, written simply as ρ if no confusion results, defined by $\rho(z) = l(x,z)$. If $z \in [x,y] \cap [x,y']$, then clearly $\rho_{x,y}(z) = \rho_{x,y'}(z)$. Also, if $x \leq z \leq u \leq y$, then $\rho_{x,y}(u) = \rho_{x,y}(z) + \rho_{z,y}(u)$.

By (II) and (III) any two n-intervals in a binomial poset contain the same number $\begin{bmatrix} n \\ k \end{bmatrix}$ of elements of rank k, where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{B(n)}{B(k)B(n-k)}, \tag{2.5}$$

whence $B(n) = \begin{bmatrix} n \\ 1 \end{bmatrix}\begin{bmatrix} n-1 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If P is a binomial poset, the set

$$\mathscr{R}(P) := \{f \in \mathbb{C}^{\mathrm{Int}(P)} : l(x,y) = l(x',y') \Rightarrow f(x,y) = f(x',y')\} \tag{2.6}$$

is a subalgebra of the incidence algebra of P, called the reduced incidence algebra of P. The mapping $\psi\colon \mathscr{R}(P) \to \mathbb{C}^{\mathbb{N}}$ defined by

$$\psi f(n) = F(n) := f(x,y), \tag{2.7}$$

where $[x,y]$ is an arbitrary n-interval, is an isomorphism from the reduced incidence algebra to the algebra of arithmetic functions $(\mathbb{C}^{\mathbb{N}}, +, \bigstar, \cdot)$ with pointwise addition and scalar multiplication, and product $\bigstar$ defined by

$$F \bigstar G(n) = \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} F(k) G(n-k). \tag{2.8}$$

If $f \in \mathscr{R}(P)$, we call the function $F = \psi f$ the reduction of f. The functions δ, ζ, and μ all belong to $\mathscr{R}(P)$, and we denote their respective reductions by D, Z, and M. Under ψ the Möbius inversion principle (2.2) reduces to

$$G = F \bigstar Z \Leftrightarrow F = G \bigstar M. \tag{2.9}$$

From (2.3) and (2.4) it is clear that when P is a binomial poset, multichain$(\cdot,k)$ and chain $(\cdot,k)$ belong to $\mathscr{R}(P)$, and that their respective reductions, MULTICHAIN$(\cdot,k)$ and CHAIN$(\cdot,k)$ are given by

$$\text{MULTICHAIN}(n,k) = Z^{\bigstar k}(n) \tag{2.10}$$

and

$$\begin{aligned}\mathrm{CHAIN}(n,k) &= (Z-D)^{\star k}(n)\\ &= \sum_{j\geq 0}(-1)^j\binom{k}{j}\{D^{\star j}\star Z^{\star(k-j)}(n)\}\\ &= \sum_{j\geq 0}(-1)^j\binom{k}{j}\sum_{i\geq 0}\begin{bmatrix} n\\ i\end{bmatrix}D^{\star j}(i)Z^{\star(k-j)}(n-i)\\ &= \sum_{j\geq 0}(-1)^j\binom{k}{j}Z^{\star(k-j)}(n). \qquad (2.11)\end{aligned}$$

In view of (2.10) there is also an obvious derivation of (2.11) by the sieve method.

Note that when P is the binomial poset (actually, lattice) of finite subsets of an infinite set ordered by inclusion, then $B(n)=n!$. So by (2.5) and (2.8), or by (2.10), $Z^{\star(k-j)}(n)=(k-j)^n$, and (2.11) reduces to (1.4).

In conclusion, we note that if $n_1+\cdots+n_k=n, n_i\in\mathbb{N}$, then

$$\begin{bmatrix} n\\ n_1\end{bmatrix}\begin{bmatrix} n-n_1\\ n_2\end{bmatrix}\cdots\begin{bmatrix} n-n_1-\cdots-n_{k-1}\\ n_k\end{bmatrix}=\frac{B(n)}{B(n_1)\cdots B(n_k)} \qquad (2.12)$$

clearly counts the multichains $x=z_0\leq z_1\leq\cdots\leq z_k=y$ in an n-interval $[x,y]$ such that $l(z_{i-1},z_i)=n_i$, $i=1,\ldots,k$. Hence

$$\mathrm{CHAIN}(n,k)=\sum_{\substack{n_1+\cdots n_k=n\\ n_i\in\mathbb{P}}}\frac{B(n)}{B(n_1)\cdots B(n_k)},\qquad 1\leq k\leq n, \quad (2.13)$$

as remarked in (1.8).

3. Surjections and differences

A different generalization of $\sigma(n,k)$ arises from its role as enumerator of surjections $f\colon[n]\to[k]$. For each such f corresponds uniquely to a sequence $(A_1,\ldots,A_n)$ of singleton subsets of $[k]$ such that $A_1\cup\cdots\cup A_n=[k]$ by the rule $A_i=\{f(x_i)\}$.

Suppose now that $[x,y]$ is an interval in the locally finite lattice L and $n\in\mathbb{N}$. Define

$$\mathrm{surj}(n,[x,y]) := |\{(a_1,\ldots,a_n): \text{each } a_i \text{ is an atom of } [x,y] \text{ and } a_1\vee\cdots\vee a_n=y\}|. \qquad (3.1)$$

In what follows we establish that if L is a binomial lattice, then $\mathrm{surj}(n,\cdot) \in \mathscr{R}(L)$, and we study its reduction $\mathrm{SURJ}(n,\cdot)$. The theory of covering algebras [5], outlined briefly below, is useful in pursuing this study.

Given a locally finite lattice L and functions $f, g \in \mathbb{C}^{\mathrm{Int}(L)}$, define the product $f\Diamond g(x,y)$ by

$$f\Diamond g(x,y) := \sum_{\substack{(w,z)\in[x,y]^2\\ w\vee z=y}} f(x,w)g(x,z). \tag{3.2}$$

That $(\mathbb{C}^{\mathrm{Int}(L)}, +, \Diamond, .)$ with pointwise addition and scalar multiplication is a commutative $\mathbb{C}$-algebra follows from the fact [5, Th. 2.1] that the mapping $f \mapsto f * \zeta$ is an isomorphism from the structure $(\mathbb{C}^{\mathrm{Int}(L)}, +, \Diamond, .)$ to the Schur algebra $(\mathbb{C}^{\mathrm{Int}(L)}, +, \Box, .)$, where $f \Box g(x,y) := f(x,y)g(x,y)$. We call the former the covering algebra of L, since $\zeta^{\Diamond n}(x,y)$ is clearly the number of ordered n-covers of y in $[x,y]$, i.e., the number of sequences $(z_1,\ldots,z_n)$ in $[x,y]$ such that $z_1 \vee \cdots \vee z_n = y$.

Suppose that every interval of the locally finite lattice L is graded. Given an interval $[x,y]$ in L with rank function ρ, and any sequence $(r_1,\ldots,r_n)$ in $\mathbb{N}$, let

$$\left\{\begin{matrix}[x,y]\\ r_1,\ldots,r_n\end{matrix}\right\} := \begin{matrix}\text{the number of ordered } n\text{-covers } (z_1,\ldots,z_n)\\ \text{of } y \text{ in } [x,y] \text{ with } \rho(z_i)=r_i.\end{matrix} \tag{3.3}$$

The "covering coefficient" (3.3) clearly reduces to $\mathrm{surj}(n,[x,y])$ when $r_1 = \cdots = r_n = 1$, so for the moment we investigate (3.3) rather than (3.1).

Recall that a locally finite lattice L in which every interval is graded is semi-modular iff the rank function ρ of each interval $[x,y]$ in L satisfies

$$\rho(z_1 \vee z_2) \le \rho(z_1) + \rho(z_2) - \rho(z_1 \wedge z_2) \tag{3.4}$$

for all $z_1, z_2 \in [x,y]$.

THEOREM 3.1. *Let L be a locally finite lattice in which every interval is graded. Then L is semi-modular iff for every interval $[x,y]$ in L, for all $n \in \mathbb{P}$, and for all $r_1,\ldots,r_n \in \mathbb{N}$,*

$$r_1 + \cdots + r_n < l(x,y) \Rightarrow \left\{\begin{matrix}[x,y]\\ r_1,\ldots,r_n\end{matrix}\right\} = 0. \tag{3.5}$$

In particular semi-modularity of L implies that $\mathrm{surj}(n,[x,y]) = 0$ *whenever* $n < l(x,y)$.

Proof: Necessity. If ρ is the rank function of $[x,y]$ and $z_1,\ldots,z_n \in [x,y]$, then by (3.4), $\rho(z_1 \vee \cdots \vee z_n) \le \rho(z_1) + \cdots + \rho(z_n)$. Hence if $\rho(z_1) + \cdots + \rho(z_n) < \rho(y) = l(x,y)$, then $z_1 \vee \cdots \vee z_n < y$.

Sufficiency. Let $[u,v]$ be any interval of L, with rank function ρ, and let $w_1, w_2 \in [u,v]$. It follows from (3.5) that $\rho(w_1 \vee w_2) \leq \rho(w_1) + \rho(w_2)$, for if not, (3.5) is contradicted when $[x,y] = [u, w_1 \vee w_2]$ and $r_i = \rho(w_i)$, $i = 1,2$.

But now if $z_1, z_2 \in [x,y]$, then

$$\begin{aligned}
\rho_{x,y}(z_1 \vee z_2) &= \rho_{x,y}(z_1 \wedge z_2) + \rho_{z_1 \wedge z_2, y}(z_1 \vee z_2) \\
&\leq \rho_{x,y}(z_1 \wedge z_2) + \rho_{z_1 \wedge z_2, y}(z_1) + \rho_{z_1 \wedge z_2, y}(z_2) \\
&= \rho_{x,y}(z_1 \wedge z_2) + \left[\rho_{x,y}(z_1) - \rho_{x,y}(z_1 \wedge z_2) \right] \\
&\quad + \left[\rho_{x,y}(z_2) - \rho_{x,y}(z_1 \wedge z_2) \right] \\
&= \rho_{x,y}(z_1) + \rho_{x,y}(z_2) - \rho_{x,y}(z_1 \wedge z_2), \qquad (3.6)
\end{aligned}$$

establishing semi-modularity.

Suppose now that L is a binomial lattice, i.e., a lattice that is a binomial poset. By earlier remarks, if $f, g \in \mathbb{C}^{\mathrm{Int}(L)}$, then

$$f \diamond g = (f \diamond g) * \zeta * \mu = [(f * \zeta) \square (g * \zeta)] * \mu. \qquad (3.7)$$

Thus if $f, g \in \mathscr{R}(L)$ then $f \diamond g \in \mathscr{R}(L)$ since $\zeta, \mu \in \mathscr{R}(L)$, which is closed under $*$, and obviously also under $\square$. This observation is the key to showing that for a binomial lattice and fixed $r_1, \ldots, r_n \in \mathbb{N}$, the covering coefficient $\left\{ \begin{matrix} [x,y] \\ r_1, \ldots, r_n \end{matrix} \right\}$ defined by (3.3) depends only on $l(x,y)$.

THEOREM 3.2. *If $[x,y]$ and $[x',y']$ are intervals in the binomial lattice L and $l(x,y) = l(x',y')$, then for all $r_1, \ldots, r_n \in \mathbb{N}$,*

$$\left\{ \begin{matrix} [x,y] \\ r_1, \ldots, r_n \end{matrix} \right\} = \left\{ \begin{matrix} [x',y'] \\ r_1, \ldots, r_n \end{matrix} \right\}. \qquad (3.8)$$

Proof: For $i = 1, \ldots, n$, define $f_i \in \mathbb{C}^{\mathrm{Int}(L)}$ by

$$f_i(u,v) = \begin{cases} 1, & \text{if } l(u,v) = r_i \\ 0, & \text{otherwise.} \end{cases} \qquad (3.9)$$

Since each $f_i \in \mathscr{R}(L)$, it follows that $f_1 \diamond \cdots \diamond f_n \in \mathscr{R}(L)$. Along with (3.2), (3.3), and (3.9) this implies that

$$\left\{ \begin{matrix} [x,y] \\ r_1, \ldots, r_n \end{matrix} \right\} = f_1 \diamond \cdots \diamond f_n(x,y) = f_1 \diamond \cdots \diamond f_n(x',y') = \left\{ \begin{matrix} [x',y'] \\ r_1, \ldots, r_n \end{matrix} \right\}.$$

Since $\operatorname{surj}(n,[x,y]) = \left\{ \begin{matrix} [x,y] \\ r_1,\ldots,r_n \end{matrix} \right\}$, where $r_1 = \cdots = r_n = 1$, Theorem 3.2 implies that $\operatorname{surj}(n,\cdot) \in \mathscr{R}(L)$ for every binomial lattice L. Denoting the reduction of $\operatorname{surj}(n,\cdot)$ by $\mathrm{SURJ}(n,\cdot)$, it follows that

$\mathrm{SURJ}(n,k)$ = the number of ordered n-covers of y by atoms of $[x,y]$ for any k-interval $[x,y]$ of the binomial lattice L.

We now establish a formula for $\mathrm{SURJ}(n,k)$ that generalizes (1.4).

THEOREM 3.3. *In any binomial lattice L, the numbers $\mathrm{SURJ}(n,k)$ are given by the formula*

$$\mathrm{SURJ}(n,k) = \sum_{j\geq 0} M(j) \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} k-j \\ 1 \end{bmatrix}^n . \tag{3.10}$$

Proof: Enumeration of all sequences $(a_1,\ldots,a_n)$ of atoms in a k-interval $[x,y]$ according to $j = \rho(a_1 \vee \cdots \vee a_n)$ yields for fixed $n \in \mathbb{N}$ the identities

$$\begin{bmatrix} k \\ 1 \end{bmatrix}^n = \sum_{j\geq 0} \begin{bmatrix} k \\ j \end{bmatrix} \mathrm{SURJ}(n,j), \qquad \forall k \in \mathbb{N}, \tag{3.11}$$

which implies (3.10) by (2.8) and (2.9).

Note that by Theorem 3.1, if L is a semi-modular binomial lattice then $\mathrm{SURJ}(n,k) = 0$ whenever $n < k$.

To see exactly how (3.10) generalizes (1.4), and to develop further properties of the numbers $\mathrm{SURJ}(n,k)$, we assume in what follows that L is a modular binomial lattice. Hence the rank function ρ of each interval $[x,y]$ of L satisfies

$$\rho(w \vee z) = \rho(w) + \rho(z) - \rho(w \wedge z), \qquad \text{for all } w,z \in [x,y]. \tag{3.12}$$

Doubilet, Rota, and Stanley [1] have shown that if L is a modular binomial lattice and one defines the characteristic q of L by

$$q := \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1, \tag{3.13}$$

then the number of atoms $\begin{bmatrix} n \\ 1 \end{bmatrix}$ in an n-interval of L is given by

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = 1 + q + \cdots + q^{n-1}, \tag{3.14}$$

and so by (2.5) the number of maximal chains $B(n)$ in an n-interval of L is given by

$$B(n) = (1+q+\cdots+q^{n-1})\cdots(1+q)(1). \tag{3.15}$$

Henceforth, we call a modular binomial lattice of characteristic q a q-lattice, and employ the notation

$$n_q := 1+q+\cdots+q^{n-1}, \qquad \text{with } 0_q := 0, \tag{3.16}$$

$$n_q^! := n_q(n-1)_q\cdots 1_q, \qquad \text{with } 0_q^! := 1, \tag{3.17}$$

$$n_q^{\underline{k}} := n_q(n_q-1_q)\cdots(n_q-(k-1)_q), \qquad \text{with } 0_q^{\underline{0}} := 1. \tag{3.18}$$

Note that $n_q^{\underline{n}} = q^{\binom{n}{2}} n_q^!$. With this notation, (2.5) and (3.15) imply that the incidence coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ of a q-lattice is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n_q^!}{k_q^!(n-k)_q^!}, \qquad 0 \le k \le n. \tag{3.19}$$

Using (3.19) it is easy to establish the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}, \qquad n,k \in \mathbb{P}. \tag{3.20}$$

The theory of modular binomial lattices provides for a unified combinatorial analysis [1, Th. 8.2] of

(1) the lattice of finite subsets of an infinite set, ordered by inclusion ($q=1$),

(2) the lattice of finite subspaces of an infinite vector space over a finite field of order $p^d (q = p^d)$, and

(3) the lattice (actually, chain) $(\mathbb{N}, \le)(q=0)$.

By [5, Th. 4.3] the Möbius function of a q-lattice L takes the nice form

$$\mu(x,y) = (-1)^{l(x,y)} q^{\binom{l(x,y)}{2}}, \tag{3.21}$$

i.e.,

$$M(n) = (-1)^n q^{\binom{n}{2}}, \tag{3.22}$$

where M is the reduction of μ. Hence in a q-lattice the identity (3.11) takes the form

$$(k_q)^n = \sum_{j \geq 0} \begin{bmatrix} k \\ j \end{bmatrix} \mathrm{SURJ}(n,j), \tag{3.23}$$

and formula (3.10) becomes

$$\mathrm{SURJ}(n,k) = \sum_{j \geq 0} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} [(k-j)_q]^n, \tag{3.24}$$

which reduces to (1.4) when $q = 1$.

To get generalizations of (1.5) and (1.6), we need q-generalizations of the falling factorial polynomials $x^{\underline{n}}$ and of the operator Δ. Let $\varphi_0(x) := 1$ and

$$\varphi_n(x) := x(x - 1_q) \cdots (x - (n-1)_q), \qquad \text{for } n \in \mathbb{P}. \tag{3.25}$$

As noted by Davis [6], the operator $\Delta_q : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by

$$\Delta_q p(x) := \frac{p(qx+1) - p(x)}{(q-1)x + 1} \tag{3.26}$$

reduces to Δ when $q = 1$ and satisfies

$$\Delta_q \varphi_n(x) = n_q \varphi_{n-1}(x). \tag{3.27}$$

by (3.25) and (3.27) every polynomial $p(x)$ may be written

$$p(x) = \sum_{k \geq 0} \frac{\Delta_q^k p(0)}{k_q^!} \varphi_k(x), \tag{3.28}$$

where Δ_q^0 is the identity operator and Δ_q^k the k-fold composition of Δ_q. In particular

$$x^n = \sum_{k \geq 0} \frac{\Delta_q^k 0^n}{k_q^!} \varphi_k(x), \tag{3.29}$$

where $\Delta_q^k 0^n := \Delta_q^k x^n |_{x=0}$.

On the other hand, we may prove the following q-generalization of (1.6) when $q \neq 0$:

THEOREM 3.4. *If $q \neq 0$, then for all $n \in \mathbb{N}$*

$$x^n = \sum_{k \geq 0} \mathrm{SURJ}(n,k) \frac{\varphi_k(x)}{k_q^{\underline{k}}}. \tag{3.30}$$

Proof: It is easy to check that if $q \neq 0$, an alternative to formula (3.19) is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n_q^{\underline{k}}}{k_q^{\underline{k}}} = \frac{\varphi_k(n_q)}{k_q^{\underline{k}}}. \tag{3.31}$$

Setting $k = r$ and $j = k$ in (3.23), we have from (3.31) that

$$(r_q)^n = \sum_{k \geq 0} \mathrm{SURJ}(n,k) \frac{\varphi_k(r_q)}{k_q^{\underline{k}}}, \qquad \text{for all } r \in \mathbb{N}. \tag{3.32}$$

Since $\mathrm{SURJ}(n,k) = 0$ whenever $n < k$ in any modular binomial lattice, (3.32) establishes the polynomial identity (3.30).

The q-generalization of (1.5) is given by the following theorem:

THEOREM 3.5. *For every q-lattice,*

$$\mathrm{SURJ}(n,k) = q^{\binom{k}{2}} \Delta_q^k 0^n, \tag{3.33}$$

where $0^0 := 1$. *In particular* $\mathrm{SURJ}(n,n) = n_q^{\underline{n}}$.

Proof: Suppose first that $q \neq 0$. Comparing coefficients of $\varphi_n(x)$ in (3.29) and (3.30) and recalling that $k(k/q) = q^{\binom{k}{2}} k_q^!$ yields (3.33). When $q = 0$, the lattice in question is $(\mathbb{N}, \leq)$ and it is clear from the definition of $\mathrm{SURJ}(n,k)$ that in this case $\mathrm{SURJ}(0,0) = 1$, $\mathrm{SURJ}(n,1) = 1$ for all $n \in \mathbb{P}$, and $\mathrm{SURJ}(n,k) = 0$, otherwise. But these are precisely the values taken by the right-hand side of (3.33), as one may easily check. Finally, (3.29) implies that $\Delta_q^n 0^n = n_q^!$, and so $\mathrm{SURJ}(n,n) = q^{\binom{n}{2}} n_q^! = n_q^{\underline{n}}$. In contrast, we recall that $\mathrm{CHAIN}(n,n) = n_q^!$.

By (3.33) and (3.24) with j replaced by $k-j$, we must of course have

$$q^{\binom{k}{2}}\Delta_q^k 0^n = \sum_{j\geq 0}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}(j_q)^n. \tag{3.34}$$

This identity offers a helpful clue in conjecturing the following formula for the kth q-difference of an arbitrary polynomial when $q\neq 0$:

THEOREM 3.6. *If $q\neq 0$, then*

$$\Delta_q^k p(x) = \frac{\Sigma_{j\geq 0}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}p\left(q^j x + j_q\right)}{q^{\binom{k}{2}}[(q-1)x+1]^k}. \tag{3.35}$$

Proof: By induction on k using the recurrence (3.20). When $q=1$, (3.35) reduces to the familiar formula for $\Delta^k p(x)$.

We conclude this section with a result on the enumeration of complements in a q-lattice and, finally, a recurrence for SURJ(n,k) that reduces to (1.7) when $q=1$.

If $[x,y]$ is an interval in any lattice and $w\in[x,y]$, we call $z\in[x,y]$ a complement of w in $[x,y]$ if $w\vee z=y$ and $w\wedge z=x$.

THEOREM 3.7. *If $[x,y]$ is an n-interval in a q-lattice and $z\in[x,y]$ has rank k, then z has $q^{k(n-k)}$ complements in $[x,y]$.*

Proof: [See [5, Th. 4.1].]

THEOREM 3.8. *For any q-lattice and for all $n,k\in\mathbb{P}$,*

$$\text{SURJ}(n,k) = k_q\left[q^{k-1}\text{SURJ}(n-1,k-1)+\text{SURJ}(n-1,k)\right]. \tag{3.36}$$

Proof: Consider the set of all sequences $(a_1,\ldots,a_n)$ of atoms in the k-interval $[x,y]$ of a q-lattice L such that $a_1\vee\cdots\vee a_n=y$. By (3.12), it must be true of any such sequence that $\rho(a_1\vee\cdots\vee a_{n-1})$ equals either $k-1$ or k (in fact, this follows just from semi-modularity). To construct a sequence of the former type one must choose an element w of rank $k-1$ in $[x,y]$, then choose atoms $a_1,\ldots,a_{n-1}$ with $a_1\vee\cdots\vee a_{n-1}=w$, then choose an atom a_n such that $w\vee a_n=y$, which by (3.12) is equivalent to choosing a complement of w in $[x,y]$. Hence by (3.19) and Theorem 3.7, there are

$$\begin{bmatrix}k\\k-1\end{bmatrix}\text{SURJ}(n-1,k-1)q^{k-1} = k_q q^{k-1}\text{SURJ}(n-1,k-1) \tag{3.37}$$

sequences of the former type. Since there are clearly

$$\mathrm{SURJ}(n-1,k)\begin{bmatrix} k \\ 1 \end{bmatrix} = k_q \mathrm{SURJ}(n-1,k) \tag{3.38}$$

sequences of the latter type, this establishes (3.36).

We note in closing that the classical q-Stirling numbers $\tilde{S}_q(n,k)$ and $S_q(n,k)$ are given by the simple formulas $\tilde{S}_q(n,k) = \mathrm{SURJ}(n,k)/k_q^{\underline{k}}$ and $S_q(n,k) = \mathrm{SURJ}(n,k)/k_q^{!}$, which yields a combinatorial interpretation of these numbers in an arbitrary q-lattice. Details will appear in a forthcoming paper [7].

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