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Enumeration of generalized weak orders

By

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1. Introduction. Let R be a binary relation on X and let

$$aR = \{(x, y) : (x, y) \in R \text{ and } (y, x) \notin R\}, \qquad cR = \{(x, y) : (x, y) \notin R\},$$

and

$$sR = \{(x, y) : (x, y) \in R \text{ and } (y, x) \in R\}.$$

We term R *negatively transitive* when cR is transitive.

R is called a *generalized weak order* if caR is transitive. Generalized weak orders were introduced by P. C. Fishburn [4], and include as special cases the *asymmetric weak orders* (asymmetric, negatively transitive relations) and *complete weak orders* (complete, reflexive, transitive relations) familiar to economists as models, respectively, of strict preference and preference-or-indifference.

Generalizing the familiar 1-1 correspondence between asymmetric weak orders, complete weak orders, and linearly ordered partitions, one may exhibit a 1-1 correspondence between generalized weak orders on a set and linearly ordered partitions of that set, the blocks of which are equipped with arbitrary symmetric relations [4, p. 165]. The essential details of this correspondence are as follows: A generalized weak order R on a set X partitions X by the equivalence relation $scaR$, and $\tilde{X} = X/scaR$ is linearly ordered by $>$, where for all $A, B \in \tilde{X}$, $A > B$ iff $(x, y) \in aR$ for all $x \in A$ and $y \in B$. The symmetric relation attached to each $A \in \tilde{X}$ is simply the restriction of R to A. Conversely, if $\tilde{X}$ is a partition of X linearly ordered by $\succ$, and each $A \in \tilde{X}$ is equipped with a symmetric relation R_A, then $R = R^{(s)} \cup R^{(a)}$ is a generalized weak order on X, where $R^{(s)} = \bigcup_{A \in \tilde{X}} R_A$ and $R^{(a)} = \bigcup_{\substack{A, B \in \tilde{X} \\ A > B}} A \times B$. In the remainder of this paper we exploit the foregoing correspondence to enumerate various classes of generalized weak orders.

2. Recurrence relations. *Let $W(n)$ denote the number of generalized weak orders on an n-set*, equivalently, the number of ordered partitions of an n-set, each block of

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which is equipped with an arbitrary symmetric relation. The number of such partitions with initial block of size k is $\binom{n}{k} 2^{\binom{k+1}{2}} W(n-k)$, and hence,

$$(1) \qquad W(n) = \sum_{k=1}^{n} \binom{n}{k} 2^{\binom{k+1}{2}} W(n-k); \qquad W(0) = 1.$$

A generalized weak order need not be transitive or negatively transitive. Indeed, the transitive (resp., negatively transitive) generalized weak orders are just those corresponding to ordered partitions with transitive (resp., negatively transitive) symmetric relations attached to each block [4, Theorems 5 and 6].

Since the map $R \to cR$ is a bijection from transitive to negatively transitive generalized weak orders, we restrict attention to determining $T(n)$, *the number of transitive generalized weak orders on an n-set.* Proceeding as in the derivation of (1) above, we need to determine the number of transitive, symmetric relations on a block of size k. Now such a relation R on a block A, when restricted to $\{x \in A: (x, x) \in R\}$, is an equivalence. As is familiar, the number of equivalence relations on a j-set is given by $B(j)$, the j-th Bell number. Hence the number of transitive, symmetric relations on a k-set is

$$\sum_{j=0}^{k} \binom{k}{j} B(j) = B(k+1),$$

by a familiar recurrence for the Bell numbers [6]. It follows that

$$(2) \qquad T(n) = \sum_{k=1}^{n} \binom{n}{k} B(k+1)\, T(n-k); \qquad T(0) = 1.$$

Next, *let $M(n)$ denote the number of generalized weak orders on an n-set which are both transitive and negatively transitive.* (Remark. Fishburn [4, Lemma 5] has proved that transitivity of R and cR imply transitivity of caR. Hence $M(n)$ is, more simply, the number of transitive, negatively transitive relations on an n-set.) It is easy to see that a transitive, negatively transitive, symmetric relation on a block is either the empty or universal relation on that block. Hence, arguing as in the case of (1) and (2) above, it follows that

$$(3) \qquad M(n) = \sum_{k=1}^{n} 2 \binom{n}{k} M(n-k); \qquad M(0) = 1.$$

Finally, *let $P(n)$ denote the number of asymmetric* (equivalently, complete) *weak orders on an n-set.* As noted above, $P(n)$ is simply the number of linearly ordered partitions on an *n-set.* (In terms of the correspondence which we have exploited above, asymmetric weak orders correspond to linearly ordered partitions whose blocks are all equipped with the empty relation, and complete weak orders with those whose blocks are all equipped with the universal relation.) By a now familiar argument, we have

$$(4) \qquad P(n) = \sum_{k=0}^{n} \binom{n}{k} P(n-k); \qquad P(0) = 1,$$

which is derived in [5] by a more elaborate argument.

3. Generating functions. Let F and G be real valued arithmetic functions on the non-negative integers, and let

$$F + G(n) = F(n) + G(n)$$

and

$$F * G(n) = \sum_{k=0}^{n} \binom{n}{k} F(k)\ G(n-k).$$

Equipped with these operations, the set of arithmetic functions is an integral domain with multiplicative identity I, where $I(0) = 1$ and $I(n) = 0$ if $n > 0$. F has a multiplicative inverse iff $F(0) \neq 0$. As is familiar, this ring of arithmetic functions is isomorphic with the ring of formal exponential series

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

equipped with ordinary addition and multiplication, the isomorphism being given by the mapping ψ, where

$$\psi(F) = \sum_{n=0}^{\infty} F(n) \frac{x^n}{n!}$$

is the exponential generating function of the arithmetic function F. We determine in this section the exponential generating functions of T, M, and P.

Define arithmetic functions Z, $\bar{B}$, and E by $Z(n) = 1$ for all $n \geqq 0$, $\bar{B}(n) = B(n+1)$, the $n+1^{st}$ Bell number, and $E(n) = 2^{\binom{n+1}{2}}$. Then we may express the recurrence relations (1), (2), (3), and (4) as equations in the ring of arithmetic functions as follows:

(5) $\qquad W = (E - I) * W + I,$

(6) $\qquad T = (\bar{B} - I) * T + I,$

(7) $\qquad M = (2Z - 2I) * M + I,$

(8) $\qquad P = (Z - I) * P + I.$

Solving the above for W, T, M, and P we have

(9) $\qquad W = I/(2I - E),$

(10) $\qquad T = I/(2I - \bar{B}),$

(11) $\qquad M = I/(3I - 2Z),$

(12) $\qquad P = I/(2I - Z).$

The numbers $W(n)$ grow too rapidly to possess a non-trivially convergent exponential generating function. In order to determine $\psi(T)$, we need to determine

$$\psi(\bar{B}) = \sum_{n=0}^{\infty} B(n+1) \frac{x^n}{n!}.$$

Now [6],

$$\psi(B) = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = e^{e^x - 1},$$

and since $\psi(\bar{B}) = D_x \psi(B)$, it follows that $\psi(\bar{B}) = e^{e^x + x - 1}$. Since $\psi(I) = 1$, (10) implies that

$$\psi(T) = \sum_{n=0}^{\infty} T(n) \frac{x^n}{n!} = (2 - e^{e^x + x - 1})^{-1}. \tag{13}$$

And, since $\psi(Z) = e^x$, (11) and (12) imply that

$$\psi(M) = \sum_{n=0}^{\infty} M(n) \frac{x^n}{n!} = (3 - 2e^x)^{-1} \tag{14}$$

and

$$\psi(P) = \sum_{n=0}^{\infty} P(n) \frac{x^n}{n!} = (2 - e^x)^{-1}, \tag{15}$$

the latter result having been derived by a more elaborate argument in [5].

4. Dobinski-type formulas. Over a century ago Dobinski [3] established the following beautiful formula for the Bell numbers:

$$B(n+1) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{(k-1)!}.$$

In this section we establish Dobinski-type formulas for $P(n)$, $M(n)$, and $T(n)$.

For $\lambda \neq 1$, define $H_n[\lambda]$ by

$$\frac{\lambda - 1}{\lambda - e^x} = \sum_{n=0}^{\infty} H_n[\lambda] \frac{x^n}{n!}. \tag{16}$$

The numbers $H_n[\lambda]$ are called *Eulerian numbers*. (See [1], [2].) From (14) and (15) it follows that $M(n) = H_n[3/2]$ and $P(n) = H_n[2]$. A Dobinski-type formula for $H_n[\lambda]$ may be derived as follows:

$$\begin{aligned} H_n[\lambda] &= (\lambda - 1) D^n (\lambda - e^x)^{-1} \Big|_{x=0} = \frac{\lambda - 1}{\lambda} D^n \left(1 - \frac{1}{\lambda} e^x\right)^{-1} \Big|_{x=0} \\ &= \frac{\lambda - 1}{\lambda} D^n \sum_{k=0}^{\infty} \left(\frac{1}{\lambda} e^x\right)^k \Big|_{x=0} = \frac{\lambda - 1}{\lambda} \sum_{k=0}^{\infty} D^n \left(\frac{1}{\lambda} e^x\right)^k \Big|_{x=0} \\ &= \frac{\lambda - 1}{\lambda} \sum_{k=0}^{\infty} \lambda^{-k} k^n. \end{aligned} \tag{17}$$

In particular,

$$M(n) = H_n[3/2] = \tfrac{1}{3} \sum_{k=0}^{\infty} (\tfrac{2}{3})^k k^n \tag{18}$$

and

$$P(n) = H_n[2] = \tfrac{1}{2} \sum_{k=0}^{\infty} (\tfrac{1}{2})^k k^n \tag{19}$$

the latter result having appeared in [5].

As for the numbers $T(n)$, we have by (13),

$$\begin{aligned} T(n) &= \tfrac{1}{2} D^n (1 - \tfrac{1}{2} e^{e^x + x - 1})^{-1}|_{x=0} = \tfrac{1}{2} D^n \sum_{k=0}^{\infty} (\tfrac{1}{2} e^{e^x + x - 1})^k|_{x=0} \\ &= \tfrac{1}{2} \sum_{k=0}^{\infty} (\tfrac{1}{2})^k D^n (e^{ke^x + kx - k})|_{x=0} \\ &= \tfrac{1}{2} \sum_{k=0}^{\infty} (\tfrac{1}{2})^k f_n(k), \end{aligned} \tag{20}$$

where $f_n(k) = D^n(e^{ke^x + kx - k})|_{x=0}$. Now, $f_0(k) = 1$, $f_1(k) = 2k$, $f_2(k) = 4k^2 + k$, $f_3(k) = 8k^3 + 6k^2 + k$, and in general, $f_n(k)$ is a polynomial in k of degree n, as seen from the recurrence

$$f_n(k) = \sum_{j=0}^{n-2} \binom{n-1}{j} k f_j(k) + 2k f_{n-1}(k), \tag{21}$$

which may be derived by writing $f_n(k) = D^{n-1}(e^{ke^x + kx - k}(ke^x + k))|_{x=0}$, and by applying Leibnitz's formula.

5. Tables. Initial values of $W(n)$, $T(n)$, $M(n)$ and $P(n)$, calculated from (1), (2), (3), and (4), appear below.

n	$W(n)$	$T(n)$	$M(n)$	$P(n)$
1	2	2	2	1
2	16	13	10	3
3	208	123	74	13
4	3,968	1,546	730	75
5	109,568	24,283	9,002	541
6	4,793,344	457,699	133,210	4,683
7	*	10,064,848	2,299,754	47,293
8	*	252,945,467	45,375,130	545,835

We remark in conclusion that the numbers $M(n)$ and $P(n)$ satisfy some interesting congruences. It follows from (16) that

$$H_n[\lambda] = \sum_{k=0}^{n} (\lambda - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \ [1]. \tag{22}$$

Hence, $H_n[\lambda]$ is integral for all n iff $(\lambda - 1)^{-1}$ is integral. If $(\lambda - 1)^{-1}$ is integral, it follows from (22) and Fermat's Theorem that, for p an odd prime,

$$H_{n+p-1}[\lambda] \equiv H_n[\lambda] \quad (\operatorname{mod} 2p(\lambda - 1)^{-2}). \tag{23}$$

Since $H_n[3/2] = M(n)$ and $H_n[2] = P(n)$, it follows from (23) that for p an odd prime,

$$M(n + p - 1) \equiv M(n) \pmod{8p} \tag{24}$$

and

$$P(n + p - 1) \equiv P(n) \pmod{2p}, \tag{25}$$

the latter result appearing in [5].

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