MOMENT ESTIMATES FOR SOME RENORMALIZED PARABOLIC ANDERSON MODELS

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The theory of regularity structures enables the definition of the following parabolic Anderson model in a very rough environment: $\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\dot{W}_t(x)$, for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, where $\dot{W}_t(x)$ is a Gaussian noise whose space time covariance function is singular. In this rough context we shall give some information about the moments of $u_t(x)$ when the stochastic heat equation is interpreted in the Skorohod as well as the Stratonovich sense. Of special interest is the critical case, for which one observes a blowup of moments for large times.

1. Introduction. The parabolic Anderson model (sometimes abbreviated as PAM in the sequel) is a linear partial differential equation in a random environment. As for other widely studied objects, many different versions of the model have been analyzed in the literature. In this paper we are concerned with the following continuous version of PAM defined for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\dot{W}_t(x), \tag{1.1}$$

where Δ stands for the Laplace operator and $\dot{W}$ is a centered Gaussian noise. Equation (1.1) is obviously a stochastic PDE, and the generalized differential element $u_t(x)\dot{W}(x)$ will be interpreted either in the Skorohod sense (for which the product $u_t(x) \diamond \dot{W}_t(x)$ is considered as a Wick product) or in the Stratonovich sense (where the product $u_t(x)\dot{W}_t(x)$ is the usual one). Notice that we not only consider those two versions of the model for the sake of generality but also because we will transfer some information from the Skorohod to the Stratonovich equation.

The study of moments for equation (1.1) is at the heart of Anderson's initial motivation in the model. Indeed, the moments of $u_t(x)$ characterize the so-called intermittency phenomenon, as described in, for example, [24]. Moments are also related to the localization of eigenvectors for the Anderson operator $L = \frac{1}{2}\Delta + \dot{W}$, since $u_t(x)$ can be seen as the Laplace transform of the spectral measure of L (see [25], Relation (2.27)). This is why quantities of the form $\mathbf{E}[|u_t(x)|^p]$, for a given $p > 1$ and for the solution u to (1.1), have been intensively analyzed in the recent past. Let us mention [11–13] when $\dot{W}$ is a white noise in time and [4, 9, 21] for fractional noises.

The current paper can be seen as an additional step toward moment estimates for the parabolic Anderson model. Namely, our study aims at giving some information about the moments of equation (1.1) when $\dot{W}$ is a very rough environment, given as the formal derivative

of a multiparametric fractional Brownian motion W. Specifically, consider a centered Gaussian process W, indexed by $\mathbb{R}_+ \times \mathbb{R}^d$ and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose covariance is expressed as

$$\mathbf{E}[W_t(x)W_s(y)] = R_0(s, t) \prod_{j=1}^{d} R_j(x_j, y_j),$$

where $\{H_j; 0 \le j \le d\}$ is a family of Hurst indices in $(0, 1)$ and the covariance function R_j is defined by

$$(1.2) \qquad R_j(u, v) = \frac{1}{2}(|u|^{2H_j} + |v|^{2H_j} - |u - v|^{2H_j}), \quad u, v \in \mathbb{R}.$$

Then, the noise $\dot{W}$ driving equation (1.1) has to be thought of as the (ill-defined) derivative $\partial_{tx_1\cdots x_d}^{d+1} W_t(x)$. In this paper we are interested in noises which are rougher than white noise in at least some directions. Otherwise stated, we wish some of the H_j's in (1.2) to be smaller than $\frac{1}{2}$. Recall that the covariance function of $\dot{W}$ is formally written as

$$(1.3) \qquad \mathbf{E}[\dot{W}_t(x)\dot{W}_s(y)] = \gamma_0(t - s)\gamma(x - y), \quad \text{with } \gamma(x - y) \equiv \prod_{j=1}^{d} \gamma_j(y_j - x_j),$$

where each γ_j is the distributional derivative $\partial_{uv}^2 R_j$. Notice that, whenever $H_j < \frac{1}{2}$, the covariance γ_j is a distribution. Therefore, we will often express γ_0 and γ in Fourier modes as

$$(1.4) \qquad \gamma_0(t) = \int_{\mathbb{R}} e^{\iota\lambda t}\mu_0(d\lambda), \quad \text{and} \quad \gamma(x) = \int_{\mathbb{R}^d} e^{\iota\xi \cdot x}\mu(d\xi),$$

where the measures μ_0 and μ on $\mathbb{R}^d$ are, respectively, defined by

$$(1.5) \qquad \mu_0(d\lambda) = c_0|\lambda|^{1-2H_0}\,d\lambda, \quad \text{and} \quad \mu(d\xi) = c_{\mathbf{H}}\prod_{j=1}^{d}|\xi_j|^{1-2H_j}\,d\xi,$$

where $\mathbf{H}$ denotes the vector $(H_1, \ldots, H_d)$ and where $c_0, c_{\mathbf{H}}$ are explicit positive constants. We should already observe at this point that the mere existence of a solution to equation (1.1) in the rough environment, given by (1.3), requires a delicate analysis of intersection local times in the Skorohod setting [6] and a cumbersome renormalization procedure for the Stratonovich case (see [15, 16, 19] for some related models).

In order to describe the main results contained in this article, let us start with the Skorohod setting for equation (1.1), for which we will assume that $H_0 > \frac{1}{2}$. Within this framework we define a family of coefficients describing the behavior of our model. Namely, set

$$(1.6) \qquad J_* = \left\{1 \le j \le d; H_j < \frac{1}{2}\right\}, \quad d_* = |J_*|, H_* = \sum_{j \in J_*} H_j, H = \sum_{j=1}^{d} H_j.$$

We also define some similar quantities J^*, d^*, H^* for the indices such that $H_j \ge \frac{1}{2}$. Then, the reference [6] exhibits a subcritical regime, for which there is existence and uniqueness of the solution to (1.1) interpreted in the Skorohod sense. This subcritical regime is characterized by the following set of conditions on J_*, d_*, H_*:

$$(1.7) \qquad d - H < 1, \quad \text{and} \quad 4(1 - H_0) + 2(d - H) + (d_* - 2H_*) < 4.$$

Denoting by $u^\diamond$ this solution, it is also proved in [6] that $u^\diamond$ admits moments of all orders, namely,

$$\mathbf{E}[|u_t^\diamond(x)|^p] < \infty, \quad \text{for all } t \ge 0, x \in \mathbb{R}^d, \text{ and } p \ge 1.$$

In contrast with this nice situation, in the current article we will focus on the so-called critical regime. This means that (1.7) is replaced by the following condition on J_*, d_*, H_*:

$$(1.8) \qquad\qquad d - H = 1, \quad \text{and} \quad 4(1 - H_0) + (d_* - 2H_*) < 2.$$

Under condition (1.8), the moments of $u_t^\diamond(x)$ blow up for large time. This assertion will be quantified precisely in our article, leading to our first contribution (which will be stated more rigorously in Theorem 3.14).

CONTRIBUTION 1. Assume that condition (1.8) is met, and recall that $u^\diamond$ designates the Skorohod solution of equation (1.1). For all $p > 1$, we define a critical time $t_0(p)$ as

$$(1.9) \qquad\qquad t_0(p) = \frac{C_{H_0, \mathbf{H}}}{(p - 1)^{1/(2H_0 - 1)}},$$

where $C_{H_0, \mathbf{H}}$ is given by an explicit variational inequality. Then, the following holds true:

(i) For any $p \geq 2$, if $t < t_0(p)$, we have $\mathbb{E}[|u_t^\diamond(x)|^p] < \infty$ for all $x \in \mathbb{R}^d$;
(ii) For all $p > 1$, if $t > t_0(p)$, the pth moment of $u_t^\diamond(x)$ blows up.

REMARK 1.1. Our Contribution 1 gives a rather complete picture of the moments problem for the Skorohod equation (1.1) in the critical regime. In addition, it is also clear from equation (1.9) that $p \mapsto t_0(p)$ decreases from $+\infty$ to 0 as p varies in $(1, \infty)$. We firmly believe that $t_0(p)$ separates well-behaved from ill-behaved pth moments for $u^\diamond$, and this is what our Contribution 1 asserts for $p \geq 2$. However, item (ii) in Contribution 1 only yields an upper bound for the critical time $t_0(p)$ when $p \in (1, 2)$. Hence, a full characterization of $t_0(p)$ for $p < 2$ is still an open problem.

REMARK 1.2. Note that Contribution 1 covers the particular and important case, in which $d = 2$ and $H_0 = 1$, $H_1 = H_2 = \frac{1}{2}$, that is, $\dot{W}$ is the spatial white noise. In this situation the constant $C_{H_0, \mathbf{H}}$ in (1.9) is given by $\kappa(2, 2)^{-4}$, where $\kappa(2, 2)$ is the Gagliardo–Nirenberg constant for $p = 2$, $d = 2$ (see Remark 3.13 below for more details).

Our second contribution focuses on equation (1.1), interpreted in the Stratonovich sense. Assuming that the coefficients $H_0, \mathbf{H} \in (0, 1)^{d+1}$ verify

$$(1.10) \qquad\qquad d + \frac{2}{3} < 2H_0 + H \leq d + 1,$$

a global Stratonovich solution u can indeed be constructed in a suitable weighted space, using the regularity structure formalism and a renormalization procedure (see Section 4.1 for further details).

As for the Skorohod case mentioned above, we are mostly interested here in getting some information about the moments of the Stratonovich solution u. This problem is challenging in a renormalized context and, to the best of our knowledge, unaddressed in the literature. We are only aware of the reference [17] about existence of moments in small time for the two-dimensional spatial white noise case. Our next contribution, which is summarized below, aims at filling this gap.

CONTRIBUTION 2. Under the assumption (1.10), let $u_t(x)$ be the (renormalized) Stratonovich solution of equation (1.1) constructed from the regularity structure formalism. Then, the following holds true:

Under the additional subcritical assumption (1.7), the random variable $u_t(x)$ admits moments of all orders. More specifically, we have

$$\mathbf{E}\big[|u_t(x)|^p\big] < \infty, \quad \text{for all } t \geq 0, x \in \mathbb{R}^d, \text{ and } p \geq 1.$$

REMARK 1.3. To the best of our knowledge, the above Contribution 2 is the first result establishing moments of any order in a renormalized Stratonovich setting.

REMARK 1.4. On top of [17], let us mention the work [1] in which the authors study equation (1.1), driven by a spatial white noise on a 2-d torus. The exponential tail bounds for the minimal eigenvalue of the Anderson operator established in [1] would certainly lead to a blowup of moments result for equation (1.1). However, their setting is restricted to the spatial white noise on a compact 2-d space, while we are considering a more general class of noises on the whole space $\mathbb{R}^d$ for all d. The 2-d white noise analyzed in [1] happens to be a critical case within our more global picture. It should also be highlighted that our approach aims at obtaining the critical exponential integrability of the solution, instead of a simple nonquantified exponential integrability.

Gathering our Contributions 1 and 2, our main goal in this paper is thus to exhibit the complexity of the moments problem for the PAM in very rough environments.

Let us say a few words about the methodology invoked in order to achieve the main contributions summarized above. We have relied on the following tools:

(a) In order to analyze the moments of the Skorohod solution in the critical case (i.e Contribution 1), we hinge on the Feynman–Kac representation of $u^\diamond$. This representation involves intersection local times of a Brownian motion weighted by the covariance function of the noise $\dot{W}$. We then proceed to a thorough analysis of those intersection local times, thanks to a delicate truncation procedure involving a Girsanov-type transform on the Brownian paths. Notice that the basic aim of the truncation mechanism is (in the end) to apply some classical asymptotic results for Feyman–Kac functionals. For the case of pth moments for $p \in (1, 2)$ (i.e., item (ii) in our Contribution 1), we use an additional ingredient based on $\mathcal{S}$-transforms from white noise analysis.

(b) The fact that the Stratonovich solution admits moments of any order (Contribution 2 item 2) stems again from Feynman–Kac representations. More precisely, the difference between the Feynman–Kac representations of u and $u^\diamond$ corresponds to some fluctuations of the Brownian intersection local times. Then, the bulk of our analysis consists in quantifying the fact that the intersection local time fluctuations are negligible with respect to the term involving $\dot{W}$.

As one can see, Feynman–Kac representations are the key ingredient in order to get our main contributions. Those are combined throughout the paper with regularity structures and Malliavin calculus elements. As a conjecture, let us also recall from Remark 1.1 that $t_0(p)$ defined by (1.9) seems to be the critical time for the moments of $u^\diamond$ when $p < 2$ (as well as for the moments of the Stratonovich solution u in the critical case). We hope to prove this claim in the future, although it currently seems to resist our method of analysis.

The study is organized as follows. First, in Section 2 we will recall some basics about the Malliavin calculus setting and the Skorohod integration procedure. This preliminary material will in fact set the stage for Section 3, where equation (1.1) will be investigated in the Skorohod sense, leading to the above-described Contribution 1. Then, in Section 4 we will turn to the Stratonovich interpretation of the model, starting with a brief reminder on how the equation is interpreted and solved in a rough regime (Section 4.1). Our second main Contribution 2 will be the topic of the subsequent Section 4.2 which will conclude the study.

NOTATION 1.5. In the sequel we set $(\Omega, \mathcal{F}, \mathbf{P})$ for the probability space related to W, with $\mathbf{E}$ for the related expected value. We denote by $(\hat{\Omega}, \mathcal{G}, \mathbb{P})$ the probability spaces corresponding to the Brownian motions in Feynman–Kac representations with a related expected value $\mathbb{E}$. The heat kernel on $\mathbb{R}^d$ is denoted by $p_t(x)$ and recall that

$$(1.11) \qquad p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

Also notice that the inner product of $a, b \in \mathbb{R}^d$ is written as $a \cdot b$ throughout the paper. In the following we will often deal with product measures and for which we adopt the following convention: for $m, n \geq 1$, a measure ν on $\mathbb{R}^m$ and $\xi = (\xi_1, \ldots, \xi_n) \in (\mathbb{R}^m)^n$, we set

$$(1.12) \qquad \nu(d\xi) = \nu^{\otimes n}(d\xi) = \prod_{k=1}^{n} \nu(d\xi_k).$$

Finally, recall that we write $\mathbf{H}$ for the vector of Hurst parameters $(H_1, \ldots, H_d)$.

2. Elements of Malliavin calculus.
In this section we recall the basic Malliavin calculus notation which will be invoked in the forthcoming computations. The reader is sent to [21] and [27] for more details.

Let us start by introducing some basic notions on Fourier transforms of functions. The space of real valued infinitely differentiable functions with compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. We write $\mathcal{S}(\mathbb{R}^d)$ or, simply, $\mathcal{S}$ for the space of Schwartz functions. Its dual, the space of tempered distributions, is $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{S}'$. If f is a vector of n tempered distributions on $\mathbb{R}^d$, then we write $f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n)$. In the sequel we will play with two different kinds of Fourier transforms on $\mathbb{R}^{d+1}$. Namely, for a function $f(t, x)$ on $\mathbb{R}^{d+1}$, the Fourier transform on the full space-time domain $\mathbb{R}^{d+1}$ is defined with the normalization

$$(2.1) \qquad \mathcal{F}f(\eta, \xi) = \int_{\mathbb{R}^{d+1}} e^{-\iota(t\eta + \xi \cdot x)} f(t, x) \, dt \, dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\eta, \xi) = \frac{\mathcal{F}f(-t, -\xi)}{(2\pi)^{d+1}}$. Our analysis will also rely on a Fourier transform denoted by $\mathcal{F}^{\mathrm{S}}$ which is defined on the spatial variables only. It is given by

$$(2.2) \qquad \mathcal{F}^{\mathrm{S}}f(t, \xi) = \int_{\mathbb{R}^d} e^{-\iota \xi \cdot x} f(t, x) \, dx.$$

2.1. *Wiener space related to W.* On a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider a Gaussian noise W, encoded by a centered Gaussian family $\{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$. As mentioned in the Introduction (see relation (1.4)), the covariance structure of W is given by

$$(2.3) \qquad \mathbf{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^d} \mathcal{F}^{\mathrm{S}}\varphi(s_1, \xi) \bar{\mathcal{F}}^{\mathrm{S}}\psi(s_2, \xi) \gamma_0(s_1 - s_2) \, ds_1 \, ds_2 \, \mu(d\xi),$$

where γ_0 and μ are, respectively, defined by (1.3) and (1.5) and where $\mathcal{F}^{\mathrm{S}}\varphi$ designates the spatial Fourier transform of φ. One can thus consider W as an isonormal Gaussian family $\{W(\varphi); \varphi \in \mathcal{H}\}$ on a space $\mathcal{H}$, obtained as the completion of Schwartz functions with respect to the inner product, given by the right-hand side of (2.3).

We refer to [27] for a detailed account of the Malliavin calculus with respect to a Gaussian process, while we focus here on the basic definitions allowing to state our results in the remainder of the article. We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \ldots, W(\phi_n)),$$

with $\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ (namely, f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$, and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = (\mathbf{E}[F^2] + \mathbf{E}[\|DF\|_{\mathcal{H}}^2])^{1/2}.$$

For any integer $n \geq 0$, we denote by $\mathcal{H}_n$ the nth Wiener chaos of W. We recall that $\mathcal{H}_0$ is simply $\mathbb{R}$, and, for $n \geq 1$, $\mathcal{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(h)); h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_n is the nth Hermite polynomial. For any $n \geq 1$, we denote by $\mathcal{H}^{\otimes n}$ (resp., $\mathcal{H}^{\odot n}$) the nth tensor product (resp., the nth symmetric tensor product) of $\mathcal{H}$. Then, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to a linear isometry between $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$.

Consider now a random variable $F \in L^2(\Omega)$ which is measurable with respect to the σ-field $\mathcal{F}^W$ generated by W. This random variable can be expressed as

$$(2.4) \qquad F = \mathbf{E}[F] + \sum_{n=1}^\infty I_n(f_n),$$

where the series converges in $L^2(\Omega)$. Moreover, the functions $f_n \in \mathcal{H}^{\odot n}$ are determined by F. Identity (2.4) is called the Wiener chaos expansion of F.

2.2. *Extended Skorohod integrals*. The standard definition of Skorohod's integral is obtained by a duality relation in $L^2(\Omega)$. Namely, we denote by $\delta^\diamond$ the L^2-adjoint of the derivative operator given by the duality formula

$$(2.5) \qquad \mathbf{E}[\delta^\diamond(u)F] = \mathbf{E}[\langle DF, u \rangle_{\mathcal{H}}]$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathcal{H})$ in the domain of $\delta^\diamond$. More specifically, the domain of $\delta^\diamond$ is defined as the set of processes $u \in L^2(\Omega; \mathcal{H})$ such that, for all cylindrical functions F, we have

$$|\mathbf{E}[\langle DF, u \rangle_{\mathcal{H}}]| \leq C_u \|F\|_{L^2(\Omega)}.$$

The operator $\delta^\diamond$ is also called the *Skorohod integral*, because in the Brownian motion case it coincides with an extension of the Itô integral introduced by Skorohod.

In the sequel we consider random variables which are only L^p-integrable with $p \in (1, 2)$. We thus need the following extension of the Skorohod integral with respect to the noise W.

DEFINITION 2.1. Fix $p > 1$. Let $u = \{u(t, y), t \geq 0, y \in \mathbb{R}^d\}$ be a random field such that

$$\|\|u(t, y)\|_{\mathcal{H}}\|_{L^p(\Omega)} < \infty.$$

Then, we say that u is Skorohod integrable in L^p, denoted by $u \in \text{Dom}_p \delta^\diamond$, if, for any smooth and cylindrical random variable $F \in \mathcal{S}$, we have

$$|\mathbf{E}[\langle DF, u \rangle_{\mathcal{H}}]| \leq c_u \|F\|_{L^q(\Omega)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. We say that u is Skorohod integrable if $u \in \text{Dom}\delta^\diamond = \bigcup_{p>1} \text{Dom}_p \delta^\diamond$. If $u \in \text{Dom}_p \delta^\diamond$, the Skorohod integral

$$\delta^\diamond(u) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} u_s(y) \, d^\diamond W_s(y)$$

is the random variable in $L^p(\Omega)$, defined by the duality relationship

$$\mathbf{E}[\langle DF, u \rangle_{\mathcal{H}}] = \mathbf{E}[\delta^\diamond(u)F], \quad \text{for all } F \in \mathcal{S}.$$

3. Skorohod case. This section focuses on equation (1.1), interpreted in the Skorohod sense. In Section 3.1 below, we first recall some existence and uniqueness results for the equation. Then, we will give some information about the moments in the subcritical case.

3.1. *Existence and uniqueness results.* In this section we first give some existence and uniqueness results for equation (1.1) considered in the Skorohod sense. Those results are mainly taken from [6], although we will also deal with extended solutions in the sense of Definition 2.1. Specifically, we consider equation (1.1), considered in the Skorohod sense, that is,

$$(3.1) \qquad \partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x) \diamond \dot{W}_t(x), \quad t \in \mathbb{R}_+, x \in \mathbb{R}^d.$$

Recall from Notation 1.5 that we denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-d/2}e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}^d$. We define the solution of equation (3.1) as follows.

DEFINITION 3.1. Let $T > 0$. Consider a random field $u = \{u_t(x); 0 \leq t \leq T, x \in \mathbb{R}^d\}$ such that there exists $p > 1$, satisfying

$$(3.2) \qquad \sup_{0 \leq s \leq T} \sup_{y \in \mathbb{R}^d} \|u_s(y)\|_{L^p(\Omega)} < \infty.$$

Then, u is said to be a mild solution to equation (3.1) up to time T with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if, for any $t \geq 0$ and $x \in \mathbb{R}^d$, the process $\{u_s(y); s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod integrable in the sense of Definition 2.1, and the following equation holds for any $t \geq 0$ and $x \in \mathbb{R}^d$:

$$(3.3) \qquad u_t(x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)u_s(y) \, d^{\diamond}W_s(y).$$

If we impose $p = 2$ in Definition 3.1, then our definition coincides with the usual notion of Skorohod solution for equation (1.1). In the subcritical regime L^p, moments are available for all $p \geq 2$, and, therefore, equation (3.1) can be solved in the usual sense. We recall this result (taken from [6]) for the sake of completeness.

PROPOSITION 3.2. *We assume that the coefficients $H_0, \ldots, H_d$ satisfy the subcritical relation* (1.7) *and that $H_0 > \frac{1}{2}$. Then, equation* (3.1) *admits a unique solution, considered in the sense of Definition* 3.1 *with $p = 2$.*

REMARK 3.3. Proposition 3.2 has been extended in [7] to the case $H_0 < 1/2$.

In contrast with the subcritical situation, in the critical case one can only guarantee the existence of a usual Skorohod solution up to a critical time. This is the content of the following proposition, also borrowed from [6].

PROPOSITION 3.4. *Assume that the critical condition* (1.8) *on H_0 and $\mathbf{H}$ holds true, where we recall that the vector $\mathbf{H}$ is introduced in Notation* 1.5. *Then, one can find a strictly positive time $t^* = t^*(H_0, \mathbf{H}, d)$ such that there exists a unique $L^2(\Omega)$-mild solution $u^{\diamond}$ of equation* (3.3) *for $t \in [0, t^*)$.*

Proposition 3.4 opens the way to a possible definition of the solution to equation (3.3) in the L^p sense for $p \in (1, 2)$. While the existence of such a solution relies heavily on moment bounds which are obtained through Feynman–Kac representations, uniqueness holds true by just invoking the fact that we are dealing with a linear equation. This is summarized in the following proposition.

PROPOSITION 3.5. *As in Proposition* 3.4, *we assume that the critical condition* (1.8) *on* H_0 *and* **H** *holds true. For* $p \in (1, 2)$, *suppose that one can find a strictly positive time* $t_p^* = t^*(p, H_0, \mathbf{H}, d)$ *such that there exists a* $L^p(\Omega)$-*mild solution* $u^\diamond$ *of equation* (3.3) *on the interval* $[0, t_p^*)$. *Then, this solution is unique.*

PROOF. Suppose that u and v are two mild solutions in the L^p sense (given by Definition 3.1) with the same initial condition. Set $w = u - v$. Then, w satisfies

$$(3.4) \qquad w_t(x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) w_s(y) \, d^\diamond W_s(y).$$

Iterating this relation, we get

$$(3.5) \qquad w_t(x) = \int_0^t \int_{\mathbb{R}^d} \left(\int_0^s \int_{\mathbb{R}^d} p_{t-s}(x - y) p_{s-r}(y - z) w_r(z) \, d^\diamond W_r(z) \right) d^\diamond W_s(y).$$

That is, $w_t(x)$ is an iterated Skorohod integral. By Proposition 2.6 and Proposition 2.7 in [28], this iterated Skorohod integral coincides with a double Skorohod integral (notice that, in the reference [28], these results are proved for Skorohod integrals in $L^2(\Omega)$, but they can be easily extended to the Skorohod integral in $L^p(\Omega)$ introduced in Definition 2.1). Moreover, using (3.4), (3.5) and iterated versions in higher order chaoses, one can prove that $w_t(x)$ is orthogonal to any element in a finite Wiener chaos. This does not imply immediately that $w_t(x) = 0$, because $w_t(x) \in L^p(\Omega)$ for some $p > 1$. However, given any random variable $F \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$, we can approximate F in L^q by a sequence F_n of elements on a finite Wiener chaos. Then, $\mathbf{E}[w_t(x)F] = \lim_{n\to\infty} \mathbf{E}[w_t(x)F_n] = 0$ which implies $w_t(x) = 0$. □

3.2. *Feynman–Kac representation.* In order to analyze the moments of $u_t^\diamond(x)$ and compare the Skorohod and Stratonovich settings for equation (1.1), we first give a Feynman–Kac representation for an approximating sequence of the solution $u^\diamond$ to equation (3.1). To this end, let p_ϵ be a $d + 1$ dimensional heat kernel given by (1.11). To separate its first coordinate from the others, we write $p_\epsilon(\hat{x}) = p_\epsilon(x_0, x)$ with $\hat{x} = (x_0, x) \in \mathbb{R}^{d+1}$. Consider now the smoothed noise

$$(3.6) \qquad W_t^\epsilon(x) = (p_\epsilon * \dot{W})_t(x) = W(p_\epsilon(t - \cdot, x - \cdot)).$$

In the same spirit as (1.3), we denote by γ_0^ϵ and γ^ϵ the corresponding covariance functions (in time and space) of W^ϵ with μ_0^ϵ and μ^ϵ their spectral measures. Similar to (1.5), it is readily checked that

$$(3.7) \qquad \mu_0^\varepsilon(d\lambda) = c_0 e^{-\epsilon|\lambda|^2} |\lambda|^{1-2H_0} \, d\lambda, \quad \text{and} \quad \mu^\varepsilon(d\xi) = c_{\mathbf{H}} e^{-\epsilon|\xi|^2} \prod_{j=1}^d |\xi_j|^{1-2H_j} \, d\xi.$$

With the above notions in hand, we set

$$(3.8) \qquad V_t^{\varepsilon, B}(x) = \int_0^t W_{t-s}^\epsilon(B_s^x) \, ds = W(\phi),$$

where the second notation $W(\phi)$ corresponds to the Wiener integral introduced in (2.3) and where the function ϕ is given by

$$(3.9) \qquad \phi(s, y) = \int_0^t p_\epsilon(u - s, B_{t-u}^x - y) \, du.$$

Moreover, in relation (3.9) B^x stands for a d-dimensional Brownian motion independent of W with initial condition $x \in \mathbb{R}^d$. The functional (3.8) gives rise to an approximate solution of the Skorohod equation, as stated in the following proposition.

PROPOSITION 3.6. *For $\varepsilon > 0$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^2$, let $V_t^{\varepsilon,B}(x)$ be defined by* (3.8), *and set*

$$(3.10) \qquad \beta_t^{\varepsilon,B} \equiv \int_{[0,t]^2} \int_{\mathbb{R}^{d+1}} e^{-\varepsilon(|\xi|^2+|\lambda|^2)} e^{\iota(\xi\cdot(B_{s_1}-B_{s_2})+\lambda(s_2-s_1))} \mu_0(d\lambda)\mu(d\xi)\,ds_1\,ds_2,$$

where we write $B^0 = B$ for notational sake, where μ^ϵ is introduced in (3.7) *and where γ_0^ϵ is the inverse Fourier transform of μ_0^ε. We define $u^{\varepsilon,\diamond}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ by*

$$(3.11) \qquad\qquad u_t^{\varepsilon,\diamond}(x) = \mathbb{E}\big[e^{V_t^{\varepsilon,B}(x)-\frac{1}{2}\beta_t^{\varepsilon,B}}\big],$$

where we recall that the expectation $\mathbb{E}$ has been introduced in Notation 1.5. *Then, the process $u^{\varepsilon,\diamond}$ is the unique solution to the following Skorohod-type equation*:

$$(3.12) \qquad \partial_t u_t^{\varepsilon,\diamond}(x) = \frac{1}{2}\Delta u_t^{\varepsilon,\diamond}(x) + u_t^{\varepsilon,\diamond}(x) \diamond \dot{W}^\varepsilon(x), \quad t \in \mathbb{R}_+, x \in \mathbb{R}^d,$$

where $\dot{W}^\varepsilon$ is the smoothed noise $\dot{W}^\varepsilon = \dot{W} * p_\varepsilon$, defined in (3.6).

PROOF. Similarly to the proof of Proposition 5.2 in [22], we can show that

$$u_t^{\varepsilon,\diamond}(x) = \mathbb{E}\bigg[\exp\bigg(V_t^{\varepsilon,B}(x) - \frac{1}{2}\mathbf{E}\big[|V_t^{\varepsilon,B}(x)|^2\big]\bigg)\bigg].$$

Now, a direct application of (2.3) reveals that

$$(3.13) \qquad \begin{aligned} \mathbf{E}\big[|V_t^{\varepsilon,B}(x)|^2\big] &= \int_{[0,t]^2} \int_{\mathbb{R}^{d+1}} e^{-\varepsilon(|\xi|^2+|\lambda|^2)} e^{\iota(\xi\cdot(B_{s_1}-B_{s_2})+\lambda(s_2-s_1))} \\ &\qquad \times \mu_0(d\lambda)\mu(d\xi)\,ds_1\,ds_2 \\ &= \beta_t^{\varepsilon,B}, \end{aligned}$$

which proves our claim. $\square$

We now prove the convergence of the regularized Feynman–Kac representation to the solution of the Skorohod equation (3.1).

PROPOSITION 3.7. *Let $u^{\varepsilon,\diamond}$ be the process defined by* (3.11), *and recall that $u^\diamond$ designates the solution to equation* (3.1), *as given in Proposition* 3.2 *or Proposition* 3.4. *We assume that one of the following situations is met*:

   (i) *The subcritical condition* (1.7) *is satisfied, and t is any positive number.*
   (ii) *The critical assumption* (1.8) *prevails, and $t \in [0, t^*)$, where t^* is defined in Proposition* 3.4.

Then, for all $x \in \mathbb{R}^d$, the random variable $u_t^{\varepsilon,\diamond}(x)$ converges to $u_t^\diamond(x)$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

PROOF. We will prove the proposition under condition (ii), the other case being handled similarly. In order to show the L^2-convergence of $u_t^{\diamond,\varepsilon}$, we consider the quantity α_t (also denoted by α_t^{12} in the sequel) defined by

$$(3.14) \qquad \alpha_t \equiv \int_{[0,t]^2} \int_{\mathbb{R}^d} e^{\iota(\xi\cdot(B_{s_1}^1-B_{s_2}^2)+\lambda(s_1-s_2))} \mu_0(d\lambda)\mu(d\xi)\,ds_1\,ds_2.$$

In the above, B^1, B^2 are two independent Brownian motions starting from x which are also independent of the noise W. Then, in order to prove that $u_t^{\varepsilon,\diamond}(x)$ converges in $L^2(\Omega)$, it is sufficient to verify that

$$(3.15) \qquad\qquad \lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbf{E}\big[u_t^{\varepsilon_1,\diamond}(x)u_t^{\varepsilon_2,\diamond}(x)\big] = \sum_{n=1}^{\infty} \frac{\mathbb{E}[\alpha_t^n]}{n!} < \infty.$$

Indeed, this follows from the fact that, for a sequence of random variable Z_n, if one has $\lim_{m,n\to\infty} \mathbb{E}Z_n Z_m$ exists, then Z_n is a Cauchy sequence in $L^2(\Omega)$. In order to prove (3.15), we resort to expression (3.11) which yields

$$\mathbf{E}\big[u_t^{\varepsilon_1,\diamond}(x)u_t^{\varepsilon_2,\diamond}(x)\big]$$
$$= \mathbf{E}\bigg[\mathbb{E}\bigg[\exp\Big(V_t^{\varepsilon_1,B^1}(x) + V_t^{\varepsilon_2,B^2}(x) - \frac{1}{2}\big(\beta_t^{\varepsilon_1,B^1} + \beta_t^{\varepsilon_2,B^2}\big)\Big)\bigg]\bigg].$$

Applying Fubini's theorem, we thus get

$$\mathbf{E}\big[u_t^{\varepsilon_1,\diamond}(x)u_t^{\varepsilon_2,\diamond}(x)\big]$$

(3.16)
$$= \mathbb{E}\bigg[\exp\Big(\frac{1}{2}\mathbf{E}\big[(V_t^{\varepsilon_1,B^1}(x) + V_t^{\varepsilon_2,B^2}(x))^2\big] - \frac{1}{2}\big(\beta_t^{\varepsilon_1,B^1} + \beta_t^{\varepsilon_2,B^2}\big)\Big)\bigg].$$

Moreover, recall that $\beta_t^{\varepsilon,B} = \mathbf{E}[|V_t^{\varepsilon,B}(x)|^2]$, according to (3.13). Plugging this identity into (3.16), we thus get

$$\mathbf{E}\big[u_t^{\varepsilon_1,\diamond}(x)u_t^{\varepsilon_2,\diamond}(x)\big] = \mathbb{E}\big[e^{\alpha_t^{\varepsilon_1,\varepsilon_2}}\big],$$

where the quantity $\alpha_t^{\varepsilon_1,\varepsilon_2}$ is defined by

(3.17)
$$\alpha_t^{\varepsilon_1,\varepsilon_2} = \mathbf{E}\big[V_t^{\varepsilon_1,B^1}(x)V_t^{\varepsilon_2,B^2}(x)\big]$$
$$= \int_{[0,t]^2}\int_{\mathbb{R}^{d+1}} e^{-\frac{(\varepsilon_1+\varepsilon_2)}{2}(|\xi|^2+|\lambda|^2)} e^{\iota(\xi\cdot(B_{s_1}^1 - B_{s_2}^2)+\lambda(s_2-s_1))}$$
$$\times \mu_0(d\lambda)\mu(d\xi)\,ds_1\,ds_2.$$

Notice that, for fixed $\varepsilon_1, \varepsilon_2 > 0$, the fact that $\alpha_t^{\varepsilon_1,\varepsilon_2}$ is well defined (and, in fact, bounded) and stems easily from the presence of the exponential term $e^{-(\varepsilon_1+\varepsilon_2)|\xi|^2}$ in the right-hand side of (3.17). In addition, [6], inequality (3.1), implies that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}[\alpha_t^n]}{n!} < \infty,$$

for $t < t^*$. Since

$$\int_{[0,t]^2} \mathbb{E}\big[e^{\iota(\xi\cdot(B_{s_1}^1 - B_{s_2}^2)+\lambda(s_1-s_2))}\big]\,ds_1\,ds_2 = \bigg|\int_0^t \mathbb{E}\big[e^{\iota(\xi\cdot B_{s_1}^1+\lambda s_1)}\big]\,ds_1\bigg|^2 \geq 0,$$

by the method of replicas, one can show that $\mathbb{E}[(\alpha_t^{\varepsilon_1,\varepsilon_2})^n] \geq 0$ for all integer $n \geq 0$ and that $\mathbb{E}[(\alpha_t^{\varepsilon_1,\varepsilon_2})^n] \uparrow \mathbb{E}[\alpha_t^n]$ as $\varepsilon_1, \varepsilon_2 \downarrow 0$. Then, taking into account that $\alpha^{\varepsilon_1,\varepsilon_2}$ is bounded for fixed $\varepsilon_1, \varepsilon_2$, an easy monotone convergence argument together with a Taylor series expansion of the exponential function yields

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbf{E}\big[u_t^{\varepsilon_1,\diamond}(x)u_t^{\varepsilon_2,\diamond}(x)\big] = \lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbb{E}\big[e^{\alpha_t^{\varepsilon_1,\varepsilon_2}}\big] = \sum_{n=1}^{\infty} \frac{\mathbb{E}[\alpha_t^n]}{n!}$$

which is our claim (3.15). We have thus obtained the L^2 convergence of $u^{\varepsilon,\diamond}$.

We now claim that the process $u_t^\diamond(x)$, defined as the limit of $u_t^{\varepsilon,\diamond}(x)$, is a mild solution to equation (3.1) in the sense of Definition 3.1 with $p = 2$. First notice that (3.2) holds for the process $u^\diamond$ because of the relation, for all small $\epsilon > 0$,

$$\sup_{\varepsilon>0}\sup_{t\leq t^*-\epsilon}\sup_{x\in\mathbb{R}^2} \big\|u_t^{\varepsilon,\diamond}(x)\big\|_{L^2(\Omega)} < \infty,$$

which stems from (3.15). Then, owing to relation (2.5), $u^\diamond$ satisfies (3.3) in the Skorohod sense if for any smooth and cylindrical random variable F, we have

$$(3.18) \qquad \mathbf{E}[F(u_t^\diamond(x) - p_t u_0(x))] = \mathbf{E}[\langle DF, G \rangle_\mathcal{H}],$$

where the process G is defined by

$$G_s(y) = p_{t-s}(x - y)u_s^\diamond(y)\mathbf{1}_{[0,t]}(s).$$

Relation (3.18) is obtained by taking limits on a similar equation for $u_t^{\varepsilon,\diamond}(x)$ (see [21], Theorem 3.6, for a similar argument). □

3.3. *Moments estimates in the subcritical case.* In this section we recall some moment estimates and representations established in [6] for the Skorohod equation. We label those results here since they will be invoked in order to get integrability results for the Stratonovich equation. We start with a bound on the L^p moments.

PROPOSITION 3.8. *Assume that the subcritical condition* (1.7) *is satisfied, and let $u^\diamond$ be the unique solution of equation* (3.1). *Then, for all $t \geq 0$ and $p \geq 1$, we have*

$$\mathbf{E}[|u_t^\diamond(x)|^p] = c_{p,t} < \infty.$$

Let us now define a generalization of the quantity α_t, defined by (3.14), which is used for moment representations. Namely, for two coordinates $1 \leq j_1, j_2 \leq p$, we set

$$(3.19) \qquad \alpha_t^{j_1 j_2} = \int_{[0,t]^2} \gamma(B_{s_1}^{j_1} - B_{s_2}^{j_2})\gamma_0(s_1 - s_2)\,ds_1\,ds_2,$$

where γ and γ_0 are the (generalized) covariance function given by (1.3) and B^j, $j = 1, \ldots, p$ are independent Brownian motions starting from x. With this notation in hand and thanks to the representation (3.11) for the approximation $u^{\varepsilon,\diamond}$ of $u^\diamond$, we also get the following representation for the moments of $u_t^\diamond(x)$.

PROPOSITION 3.9. *We suppose that the subcritical condition* (1.7) *is fulfilled, and recall that $u^\diamond$ is the unique solution of equation* (3.1). *Then, for all $t \geq 0$ and any integer $p \geq 2$, we have*

$$(3.20) \qquad \mathbf{E}[|u_t^\diamond(x)|^p] = \mathbb{E}\left[\exp\left(\sum_{1 \leq j_1 < j_2 \leq p} \alpha_t^{j_1 j_2}\right)\right],$$

where $\alpha_t^{j_1, j_2}$ is introduced in (3.19). Moreover, for any $\lambda > 0$, one has

$$\mathbb{E}[\exp(\lambda \alpha_t^{j_1 j_2})] < \infty.$$

3.4. *Critical moments for* $\mathbf{p} \geq 2$. We assume throughout this section that the critical condition (1.8) is satisfied. In this case we will see that the moments of $u^\diamond$ blow up after a critical time t^*, and we shall identify the time t^* in some cases. Before we proceed to the proof of this fact, let us introduce some functional spaces and inequalities of interest.

DEFINITION 3.10. Let $W^{1,2}(\mathbb{R}^d)$ be the usual Sobolev space of order $(1, 2)$ in $\mathbb{R}^d$. We define a subset $\mathcal{A}_d$ of functions, defined on $[0, 1] \times \mathbb{R}^d$, as follows:

$$\mathcal{A}_d = \left\{ g : [0, 1] \times \mathbb{R}^d \to \mathbb{R}; \, g(s, \cdot) \in W^{1,2}(\mathbb{R}^d), \text{ and } \int_{\mathbb{R}^d} g^2(s, x)dx = 1 \text{ for all } s \in [0, 1] \right\}.$$

It is readily checked that there exists a constant C such that the following inequality holds for all $g \in \mathcal{A}_d$:

(3.21)
$$\int_{[0,1]^2} \int_{(\mathbb{R}^d)^2} \gamma_0(s-t)\gamma(x-y)g^2(s,x)g^2(t,y)\,dx\,dy\,ds\,dt$$
$$\leq C^4 \int_{[0,1]\times\mathbb{R}^d} |\nabla_x g(s,x)|^2\,ds\,dx.$$

REMARK 3.11. When the noise W is time independent, that is, when $H_0 = 1$ (hence, $\gamma_0 \equiv 1$), equality (3.21) can be established using the same method as in [2]. The general case then follows from [5], Lemma 5.2.

NOTATION 3.12. We call $\kappa = \kappa(H_0, \mathbf{H})$ the best constant in inequality (3.21).

REMARK 3.13. The classical Gagliardo–Nirenberg inequality asserts that for $p(d-2) \leq d$ we have

(3.22)
$$\|f\|_{L^{2p}(\mathbb{R}^d)} \leq \kappa \|\nabla f\|_{L^2(\mathbb{R}^d)}^{\frac{d(p-1)}{2p}} \cdot \|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d(p-1)}{2p}}.$$

The best constant κ in the above inequality is called the Gagliardo–Nirenberg constant. Now in equation (3.21), consider the special case when $d = 2$, $p = 2$, and $H_0 = 1$, $H_1 = H_2 = 1/2$. In this situation it is clear that the quantity κ defined in Notation 3.12 coincides with the Gagliardo–Nirenberg constant in relation (3.22).

We can now state the main result of this section which identifies the exact time of blowup for the integer moments of $u_t(x)$ in the critical case.

THEOREM 3.14. *We suppose that condition* (1.8) *is met for our indices H_0 and H. For any integer $p \geq 2$, we define a critical time $t_0(p)$ by*

$$t_0(p) = \left(\frac{1}{\kappa^4(p-1)}\right)^{1/(2H_0-1)},$$

where we recall that κ is the constant introduced in Notation 3.12. For $\epsilon > 0$, let $u^{\epsilon,\diamond}$ be the unique solution to equation (3.12). *Then, the following assertions hold true:*

(i) *If $t < t_0 \equiv t_0(2) = \kappa^{-4/(2H_0-1)}$, then, for every $x \in \mathbb{R}^d$, the sequence $\{u_t^{\epsilon,\diamond}(x), \epsilon > 0\}$ converges in $L^2(\Omega)$ to an element $u_t^\diamond(x)$. The process $\{u_t^\diamond(x) : t \in [0,t_0), x \in \mathbb{R}^d\}$ is said to be the solution of equation* (3.1).

(ii) *For any $p \geq 2$, if $t < t_0(p)$, the sequence $\{u_t^{\epsilon,\diamond}(x); \epsilon > 0\}$ also converges in $L^p(\Omega)$ to $u_t^\diamond(x)$.*

(iii) *For $p \in (1,\infty)$, if $t > t_0(p)$ and $x \in \mathbb{R}^d$, we have*

$$\lim_{\epsilon\to 0} \|u_t^{\epsilon,\diamond}(x)\|_{L^p(\Omega)} = \infty.$$

REMARK 3.15. In the critical case when $p \in [1, 2)$, the notion of solution to equation (3.1) or (3.3) is not as clear as in Section 3.2. However, let us mention the following facts:

(a) In Theorem 3.14 we are able to identify the optimal value of t^* in Proposition 3.4 (also appeared later in Proposition 3.7) to be $t_0(2)$ which has an explicit expression given above.

(b) If $p \geq 2$ and $t < t_0(p)$, then we can take limits in equation (3.12) and show that the limit $\{u_t^\diamond(x); t \geq 0, x \in \mathbb{R}^d\}$ still solves the Skorohod equation (3.3). We have not provided details for the sake of conciseness.

(c) Whenever $p \in (1, \infty)$ and $t > t_0(p)$, one can not take limits in equation (3.3), even if one resorts to the extended Skorohod setting of Definition 2.1. However, item (iii) in Theorem 3.14 asserts that if we could give a meaning to equation (3.3) by a regularization procedure, then its solution $u_t(x)$ would not belong to $L^p(\Omega)$.

(d) Our current techniques do not allow to assert the L^p-convergence of u^ϵ when $p \in (1, 2)$ and $t < t_0(p)$. Neither are we able to prove the existence of a L^p-solution of (3.3) for $p < 2$, even if we invoke the extended Skorohod setting. On the other hand, the definition of $t_0(p)$ in Theorem 3.14 clearly allows p to be less than 2. It is, therefore, reasonable to conjecture that, when $p \in (1, 2)$ and $t < t_0(p)$, u^ϵ converges in L^p to the unique solution of equation (3.3) in the extended Skorohod setting. Provided this can be achieved, Proposition 3.5 establishes the uniqueness of the aforementioned extended solution.

The proof of Theorem 3.14 is split in the sections below.

3.5. *Proof of the* $\mathbf{L^p}$-*convergence in Theorem* 3.14. In this section we work under the critical assumption (1.8). Before we proceed to the proof of the relation $\mathbf{E}[|u_t^\diamond(x)|^p] < \infty$ for t small enough, we first state a general subadditivity result which is useful for our next computations. It is borrowed from [6], Lemma 6.1.

LEMMA 3.16. *Let* $\nu(d\xi)$ *be a measure on* $\mathbb{R}^d$, *and consider two* $\mathbb{R}^d$-*valued independent Brownian motions* B *and* $\widetilde{B}$. *For* $t \geq 0$, *we introduce the random variable* $\mathcal{H}_t$, *defined by*

$$\mathcal{H}_t = \int_{\mathbb{R}^d} \nu(d\xi) \left[\int_0^t e^{\iota \xi \cdot B_s} \, ds \right] \left[\int_0^t e^{-\iota \xi \cdot \widetilde{B}_s} \, ds \right].$$

Then, for any $t_1, t_2 > 0$ *and* $\theta \in \mathbb{R}_+$, *the following inequality holds true*:

(3.23)
$$\sum_{n=0}^\infty \frac{\theta^n}{n!} \left\{ \frac{1}{(t_1 + t_2)^n} \mathbb{E}[\mathcal{H}_{t_1+t_2}^n] \right\}$$
$$\leq \left(\sum_{n=0}^\infty \frac{\theta^n}{n!} \left\{ \frac{1}{t_1^n} \mathbb{E}[|\mathcal{H}_{t_1}|^n] \right\} \right) \left(\sum_{n=0}^\infty \frac{\theta^n}{n!} \left\{ \frac{1}{t_2^n} \mathbb{E}[\mathcal{H}_{t_2}^n] \right\} \right),$$

whenever the right-hand side is finite.

REMARK 3.17. Although stated in a different way, [6], Lemma 6.1, is indeed proved by showing (3.23) above (see inequality (6.5) in [6]).

We will now separate the upper bound computations for $u_t^\diamond(x)$ into a L^2 bound and L^p bounds for $p > 2$.

3.5.1. *Convergence for* $p = 2$. Our aim in this section is to prove item (i) in Theorem 3.14. According to the Feynman–Kac representation (3.20), an upper bound on the L^2 moments of $u_t^\diamond(x)$ amounts to prove the following relation:

(3.24)
$$\mathbb{E}[\exp(\alpha_t^{12})] < \infty, \quad \text{for all } t < \frac{1}{\kappa^{4/(2H_0-1)}} \equiv t_0(2),$$

where we recall that the constant κ is defined in Notation 3.12. Also, notice that the random variable α_t^{12}, introduced in (3.19), will be written as

(3.25)
$$\alpha_t^{12} = \int_{[0,t]^2} \gamma_0(s - r) \gamma(B_s - \widetilde{B}_r) \, ds \, dr,$$

for two independent Brownian motions B and $\widetilde{B}$. In addition, notice that only the large positive values of α_t^{12} might be responsible for the blowup of exponential moments. Therefore, relation (3.24) can be easily deduced from the following asymptotic tail behavior

$$(3.26) \qquad \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{P}\{\alpha_t^{12} \geq b\} \leq -\frac{1}{\kappa^4 t^{2H_0-1}}.$$

We now proceed to prove (3.26), and we divide our proof in several steps:

Step 1: Scaling arguments. We recall once again that γ_0 and γ are introduced in (1.3). Formally, they are given by

$$(3.27) \qquad \gamma_0(u) = C_{H_0}|u|^{-\alpha_0}, \quad \text{and} \quad \gamma(x) = C_{\mathbf{H}} \prod_{j=1}^{d} |x_j|^{-\alpha_j}, \quad \text{where } \alpha_j = 2 - 2H_j.$$

With this formal expression in mind, we consider $b > 0$ and set $\sigma = \frac{b}{t}s$, $\rho = \frac{b}{t}r$ in expression (3.25). Invoking the usual Brownian scaling and the fact that $\sum_{j=1}^{d}(H_j - 1) = -1$ under our critical assumption (1.8), we let the patient reader check that

$$(3.28) \qquad \alpha_t^{12} \overset{(d)}{=} b^{-1}t^{2H_0-1} \int_{[0,b]^2} \gamma_0(b^{-1}(\sigma - \rho))\gamma(B_\sigma - \widetilde{B}_\rho)\, d\sigma\, d\rho.$$

As mentioned in the Introduction, expressions like (3.28) are better expressed in Fourier modes, thanks to (1.4) and (1.5). We get

$$\alpha_t^{12} \overset{(d)}{=} b^{-1}t^{2H_0-1} \times \int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi) \left[\int_0^b e^{\iota(\lambda b^{-1}s + \xi \cdot B_s)}\, ds\right]\left[\int_0^b e^{-\iota(\lambda b^{-1}s + \xi \cdot \widetilde{B}_s)}\, ds\right].$$

In order to write the above expression in a more compact way, let us define the following random variables:

$$(3.29) \quad \eta_b(\lambda, \xi) = \int_0^b e^{\iota(\lambda b^{-1}s + \xi \cdot B_s)}\, ds, \quad \text{and} \quad \tilde{\eta}_b(\lambda, \xi) = \int_0^b e^{-\iota(\lambda b^{-1}s + \xi \cdot \widetilde{B}_s)}\, ds,$$

and denote by Z_b the weighted integral of η_b and $\tilde{\eta}_b$, namely,

$$Z_b = \int_{\mathbb{R}^{d+1}} \eta_b(\lambda, \xi)\tilde{\eta}_b(\lambda, \xi)\mu_0(d\lambda)\mu(d\xi).$$

We end up with the following identity in law:

$$(3.30) \qquad \alpha_t^{12} \overset{(d)}{=} b^{-1}t^{2H_0-1}Z_b.$$

We now go back to our main objective. Plugging (3.30) into (3.26), a few elementary algebraic manipulations yield that (3.24) can be reduced to the following bound:

$$(3.31) \qquad \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{P}\left\{|Z_b|^{1/2} \geq \frac{b}{t^{H_0-1/2}}\right\} \leq -\frac{1}{\kappa^4 t^{2H_0-1}}.$$

Moreover, since t above is a positive number, we simply set $u = t^{-(H_0-1/2)}$. We get that (3.24) is implied by the inequality

$$(3.32) \qquad \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{P}\{|Z_b|^{1/2} \geq ub\} \leq -\frac{u^2}{\kappa^4}$$

which should hold for all $u > 0$.

As a last preliminary step, we set up a cutoff procedure on the random variable Z_b. Namely, for $\delta > 0$, we define

$$(3.33) \qquad \mu_0^\delta(d\lambda) = e^{-\delta|\lambda|^2}\mu_0(d\lambda), \quad \text{and} \quad \bar{\mu}_0^\delta(d\lambda) = (1 - e^{-\delta|\lambda|^2})\mu_0(d\lambda).$$

Accordingly, we also set

$$
(3.34) \qquad
\begin{aligned}
Z_b^\delta &= \int_{\mathbb{R}^{d+1}} \eta_b(\lambda, \xi) \tilde\eta_b(\lambda, \xi) \mu_0^\delta(d\lambda) \mu(d\xi), \\
\bar Z_b^\delta &= \int_{\mathbb{R}^{d+1}} \eta_b(\lambda, \xi) \tilde\eta_b(\lambda, \xi) \bar\mu_0^\delta(d\lambda) \mu(d\xi)
\end{aligned}
$$

and notice that $Z_b = Z_b^\delta + \bar Z_b^\delta$. We will treat Z_b^δ and $\bar Z_b^\delta$ separately in (3.32).

Step 2: Identification of a negligible term. In this step we prove that the contribution of $\bar Z_b^\delta$ in (3.32) is negligible. Specifically, recalling that $\bar Z_b^\delta$ is defined by (3.34), given any $\epsilon > 0$ and any (large) $L > 0$, we prove that

$$
(3.35) \qquad \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{P}\{|\bar Z_b^\delta|^{1/2} \geq \epsilon b\} \leq -L,
$$

when δ is small enough. As a first step in this direction, introduce an additional parameter $\hat\alpha_0 \in (\alpha_0, 1)$, where we recall that α_0 is defined by (3.27). According to the definition (3.33) of $\bar\mu_0^\delta$, we have

$$
\bar\mu_0^\delta(d\lambda) = \left(1 - e^{-\delta|\lambda|^2}\right) \mu_0(d\lambda) \leq \left(1 - e^{-\delta|\lambda|^2}\right)^{\frac{\hat\alpha_0 - \alpha_0}{2}} \mu_0(d\lambda).
$$

Owing to the elementary relation $1 - e^{-x} \leq x$ for $x \geq 0$, recalling from (1.5) that $\mu_0(d\lambda) = c_0 |\lambda|^{-(1-\alpha_0)} \, d\lambda$, and setting

$$
\hat\mu_0(d\lambda) = c_0 |\lambda|^{-(1-\hat\alpha_0)} \, d\lambda,
$$

we get

$$
(3.36) \qquad \bar\mu_0^\delta(d\lambda) \leq \delta^{\frac{\hat\alpha_0 - \alpha_0}{2}} |\lambda|^{\hat\alpha_0 - \alpha_0} \mu_0(d\lambda) = \delta^{\frac{\hat\alpha_0 - \alpha_0}{2}} \hat\mu_0(d\lambda).
$$

We now compute the nth moment of the random variable $\bar Z_b^\delta$. Starting from definition (3.34) and invoking our convention (1.12) on product measures, for all $n \geq 1$, we have

$$
(3.37) \qquad \mathbb{E}[(\bar Z_b^\delta)^n] = \int_{(\mathbb{R}^{d+1})^n} \mathbb{E}\left[\prod_{k=1}^n \eta_b(\lambda_k, \xi_k) \tilde\eta_b(\lambda_k, \xi_k)\right] \bar\mu_0^\delta(d\lambda) \mu(d\xi).
$$

In addition, the families $\{\eta_b(\lambda_k, \xi_k); k \leq n\}$ and $\{\tilde\eta_b(\lambda_k, \xi_k); k \leq n\}$ above are independent and have the same law up to conjugation, due to definition (3.29), and the fact that B, $\widetilde B$ are two independent Brownian motion. Hence, relation (3.37) can be written as

$$
(3.38) \qquad \mathbb{E}[(\bar Z_b^\delta)^n] = \int_{(\mathbb{R}^{d+1})^n} \left| \mathbb{E}\left[\prod_{k=1}^n \eta_b(\lambda_k, \xi_k)\right] \right|^2 \bar\mu_0^\delta(d\lambda) \mu(d\xi).
$$

Note that from the right-hand side of the above identity, the nth moment of $\bar Z_b^\delta$ is nonnegative. Plugging inequality (3.36) in the above identity, we have

$$
(3.39) \qquad \mathbb{E}[(\bar Z_b^\delta)^n] \leq \delta^{\frac{\hat\alpha_0 - \alpha_0}{2} n} \int_{(\mathbb{R}^{d+1})^n} \left| \mathbb{E}\left[\prod_{k=1}^n \eta_b(\lambda_k, \xi_k)\right] \right|^2 \hat\mu_0(d\lambda) \mu(d\xi).
$$

In order to bound the right-hand side of (3.39), we first go back to the definition (3.29) of η_b and set $\lambda := b^{-1}\lambda$ therein. We let the patient reader check that the scaling can be read in (3.39) as

$$
(3.40) \qquad \mathbb{E}[(\bar Z_b^\delta)^n] \leq \delta^{\frac{\hat\alpha_0 - \alpha_0}{2} n} b^{n\hat\alpha_0} \int_{(\mathbb{R}^{d+1})^n} \left| \mathbb{E}\left[\prod_{k=1}^n \int_0^b e^{\iota(\lambda_k s_k + \xi_k \cdot \widetilde B_{s_k})} \, ds_k\right] \right|^2 \hat\mu_0(d\lambda) \mu(d\xi).
$$

Next, we notice that the right-hand side of (3.40) is upper bounded in [6], relation (3.1). Indeed, with the correspondence $\hat{H}_0 \equiv 2^{-1}(2 - \hat{\alpha}_0)$ and with relation (1.8) in mind, we choose $\hat{\alpha}_0 > \alpha_0$ such that $4(1 - \hat{H}_0) + (d_* - 2H_*) < 2$. Then, a direct application of [6], relation (3.1), yields

$$(3.41) \qquad \mathbb{E}\big[(\bar{Z}_b^\delta)^n\big] \le C^n \delta^{\frac{(\hat{\alpha}_0 - \alpha_0)n}{2}} (n!) b^{n\hat{\alpha}_0} b^{n(1-\hat{\alpha}_0)} = C^n (n!) \delta^{\frac{(\hat{\alpha}_0 - \alpha_0)n}{2}} b^n.$$

Starting from this inequality, we can easily get a similar bound for $\mathbb{E}[|\bar{Z}_b^\delta|^n]$ by changing the constant C in the right-hand side of (3.41). Namely, when n is even, note that $\mathbb{E}[(\bar{Z}_b^\delta)^n] = \mathbb{E}[|\bar{Z}_b^\delta|^n]$, due to the fact that our measures μ and μ_0 are symmetric. For any odd number $n = 2k + 1$, we just invoke the Cauchy–Schwartz inequality which yields

$$(3.42) \qquad \mathbb{E}\big[|\bar{Z}_b^\delta|^{2k+1}\big] \le \big(\mathbb{E}[(\bar{Z}_b^\delta)^{2k}]\big)^{1/2} \big(\mathbb{E}[(\bar{Z}_b^\delta)^{2k+2}]\big)^{1/2}.$$

We let the reader check that this slight elaboration yields (3.41) with $\mathbb{E}[|\bar{Z}_b^\delta|^n]$ on the left-hand side. Therefore, inserting (3.41) into a Taylor expansion for $x \mapsto e^x$, we get that, for some constant $C > 0$,

$$(3.43) \qquad Q \equiv \sup_{b \ge 1} \mathbb{E}\bigg[\exp\bigg(\frac{|\bar{Z}_b^\delta|}{C\delta^{\frac{\hat{\alpha}_0 - \alpha_0}{2}} b}\bigg)\bigg] < \infty.$$

We can now go back to our claim (3.35). Indeed, plugging (3.43) into a standard application of Chebyshev's inequality, we obtain

$$\mathbb{P}\{|\bar{Z}_b^\delta|^{1/2} \ge \epsilon b\} = \mathbb{P}\{|\bar{Z}_b^\delta| \ge \epsilon^2 b^2\} \le Q \exp\bigg(-\frac{\epsilon^2 b}{C\delta^{(\hat{\alpha}_0 - \alpha_0)/2}}\bigg),$$

from which (3.35) is easily deduced by picking a small enough δ. Taking the decomposition $Z_b = Z_b^\delta + \bar{Z}_b^\delta$, given by (3.34), our objective (3.32) is thus reduced to show the following bound for all $u, \delta > 0$ (see, e.g., [14], Lemma 1.2.15, for a general result yielding a proper identification of exponentially negligible terms):

$$(3.44) \qquad \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{P}\{|Z_b^\delta|^{1/2} \ge ub\} \le -\frac{u^2}{\kappa^4}.$$

Step 3: Cutoff procedure in space. In order to prove (3.44), we further decompose Z_b^δ as follows:

$$(3.45) \qquad Z_b^\delta = Z_b^{\delta, M} + \bar{Z}_b^{\delta, M},$$

where $M \ge 1$ is an additional parameter and $Z_b^{\delta, M}$, $\bar{Z}_b^{\delta, M}$ are, respectively, defined by

$$(3.46) \qquad Z_b^{\delta, M} = \int_{\mathbb{R} \times [-M, M]^d} \eta_b(\lambda, \xi) \tilde{\eta}_b(\lambda, \xi) \mu_0^\delta(d\lambda) \mu(d\xi)$$

and

$$(3.47) \qquad \bar{Z}_b^{\delta, M} = \int_{\mathbb{R} \times ([-M, M]^d)^c} \eta_b(\lambda, \xi) \tilde{\eta}_b(\lambda, \xi) \mu_0^\delta(d\lambda) \mu(d\xi).$$

Similarly to the previous step, we will now prove that, for any (large) $l > 0$, we have

$$(3.48) \qquad \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{P}\{|\bar{Z}_b^{\delta, M}|^{1/2} \ge \epsilon b\} \le -l.$$

To this aim, we will first upper bound the moments of $\bar{Z}_b^{\delta, M}$, as in (3.41). Namely, along the same lines as for (3.38), we have

$$(3.49) \qquad \mathbb{E}\big[(\bar{Z}_b^{\delta, M})^n\big] = \int_{(\mathbb{R} \times ([-M, M]^d)^c)^n} \bigg|\mathbb{E}\bigg[\prod_{k=1}^n \eta_b(\lambda_k, \xi_k)\bigg]\bigg|^2 \mu_0^\delta(d\lambda) \mu(d\xi)$$

which shows, in particular, that $\mathbb{E}[(\bar{Z}_b^{\delta,M})^n] \geq 0$, for all $n \geq 1$. Furthermore, recall from (3.29) that

$$\mathbb{E}\left[\prod_{k=1}^n \eta_b(\lambda_k, \xi_k)\right] = \mathbb{E}\left[\int_{[0,b]^n} \prod_{k=1}^n e^{-(\iota\lambda_k b^{-1} s_k + \xi_k \cdot B_{s_k})} ds\right]$$

$$= \int_{[0,b]^n} e^{-\iota\lambda_k b^{-1} s_k} \mathbb{E}\left[\prod_{k=1}^n e^{-\iota\xi_k \cdot B_{s_k}}\right] ds.$$

One can then trivially bound the terms $|e^{-\iota\lambda_k s_k}|$ by 1 in the right-hand side above and resort to the fact that $\mathbb{E}[\prod_{k=1}^n e^{-\iota\xi_k B_{s_k}}] \geq 0$. This yields

$$\left|\mathbb{E}\left[\prod_{k=1}^n \eta_b(\lambda_k, \xi_k)\right]\right| \leq \int_{[0,b]^n} \mathbb{E}\left[\prod_{k=1}^n e^{-\iota\xi_k \cdot B_{s_k}}\right] ds.$$

Reporting this inequality into (3.49), we get

(3.50) $$\mathbb{E}[(\bar{Z}_b^{\delta,M})^n] \leq [\mu_0^\delta(\mathbb{R})]^n \mathbb{E}[(\mathcal{H}_b^c(M))^n],$$

where we have set

(3.51) $$\mathcal{H}_b^c(M) = \int_{([-M,M]^d)^c} \left(\int_0^b e^{\iota\xi \cdot B_s} ds\right)\left(\int_0^b e^{-\iota\xi \cdot \tilde{B}_s} ds\right)\mu(d\xi).$$

We now wish to apply subadditivity properties of $\mathcal{H}_b^c(M)$, such as (3.23), in order to obtain relation (3.48). A first step in this direction is to apply Chebyshev's inequality, which asserts that, for all $\epsilon, b > 0$ and $k \geq 1$, we have

(3.52) $$(\epsilon^2 b^2 N)^{2k} \mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq N^{2k} \mathbb{E}[|\bar{Z}_b^{\delta,M}|^{2k}] = N^{2k} \mathbb{E}[(\bar{Z}_b^{\delta,M})^{2k}],$$

where we have introduced an additional parameter $N \geq 0$ to be specified later on. We sum inequality (3.52) over k and resort to the elementary inequality

$$\frac{1}{2}\exp(x) \leq \frac{e^x + e^{-x}}{2} = \sum_{k=0}^\infty x^{2k}/(2k)!,$$

in order to get

$$\frac{1}{2}e^{\epsilon^2 b^2 N}\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq \sum_{k=0}^\infty \frac{(\epsilon^2 b^2 N)^{2k}}{(2k)!}\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq \sum_{k=0}^\infty \frac{N^{2k}}{(2k)!}\mathbb{E}[(\bar{Z}_b^{\delta,M})^{2k}].$$

Therefore, invoking the fact that $\mathbb{E}[(\bar{Z}_b^{\delta,M})^k] \geq 0$, for all $k \geq 0$, we have

$$\frac{1}{2}\exp\{\epsilon^2 b^2 N\}\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq \sum_{k=0}^\infty \frac{N^k}{k!}\mathbb{E}[(\bar{Z}_b^{\delta,M})^k],$$

and, owing to inequality (3.50), the above becomes

$$\frac{1}{2}\exp\{\epsilon^2 b^2 N\}\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq \sum_{k=0}^\infty \frac{(N\mu_0^\delta(\mathbb{R}))^k}{k!}\mathbb{E}[(\mathcal{H}_b^c(M))^k],$$

where we recall that $\mathcal{H}_b^c(M)$ is defined by (3.51). Summarizing our considerations for this step, we have found that, for all $\epsilon, b > 0$ and $N > 0$, we have

$$\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq 2\exp\{-\epsilon^2 b^2 N\} \sum_{k=0}^\infty \frac{(N\mu_0^\delta(\mathbb{R}))^k}{k!}\mathbb{E}[(\mathcal{H}_b^c(M))^k]$$

$$= 2\exp\{-\epsilon^2 b^2 N\} \sum_{k=0}^\infty \frac{(bN\mu_0^\delta(\mathbb{R}))^k}{k!}\left\{\frac{1}{b^k}\mathbb{E}[(\mathcal{H}_b^c(M))^k]\right\}.$$

We can now apply Lemma 3.16 in the following way: we set $\theta = bN\mu_0^\delta(\mathbb{R})$, and we assume that b is an integer (generalizations to an arbitrary positive b are left to the reader). Then, iterating (3.23) b times and writing $R = Nb$, we end up with

$$(3.53) \qquad \mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq 2\exp\{-\epsilon^2 bR\}(\mathcal{A}^\delta(M,R))^b,$$

where the quantity $\mathcal{A}^\delta(M,R)$ is defined by

$$(3.54) \qquad \mathcal{A}^\delta(M,R) = \sum_{k=0}^\infty \frac{(R\mu_0^\delta(\mathbb{R}))^k}{k!}\mathbb{E}[|\mathcal{H}_1^c(M)|^k].$$

Moreover, first note that the bound (3.1) in [6], already used for our relation (3.41), can be extended to the case when $H_0 = 1$ (i.e., the setting without time dependence). A careful examination of the proof of this bound in [6] yields that, for all $k \geq 1$, we have

$$\mathbb{E}[(\mathcal{H}_1^c(M))^k]] \leq C_M^k k!,$$

where C_M is a constant depending on M such that $C_M \downarrow 0$ when $M \uparrow \infty$. For even k, it gives the bound for $\mathbb{E}[|\mathcal{H}_1^c(M)|^k]]$. Furthermore, the same bound can be extended to odd k by the Cauchy–Schwartz inequality, as in (3.42). (Hence, for any fixed R, one can always pick M large enough so that the right-hand side of (3.54) is finite.) Plugging this bound into (3.54), inequality (3.53) becomes

$$\frac{1}{b}\log\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq \frac{\log 2}{b} - \epsilon^2 R - \log(1 - R\mu_0^\delta(\mathbb{R})C_M).$$

In the right-hand side above, one can fix any $R > 0$ and then take M large enough so that

$$|\log(1 - R\mu_0^\delta(\mathbb{R})C_M)| \leq \epsilon^4.$$

Therefore for ϵ small enough we get

$$\frac{1}{b}\log\mathbb{P}\{|\bar{Z}_b^{\delta,M}|^{1/2} \geq \epsilon b\} \leq \frac{\log 2}{b} - \frac{1}{2}\epsilon^2 R,$$

from which (3.48) is easily deduced.

Summarizing our considerations from this step, having (3.44), (3.45) and (3.48) in mind, we get that our claim (3.32) is achieved as soon as we can prove

$$(3.55) \qquad \limsup_{b\to\infty} \frac{1}{b}\log\mathbb{P}\{|Z_b^{\delta,M}|^{1/2} \geq ub\} \leq -\frac{u^2}{\kappa^4},$$

where we recall that $Z_b^{\delta,M}$ is defined by (3.46).

Step 4: Expression in terms of the moments of $Z_b^{\delta,M}$. Thanks to a standard use of Chebyshev's inequality, relation (3.55) is achieved as long as we can prove the following Gaussian-type bound for $|Z_b^{\delta,M}|^{1/2}$:

$$(3.56) \qquad \limsup_{b\to\infty} \frac{1}{b}\log\mathbb{E}[\exp(\theta|Z_b^{\delta,M}|^{1/2})] \leq \frac{\kappa^4\theta^2}{4}$$

which should be valid for all $\theta \geq 0$. In order to separate the variables η_b and $\tilde{\eta}_b$ in the definition (3.46) of $Z_b^{\delta,M}$, we first apply Cauchy–Schwarz inequality and then the elementary inequality $2\sqrt{ab} \leq a + b$ for all $a, b \geq 0$. We get

$$|Z_b^{\delta,M}|^{1/2} \leq \frac{1}{2}\Bigg\{\Bigg(\int_{\mathbb{R}\times[-M,M]^d}|\eta_b(\lambda,\xi)|^2\mu_0^\delta(d\lambda)\mu(d\xi)\Bigg)^{1/2}$$

$$+ \Bigg(\int_{\mathbb{R}\times[-M,M]^d}|\tilde{\eta}_b(\lambda,\xi)|^2\mu_0^\delta(d\lambda)\mu(d\xi)\Bigg)^{1/2}\Bigg\}.$$

Taking into account the fact that η and $\tilde{\eta}$ are independent, this entails

$$(3.57) \qquad \mathbb{E}[\exp(\theta|Z_b^{\delta,M}|^{1/2})] \leq \left\{ \mathbb{E}\left[\exp\left(\frac{\theta}{2}|X_b^{\delta,M}|^{1/2}\right)\right]\right\}^2,$$

where the random variable $X_b^{\delta,M}$ is defined by

$$(3.58) \qquad X_b^{\delta,M} = \int_{\mathbb{R}\times[-M,M]^d} |\eta_b(\lambda,\xi)|^2 \mu_0^\delta(d\lambda)\mu(d\xi).$$

Putting together (3.56) and (3.57) and setting $\theta = 2\theta$, we are reduced to prove

$$(3.59) \qquad \limsup_{b\to\infty} \frac{1}{b} \log \mathbb{E}[\exp(\theta|X_b^{\delta,M}|^{1/2})] \leq \frac{\kappa^4\theta^2}{2}.$$

In addition, condition (3.59) can be expressed in terms of the moments of $X_b^{\delta,M}$. Indeed, taking into account the fact that $X_b^{\delta,M}$ is a positive random variable, a direct application of [3], Lemma 1.2.6, asserts that (3.59) is equivalent to the following property:

$$(3.60) \qquad \limsup_{b\to\infty} \frac{1}{b} \log \sum_{n=0}^\infty \frac{\theta^n}{n!} (\mathbb{E}[(X_b^{\delta,M})^n])^{1/2} \leq \kappa^4\theta^2.$$

We will use the formulation (3.60) below in order to replace the Brownian motion B in $X_b^{\delta,M}$ by a Ornstein–Uhlenbeck process.

Step 5: Expression in terms of an Ornstein–Uhlenbeck process. Similarly to a strategy borrowed from [23], we now introduce a family of $\mathbb{R}^d$-valued Ornstein–Uhlenbeck processes indexed by $\alpha > 0$ and denoted by B^α. The process B^α solves the equation

$$(3.61) \qquad dB_t^\alpha = -\alpha B_t^\alpha dt + dB_t,$$

where B is our standing d-dimensional Wiener process. Now notice that B can also be seen as an Ornstein–Uhlenbeck-type process of the form

$$dB_t = -\alpha B_t dt + dW_t, \quad \text{with } W_t = B_t + \alpha \int_0^t B_s\, ds.$$

Therefore, if we denote by $\mathbb{P}^\alpha$ the law of B^α, a standard application of Girsanov's theorem yields

$$(3.62) \qquad \frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\bigg|_{[0,\tau]} = \exp\left(-\alpha \int_0^\tau B_s \cdot dB_s - \frac{\alpha^2}{2} \int_0^\tau |B_s|^2\, ds\right).$$

In addition, Itô's formula applied to B entails that, for any $\tau > 0$, we have

$$|B_\tau|^2 = 2 \int_0^\tau B_s \cdot dB_s + \tau d.$$

Therefore, the exponential term in (3.62) can be recast as

$$(3.63) \qquad \frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\bigg|_{[0,\tau]} = \exp\left(\frac{1}{2}\left(\alpha\tau d - \alpha|B_\tau|^2 - \alpha^2 \int_0^\tau |B_s|^2\, ds\right)\right).$$

In particular, it is readily checked that

$$(3.64) \qquad \frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\bigg|_{[0,\tau]} \leq \exp\left(\frac{1}{2}\alpha\tau d\right).$$

Let us also recall a moment comparison inequality which is obtained in [6], relation (6.20). Namely, for $n \geq 1$ and the random variables $X_b^{\delta,M}$ defined by (3.58), we have

$$\mathbb{E}[(X_b^{\delta,M})^n] \leq \mathbb{E}^\alpha[(X_b^{\delta,M})^n],$$

for all $\alpha > 0$, where $\mathbb{E}^\alpha$ denotes the expectation under $\mathbb{P}^\alpha$. Hence in order to prove (3.60), it will be enough to show a uniform bound in α, namely,

$$\limsup_{\alpha \to 0} \limsup_{b \to \infty} \frac{1}{b} \log \sum_{n=0}^\infty \frac{\theta^n}{n!} (\mathbb{E}^\alpha[(X_b^{\delta,M})^n])^{1/2} \leq \kappa^4 \theta^2.$$

Therefore, invoking again the equivalence between (3.59) and (3.60) for positive random variables, we are reduced to prove

$$(3.65) \qquad \limsup_{\alpha \to 0} \limsup_{b \to \infty} \frac{1}{b} \log \mathbb{E}^\alpha[\exp(\theta |X_b^{\delta,M}|^{1/2})] \leq \frac{\kappa^4 \theta^2}{2}$$

for every $\theta > 0$. We now focus on inequality (3.65) for a generic $\theta > 0$.

   Step 6: Space-time cutoff for the Ornstein–Uhlenbeck process. Let us introduce an additional parameter $N > 0$, and write

$$X_b^{\delta,M} = X_b^{\delta,M,N} + \bar{X}_b^{\delta,M,N},$$

with

$$(3.66)
\begin{aligned}
X_b^{\delta,M,N} &= \int_{[-N,N] \times [-M,M]^d} |\eta_b(\lambda,\xi)|^2 \mu_0^\delta(d\lambda) \mu(d\xi), \\
\bar{X}_b^{\delta,M,N} &= \int_{[-N,N]^c \times [-M,M]^d} |\eta_b(\lambda,\xi)|^2 \mu_0^\delta(d\lambda) \mu(d\xi).
\end{aligned}$$

Trivially bounding the oscillating exponential terms by 1 in the definition (3.29) of $\eta_b(\lambda,\xi)$, we obtain

$$(3.67) \qquad |\bar{X}_b^{\delta,M,N}| \leq b^2 \mu_0^\delta([-N,N]^c) \mu([-M,M]^d).$$

Therefore, thanks to the fact that $\delta > 0$, the right-hand side of (3.67) can be made as small as desired by picking N large enough. Similarly to what has been done in Step 2, one can thus prove an inequality of the same form as (3.35) for $\bar{X}_b^{\delta,M,N}$. We are now reduced to show that (3.65) holds true with $X_b^{\delta,M}$ replaced by $X_b^{\delta,M,N}$.

   Next, we consider a new parameter $K > 0$, and we fix a value $\alpha > 0$. We further decompose $X_b^{\delta,M,N}$ and write

$$(3.68) \qquad X_b^{\delta,M,N} = X_b^{\delta,M,N} \mathbf{1}_{\Omega_{b,K}} + X_b^{\delta,M,N} \mathbf{1}_{\Omega_{b,K}^c},$$

where

$$\Omega_{b,K} = \left\{ \frac{1}{b} \int_0^b |B_s| \, ds \leq K \right\}.$$

We will prove that for K large enough, the quantity $X_b^{\delta,M,N} \mathbf{1}_{\Omega_{b,K}^c}$ is negligible with respect to $X_b^{\delta,M,N} \mathbf{1}_{\Omega_{b,K}}$. Indeed, resorting to (3.63), we have

$$\mathbb{E}^\alpha\left[\exp\left(\int_0^b |B_s| \, ds\right)\right] \leq \mathbb{E} \exp\left\{ \int_0^b \left(|B_s| - \frac{\alpha^2}{2}|B_s|^2\right) ds + \frac{\alpha d}{2} b \right\}.$$

Now we can use the elementary inequality $x - \alpha^2 x^2/2 \leq 1/2\alpha^2$, valid for all $x > 0$, in order to get

$$\mathbb{E}^\alpha\left[\exp\left(\int_0^b |B_s| \, ds\right)\right] \leq e^{C_\alpha b}, \quad \text{where } C_\alpha = \frac{1}{2\alpha^2} + \frac{\alpha d}{2}.$$

Hence, given $l > 0$ one can choose $K > 0$ large enough (say $K = l + 2C_\alpha$) such that uniformly in b we have

$$(3.69) \qquad \mathbb{P}^\alpha\big(\Omega_{b,K}^c\big) \le e^{-lb},$$

where we recall that $\Omega_{b,K}$ is defined by (3.68). In addition, one can trivially bound the quantity $|\eta_b(\lambda, \xi)|^2$ in the definition (3.66) of $X_b^{\delta, M, N}$ by b^2. This yields

$$\mathbb{E}^\alpha\big[\exp\big(\theta|X_b^{\delta, M, N}|^{1/2}\big)\mathbf{1}_{\Omega_{b,K}^c}\big] \le \exp\big(C_{\delta, M, N}^{1/2}\theta b - lb\big),$$

where l above can be made arbitrarily large. In conclusion of this step, it is enough to prove that, for all $\theta, \delta, M, N > 0$, we have

$$(3.70) \qquad \limsup_{\alpha \to 0}\limsup_{b \to \infty}\frac{1}{b}\log\mathbb{E}^\alpha\big[\exp\big(\theta|X_b^{\delta, M, N}|^{1/2}\big)\mathbf{1}_{\Omega_{b,K}}\big] \le \frac{\kappa^4\theta^2}{2}.$$

Step 7: Linearization procedure. One way to recast the definition (3.66) of $|X_b^{\delta, M, N}|^{1/2}$ is to write

$$(3.71) \qquad |X_b^{\delta, M, N}|^{1/2} = \|\eta_b\|_{\mathcal{G}},$$

where the functional space $\mathcal{G}$ is defined by

$$(3.72) \qquad \mathcal{G} = \big\{h \in L^2([-N, N] \times [-M, M]^d; \mu_0^\delta \otimes \mu); h(-\lambda, -\xi) = \bar{h}(\lambda, \xi)\big\}.$$

In this step we show how to linearize the $\mathcal{G}$-norm above when $\omega \in \Omega_{b,K}$. To this aim, going back to (3.29), notice that, for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^d$, we have

$$\big|\eta_b(\lambda_2, \xi_2) - \eta_b(\lambda_1, \xi_1)\big| \le \int_0^b \big|e^{\iota(b^{-1}\lambda_2 + \xi_2 \cdot B_s)} - e^{\iota(b^{-1}\lambda_1 + \xi_1 \cdot B_s)}\big|\,ds$$

$$\le |\lambda_2 - \lambda_1| + \bigg(\int_0^b |B_s|\,ds\bigg)|\xi_2 - \xi_1|$$

$$\le |\lambda_2 - \lambda_1| + Kb|\xi_2 - \xi_1|,$$

where we have invoked the fact that $\omega \in \Omega_{b,K}$, with $\Omega_{b,K}$ given by (3.68), for the last inequality. Therefore, setting

$$(3.73) \qquad \begin{aligned} \mathcal{C} = \big\{h(\cdot, \cdot) \in \mathcal{G}; \,&|h(\lambda, \xi)| \le 1, \\ &|h(\lambda_1, \xi_1) - h(\lambda_2, \xi_2)| \le (1 + K)|(\lambda_1, \xi_1) - (\lambda_2, \xi_2)|\big\}, \end{aligned}$$

it is easily seen that $b^{-1}\eta_b \in \mathcal{C}$ for all $b \ge 1$, whenever $\omega \in \Omega_{b,K}$. Moreover, some standard uniform continuity arguments show that the closure $\mathcal{K}$ of $\mathcal{C}$ in $\mathcal{G}$ is a compact set. We have thus proved that, for all $b \ge 1$ and $\omega \in \Omega_{b,K}$, we have

$$b^{-1}\eta_b \in \mathcal{K}.$$

Our next step will be to construct a finite cover of $\mathcal{K}$. To this aim, consider the unit ball in $\mathcal{G}$, denoted by $B_{\mathcal{G}}(0, 1)$. We also consider the following set for any $f \in B_{\mathcal{G}}(0, 1)$:

$$(3.74) \qquad \mathcal{O}_f = \big\{h \in \mathcal{G}; \langle f, h\rangle_{\mathcal{G}} > \|h\|_{\mathcal{G}} - \epsilon\big\}.$$

Then, we claim that the family $\{\mathcal{O}_f; f \in B_{\mathcal{G}}(0, 1)\}$ covers $\mathcal{K}$. Indeed, consider $f \in B_{\mathcal{G}}(0, 1)$ such that $\|f\|_{\mathcal{G}} = 1$ and a constant $a > 0$. We set $h = af$. Then, it is readily checked that $\langle f, h\rangle_{\mathcal{G}} = \|h\|_{\mathcal{G}}$. Therefore, we get

$$\mathcal{K} \subset \big\{af; \|f\|_{\mathcal{G}} = 1, a > 0\big\} \subset \big\{\mathcal{O}_f; f \in B_{\mathcal{G}}(0, 1)\big\}.$$

Otherwise stated, $\{\mathcal{O}_f; f \in B_{\mathcal{G}}(0, 1)\}$ covers $\mathcal{K}$. In the sequel we will extract a finite family $\{\mathcal{O}_{f_i}, 1 \leq i \leq m\}$ which still covers $\mathcal{K}$ and which is possible since $\mathcal{K}$ is a compact set.

Let us now go back to the random function $b^{-1}\eta_b$, seen as an element of $\mathcal{K}$. From the previous consideration, we know that, for almost every $\omega \in \Omega_{b,K}$, there exists $j = j(\omega) \in \{1, \ldots, m\}$ such that $b^{-1}\eta_b \in \mathcal{O}_{f_j}$. Hence, from the very definition (3.74) of $\mathcal{O}_{f_j}$, we get

$$\|b^{-1}\eta_b(\cdot, \cdot)\|_{\mathcal{G}} \leq \epsilon + \max_{1 \leq j \leq m} \langle f_j, b^{-1}\eta_b \rangle_{\mathcal{G}}.$$

Plugging this inequality into (3.71), we end up with

$$\mathbb{E}^{\alpha}\big[\exp(\theta|X_b^{\delta, M, N}|^{1/2})\mathbf{1}_{\Omega_{b,K}}\big] = \mathbb{E}^{\alpha}\big[\exp(\theta\|\eta_b\|_{\mathcal{G}})\mathbf{1}_{\Omega_{b,K}}\big]$$

$$\leq \mathbb{E}^{\alpha}\Big[\exp\Big(\theta \max_{1 \leq j \leq m} \langle f_j, \eta_b \rangle_{\mathcal{G}} + b\theta\epsilon\Big)\Big]$$

$$\leq e^{b\theta\epsilon} \sum_{j=1}^{m} \mathbb{E}^{\alpha}\big[\exp(\theta\langle f_j, \eta_b \rangle_{\mathcal{G}})\big].$$

Hence, some elementary properties of the logarithmic function (see, e.g., [14], Lemma 1.2.15) entail

$$(3.75) \quad \begin{aligned} &\limsup_{b \to \infty} \frac{1}{b} \log \mathbb{E}^{\alpha}\big[\exp(\theta|X_b^{\delta, M, N}|^{1/2})\mathbf{1}_{\Omega_{b,K}}\big] \\ &\leq \theta\epsilon + \max_{1 \leq j \leq m}\Big\{\limsup_{b \to \infty} \frac{1}{b} \log(\mathbb{E}^{\alpha}\big[\exp(\theta\langle f_j, \eta_b \rangle_{\mathcal{G}})\big])\Big\}. \end{aligned}$$

We will now treat each term in the right-hand side of (3.75) separately.

Step 8: Feynman–Kac-type asymptotics. Due to the fact that $\mathcal{G}$ is a subspace of $L^2([-N, N] \times [-M, M]^d : \mu_0^{\delta} \otimes \mu)$ for $j = 1, \ldots, m$, we have

$$\langle f_j, \eta_b \rangle_{\mathcal{G}} = \int_{[-N,N] \times [-M,M]^d} f_j(\lambda, \xi)\bar{\eta}_b(\lambda, \xi)\mu_0^{\delta}(d\lambda)\mu(d\xi).$$

Recalling once again the definition (3.29) of η_b, this yields

$$\langle f_j, \eta_b \rangle_{\mathcal{G}} = \int_0^b \tilde{f}_j\Big(\frac{s}{b}, B_s\Big) ds,$$

where the functions $\tilde{f}_j$ are given by the following relation:

$$(3.76) \qquad \tilde{f}_j(s, x) = \int_{[-N,N] \times [-M,M]^d} f_j(\lambda, \xi)e^{\iota(\lambda s + \xi \cdot x)}\mu_0^{\delta}(d\lambda)\mu(d\xi).$$

Also, notice that, according to the definition (3.72) of $\mathcal{G}$, we have $f_j(\lambda, \xi) = \overline{f_j}(-\lambda, -\xi)$ for all $j = 1, \ldots, m, \lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. With the expression (3.76) in mind, this yields that $\tilde{f}_j$ is real valued. Thus, inequality (3.64) entails

$$(3.77) \qquad \mathbb{E}^{\alpha}\big[\exp(\theta\langle f_j, \eta_b \rangle_{\mathcal{G}})\big] \leq \exp\Big(\frac{\alpha d}{2}b\Big)\mathbb{E}\Big[\exp\Big(\theta \int_0^b \tilde{f}_j\Big(\frac{s}{b}, B_s\Big) ds\Big)\Big],$$

where the expectation on the right-hand side is now taken with respect to a $\mathbb{R}^d$-valued Brownian motion B.

With (3.77) in hand, we are now back to a more classical Feynman–Kac computation. Specifically, $\tilde{f}_j$ can be seen as the Fourier transform of a finite and compactly supported

measure, as is apparent from (3.76). Therefore it fulfills all the regularity assumptions allowing to apply [10], Proposition 3.1. Applying this proposition, we thus get

$$
\begin{aligned}
(3.78) \quad & \lim_{b \to \infty} \frac{1}{b} \log \mathbb{E}\left[\exp\left(\theta \int_0^b \tilde{f}_j\left(\frac{s}{b}, B_s\right) ds\right)\right] \\
& = \sup_{g \in \mathcal{A}_d}\left\{\theta \int_0^1 \int_{\mathbb{R}^d} \tilde{f}_j(s, x) g^2(s, x)\, dx\, ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2\, dx\, ds\right\},
\end{aligned}
$$

where the space $\mathcal{A}_d$ is introduced in Definition 3.10.

Step 9: Evaluation of the Feynman–Kac asymptotics. Let us analyze the right-hand side of relation (3.78). Owing to the definition of $\tilde{f}_j$ in (3.76), it is readily checked that

$$
\int_0^1 \int_{\mathbb{R}^d} \tilde{f}_j(s, x) g^2(s, x)\, dx\, ds = \langle f_j, \tilde{\mathcal{F}}(g^2)\rangle_{\mathcal{G}},
$$

where we recall that the space $\mathcal{G}$ is given by (3.72) and where we have set

$$
(3.79) \quad \tilde{\mathcal{F}}(g^2)(\lambda, \xi) = \int_0^1 \int_{\mathbb{R}^d} g^2(s, x) e^{-\iota(\lambda s + \xi \cdot x)}\, dx\, ds.
$$

Furthermore, each f_j sits in the unit ball $B_{\mathcal{G}}(0, 1)$, and, thus,

$$
\left|\int_0^1 \int_{\mathbb{R}^d} \tilde{f}_j(s, x) g^2(s, x)\, dx\, ds\right| \leq \|\tilde{\mathcal{F}}(g^2)\|_{\mathcal{G}}.
$$

Now it is easily seen that

$$
\begin{aligned}
\|\tilde{\mathcal{F}}(g^2)\|_{\mathcal{G}}^2 &= \int_{[-N,N] \times [-M,M]^d} |\tilde{\mathcal{F}}(g^2)(\lambda, \xi)|^2 \mu_0^\delta(d\lambda) \mu(d\xi) \\
&\leq \int_{\mathbb{R}^{d+1}} |\tilde{\mathcal{F}}(g^2)(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(d\xi) \\
&= \int_{[0,1]^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_0(s - r) \gamma(x - y) g^2(s, x) g^2(r, y)\, dx\, dy\, dr\, ds.
\end{aligned}
$$

Plugging this information into (3.78), we have obtained that

$$
\begin{aligned}
& \lim_{b \to \infty} \frac{1}{b} \log \mathbb{E}\left[\exp\left(\theta \int_0^b \tilde{f}_j\left(\frac{s}{b}, B_s\right) ds\right)\right] \\
(3.80) \quad & \leq \sup_{g \in \mathcal{A}_d}\left\{\theta\left(\int_{[0,1]^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_0(s - r) \gamma(x - y) g^2(s, x) g^2(r, y)\, dx\, dy\, dr\, ds\right)^{1/2}\right. \\
& \quad \left. - \frac{1}{2}\|\nabla_x g(s, x)\|_{L^2([0,1] \times \mathbb{R}^d)}^2\right\}.
\end{aligned}
$$

Now, resorting to (3.21), the right-hand side of (3.80) can be upper bounded by

$$
\sup_{g \in \mathcal{A}_d}\left\{\theta \kappa^2 \|\nabla_x g\|_{L^2([0,1] \times \mathbb{R}^d)} - \frac{1}{2}\|\nabla_x g\|_{L^2([0,1] \times \mathbb{R}^d)}^2\right\} = \sup_{z > 0}\left\{\theta \kappa^2 z - \frac{1}{2} z^2\right\} = \frac{1}{2} \kappa^4 \theta^2,
$$

where the last equality stems from a trivial optimization procedure. Summarizing our computations for this step, we have found that, for $j = 1, \ldots, m$, we have

$$
(3.81) \quad \lim_{b \to \infty} \frac{1}{b} \log \mathbb{E}\left[\exp\left(\theta \int_0^b \tilde{f}_j\left(\frac{s}{b}, B_s\right) ds\right)\right] \leq \frac{1}{2} \kappa^4 \theta^2.
$$

Step 10: Conclusion. Recall that our desired upper bound has been successively reduced to (3.70). Moreover, we have seen that the left-hand side of (3.70) is bounded by the right-hand side of (3.75) for any arbitrary small ϵ. Therefore, we are left with the evaluation of

$$A_{\theta,\epsilon,\alpha} \equiv \theta\epsilon + \max_{1 \leq j \leq m} \left\{ \limsup_{b \to \infty} \frac{1}{b} \log(\mathbb{E}^{\alpha}[\exp(\theta \langle f_j, \eta_b \rangle_{\mathcal{G}})]) \right\}.$$

Now, putting together (3.77) and (3.81), we obtain that

$$A_{\theta,\epsilon,\alpha} \leq \theta\epsilon + \frac{\alpha d}{2} + \frac{1}{2}\kappa^4\theta^2.$$

Since ϵ and α can be made arbitrarily small, we have shown that relation (3.70) holds true. This finishes the proof of relation (3.24). With (3.24) in hand, Theorem 3.14-(i) is obtained along the same lines as for Proposition 3.7. $\square$

REMARK 3.18. According to Remark 3.15(b), we could prove that the limit of $u^{\varepsilon,\diamond}$ in Theorem 3.14-(i) also solves the mild Skorohod equation (3.1). Moreover, thanks to a slight elaboration of Proposition 3.7, we could also obtain the uniqueness of the solution. We have not included those details for sake of conciseness.

3.5.2. *Convergence for $p > 2$.* Recall that we have proved item (i) in Theorem 3.14. Namely, we have shown the L^2 convergence of $u_t^{\varepsilon,\diamond}$ to the solution $u_t^\diamond$ of (3.1) (see Remark 3.18 about a notion of solution) for $t < t_0 = t_0(2)$. In this section we extend this result to a general $p > 2$, that is, we prove item (ii) in Theorem 3.14. As in Section 3.5.1, we will first focus on showing that $\mathbb{E}[|u_t^\diamond(x)|^p]$ is finite whenever $t < t_0(p)$.

In order to prove the L^p-boundedness of $u_t^\diamond(x)$, we introduce an additional intensity parameter $\zeta > 0$ and consider the process $u^{(\zeta)}$, solution of the following slight extension of (3.1):

$$\partial_t u_t^{(\zeta)}(x) = \frac{1}{2}\Delta u_t^{(\zeta)}(x) + \sqrt{\zeta} u_t^{(\zeta)}(x) \diamond \dot{W}_t(x),$$

with initial condition $u_0(x) = 1$. According to an hypercontractivity inequality shown in [26], Theorem 1, the following inequality holds true for all $p \geq 2$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$(3.82) \qquad \left\| u_t^{(\zeta)}(x) \right\|_{L^p(\Omega)} \leq \left\| u_t^{((p-1)\zeta)}(x) \right\|_{L^2(\Omega)},$$

or otherwise stated for $\zeta = 1$,

$$(3.83) \qquad \mathbb{E}[|u_t^\diamond(x)|^p] \leq (\mathbb{E}[|u_t^{(p-1)}(x)|^2])^{p/2}.$$

In addition, writing formula (3.20) for $p = 2$ and for a noise $\sqrt{p-1}\dot{W}$, we get

$$\mathbb{E}[|u_t^{(p-1)}(x)|^2] = \mathbb{E}\left[\exp\left((p-1)\int_{[0,t]^2} \gamma_0(s-r)\gamma(B_s - \tilde{B}_r)\,ds\,dr\right)\right].$$

Now, resorting to a simple change of variables and invoking formula (3.20) again, we end up with

$$(3.84) \qquad \begin{aligned} \mathbb{E}[|u_t^{(p-1)}(x)|^2] &= \mathbb{E}\left[\exp\left(\int_{[0,(p-1)^{1/(2H_0-1)}t]^2} \gamma_0(s-r)\gamma(B_s - \tilde{B}_r)\,ds\,dr\right)\right] \\ &= \mathbb{E}[|u_{(p-1)^{1/(2H_0-1)}t}^\diamond(x)|^2]. \end{aligned}$$

Gathering (3.83) and (3.84), we have thus obtained

$$(3.85) \qquad \mathbb{E}[|u_t^\diamond(x)|^p] \leq (\mathbb{E}[|u_{(p-1)^{1/(2H_0-1)}t}^\diamond(x)|^2])^{p/2}.$$

Applying Theorem 3.14-item (i) to the right-hand side of (3.85), we obtain that $u_t^\diamond(x) \in L^p(\Omega)$ when $t < t_0(p)$.

We now turn to the proof of the fact that $u_t^{\epsilon,\diamond}(x)$ converges to $u_t^\diamond(x)$ in $L^p(\Omega)$ for $t < t_0(p)$. To this aim, we fix a $p' > p$ such that $t < t_0(p') < t_0(p)$. By the same argument that we used to get (3.85), we obtain the following inequality for $u_t^{\epsilon,\diamond}(x)$:

$$(3.86) \qquad \mathbf{E}\big[|u_t^{\epsilon,\diamond}(x)|^{p'}\big] \leq \big(\mathbf{E}\big[|u_{(p'-1)^{1/(2H_0-1)}t}^{\epsilon,\diamond}(x)|^2\big]\big)^{p'/2}.$$

Furthermore, it is easily seen from the proof of Proposition 3.7 that the $L^2(\Omega)$-norm of $u_t^{\epsilon,\diamond}(x)$ is dominated by that of $u_t^\diamond(x)$. Therefore, relation (3.86) implies

$$\mathbf{E}\big[|u_t^{\epsilon,\diamond}(x)|^{p'}\big] \leq \big(\mathbf{E}\big[|u_{(p'-1)^{1/(2H_0-1)}t}^\diamond(x)|^2\big]\big)^{p'/2}.$$

Now, by our choice of p' (such that $t < t_0(p')$), we conclude

$$\sup_{\epsilon > 0} \mathbf{E}\big[|u_t^{\epsilon,\diamond}(x)|^{p'}\big] < \infty.$$

Therefore, the family $\{|u_t^{\epsilon,\diamond}(x)|^p\}_{\epsilon > 0}$ is uniformly integrable. This, together with the fact that $u_t^{\epsilon,\diamond}(x)$ converges to $u_t^\diamond(x)$ in $L^2(\Omega)$, gives us the convergence in $L^p(\Omega)$. The proof of Theorem 3.14-item (ii) is thus completed. $\square$

3.6. *Proof of Theorem* 3.14-*item* (iii). Recall that $u_t^{\epsilon,\diamond}(x)$ is defined in Proposition 3.6. In the previous section we essentially proved that $u_t^{\epsilon,\diamond}(x)$ converges to $u_t^\diamond(x)$ in L^p when $t < t_0(p)$. In this section we show that $\|u_t^{\epsilon,\diamond}(x)\|_p$ diverges as $\epsilon \to 0$ when $t > t_0(p)$.

In order to state a lower bound on the $L^p(\Omega)$ moments of $u^{\epsilon,\diamond}$, we first define a functional space $\mathcal{G}_0$ which generalizes the space $\mathcal{G}$ introduced in (3.72).

DEFINITION 3.19. Let $C_{0,b}(\mathbb{R} \times \mathbb{R}^d)$ be the space of compactly supported and bounded functions on $\mathbb{R} \times \mathbb{R}^d$. We set

$$\mathcal{G}_0 = \{h \in L^2(\mathbb{R} \times \mathbb{R}^d, \mu_0 \otimes \mu) \cap C_{0,b}(\mathbb{R} \times \mathbb{R}^d); h(-\lambda, -\xi) = \bar{h}(\lambda, \xi), \mu_0 \otimes \mu - \text{a.e.}\}.$$

We can now state a first lower bound on the moments of $u^{\epsilon,\diamond}$ in terms of a variational quantity.

PROPOSITION 3.20. *For $\epsilon > 0$, consider the solution $u^{\epsilon,\diamond}$ to the regularized Skorohod equation* (3.12). *We assume that the critical conditions* (1.8) *are met. Let $p > 1$, and denote its conjugate exponent by q. Then, for $t > 0$ and $x \in \mathbb{R}^d$, we have*

$$
\begin{aligned}
(3.87) \qquad &\liminf_{\varepsilon \to 0} \|u_t^{\varepsilon,\diamond}(x)\|_p \\
&\geq \sup_{h \in \mathcal{G}_0} \mathbb{E}\bigg[\exp\bigg(\int_0^t \tilde{h}(s, B_s)\, ds - \frac{q-1}{2} \int_{\mathbb{R}^{d+1}} |h(\lambda, \xi)|^2 \mu_0(d\lambda)\mu(d\xi)\bigg)\bigg],
\end{aligned}
$$

where the function $\tilde{h}$ is defined similarly to $\tilde{f}_j$ in (3.76), *namely,*

$$(3.88) \qquad \tilde{h}(s, x) = \int_{\mathbb{R} \times \mathbb{R}^d} h(\lambda, \xi) e^{\iota(\lambda s + \xi \cdot x)} \mu_0(d\lambda)\mu(d\xi).$$

PROOF. Recall that $u_t^{\epsilon,\diamond}(x)$ is also given by expression (3.11), where $V_t^{\epsilon,B}(x)$ is introduced in (3.8). Next, we recall the definition of $\mathcal{S}$-transform on the Wiener space related to

our noise W (see [20] for more details about the $\mathcal{S}$-transform). Having in mind the notation introduced in Section 2.1, the $\mathcal{S}$-transform of $F \equiv u_t^{\epsilon,\diamond}(x)$ is defined for $\varphi \in \mathcal{H}$ by

$$\mathcal{S}F(\varphi) = \mathbf{E}[F\mathcal{E}_\varphi],$$

where the martingale exponential $\mathcal{E}_\varphi$ is given by

$$(3.89) \qquad \mathcal{E}_\varphi = \exp\left(W(\varphi) - \frac{1}{2}\|\varphi\|_{\mathcal{H}}^2 \right).$$

As highlighted in [20], Chapter 2, the $\mathcal{S}$-transform has to be considered as the equivalent of the Fourier transform on a Wiener space. However, in our context we will just use the following basic estimate for $F = u_t^{\epsilon,\diamond}(x)$:

$$(3.90) \qquad \|F\|_p \geq \sup\left\{ \frac{\mathcal{S}F(\varphi)}{\|\mathcal{E}_\varphi\|_q}; \varphi \in \mathcal{H} \right\},$$

where we recall that q is the conjugate of p. We will now analyze the right-hand side of relation (3.90).

In order to evaluate the $\mathcal{S}$-transform $\mathcal{S}F(\varphi)$ in (3.90), We resort to the Feynman–Kac formula (3.11) for $u_t^{\epsilon,\diamond}(x)$, Fubini's theorem and the isometry (2.3) on our standing Wiener space. Similarly to (3.16)–(3.17), albeit with a time-space Fourier transform, we get

$$(3.91) \qquad \mathcal{S}F(\varphi) = \mathbb{E}\left[\exp\left(\int_{\mathbb{R}\times\mathbb{R}^d} \mathcal{F}\varphi(\lambda,\xi)\bar{\mathcal{F}}\psi_t(\lambda,\xi)\mu_0(d\lambda)\mu(d\xi) \right) \right],$$

where the function ψ_t is given by

$$\psi_t(\tau,x) = \int_0^t p_\epsilon\left(\tau - (t-s), x - B_s\right) ds.$$

Evaluating the Fourier transform of ψ_t and plugging into (3.91), we thus get

$$(3.92) \qquad \mathcal{S}F(\varphi) = \int_0^t \left[\int_{\mathbb{R}\times\mathbb{R}^d} e^{-\epsilon^2(\lambda^2+|\xi|^2)/2} e^{i(\lambda(t-s)+\xi\cdot B_t)} \mathcal{F}\varphi(\lambda,\xi)\mu_0(d\lambda)\mu(d\xi) \right] ds.$$

Let us now compute the quantity $\|\mathcal{E}_\varphi\|_q$ in (3.90). Owing to the fact that $\|\mathcal{E}_g\|_1 = 1$ for any $g \in \mathcal{H}$, we easily get that

$$(3.93) \qquad \|\mathcal{E}_\varphi\|_q = \exp\left(\frac{q-1}{2} \int_{\mathbb{R}\times\mathbb{R}^d} |\mathcal{F}\varphi(\lambda,\xi)|^2 \mu_0(d\lambda)\mu(d\xi) \right).$$

Therefore, gathering (3.92) and (3.93) into (3.90), taking limit $\epsilon \to 0$ in (3.92) and observing that $\mathcal{F}\varphi \in L^2(\mathbb{R}\times\mathbb{R}^d; \mu_0 \otimes \mu)$ whenever $\varphi \in \mathcal{H}$, we end up with our claim (3.87). $\quad\square$

REMARK 3.21. By the local finiteness and the symmetry of μ_0 and μ, the function $\tilde{h}$ is real-valued, bounded and uniformly continuous on $[0,1] \times \mathbb{R}^d$ whenever $h \in \mathcal{G}_0$.

Our next step is to relate the lower bound (3.87) to the space $\mathcal{A}_d$ introduced in Definition 3.10. This is summarized in the following lemma.

LEMMA 3.22. *Let us assume that the conditions of Proposition 3.20 are met. We also suppose that the following condition is satisfied*:

$$(3.94) \qquad \sup_{h\in\mathcal{G}_0} \sup_{g\in\mathcal{A}_d} \left\{ \phi_t(h,g) - \frac{1}{2}\int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2 \, dx \, ds \right\} > 0,$$

where we recall that $\mathcal{G}_0$ is the space introduced in Definition 3.19 *and where the variational quantity $\phi_t(h, g)$ is defined by*

$$(3.95) \quad \phi_t(h, g) = \int_{\mathbb{R} \times \mathbb{R}^d} h(\lambda, \xi) \tilde{\mathcal{F}} g^2(\lambda, \xi) \mu_0(d\lambda) \mu(d\xi)$$
$$- \frac{1}{2t^{2H_0 - 1}(p - 1)} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(d\xi),$$

with $\tilde{\mathcal{F}} g^2$ being the truncated Fourier transform given in (3.79). *Then, we have that*

$$(3.96) \qquad \lim_{\epsilon \to 0} \| u_t^{\epsilon, \diamond}(x) \|_p = \infty.$$

PROOF. We start from the right-hand side of (3.87) and consider a generic $h \in \mathcal{G}_0$. Then, for $b, t > 0$, we also introduce a family of rescaled functions $\{ h_{t,b}; t, b > 0 \}$ given by

$$h_{t,b}(\lambda, \xi) = \frac{1}{t^{2H_0 - 1}} h \left(t\lambda, \left(\frac{t}{b} \right)^{1/2} \xi \right).$$

Each $h_{t,b}$ is an element of $\mathcal{G}_0$, and we will now evaluate the expression (3.87) for those functions. We will take the next two observations into account:

(i) The function $\tilde{h}_{t,b}$, defined by (3.88), can be computed thanks to an elementary change of variable. Owing to an additional Brownian scaling argument, we get

$$(3.97) \qquad \int_0^t \tilde{h}_{t,b}(s, B_s) \, ds \stackrel{(d)}{=} \int_0^b \tilde{h} \left(\frac{s}{b}, B_s \right) ds.$$

(ii) Under our standing assumption $d - H = 1$, we also have

$$(3.98) \quad \int_{\mathbb{R} \times \mathbb{R}^d} |h_{t,b}(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(d\xi) = \frac{b}{t^{2H_0 - 1}} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(d\xi).$$

Therefore, plugging (3.97) and (3.98) into (3.87), we get that

$$(3.99) \quad \liminf_{\epsilon \to 0} \| u_t^{\epsilon, \diamond}(x) \|_p$$
$$\geq \sup_{b > 0} \sup_{h \in \mathcal{G}_0} \mathbb{E} \left[\exp \left(\int_0^b \tilde{h} \left(\frac{s}{b}, B_s \right) ds - \frac{(q - 1)b}{2t^{2H_0 - 1}} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(ds) \right) \right].$$

We are now in a position to apply [6], Proposition 3.1, and take limits as $b \to \infty$ in (3.99). Indeed, it is readily checked that $\tilde{h}$ is bounded and uniformly continuous whenever $h \in \mathcal{G}_0$. Hence [6], Proposition 3.1, reads

$$(3.100) \quad \lim_{b \to \infty} \frac{1}{b} \log \mathbb{E} \left[\exp \left(\int_0^b \tilde{h} \left(\frac{s}{b}, B_s \right) ds \right) \right]$$
$$= \sup_{g \in \mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d} \tilde{h}(s, x) g^2(s, x) \, dx \, ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 \, dx \, ds \right\}.$$

We also trivially have

$$(3.101) \quad \lim_{b \to \infty} \frac{1}{b} \log \left(\exp \left(\frac{(q - 1)b}{2t^{2H_0 - 1}} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(ds) \right) \right)$$
$$= \frac{(q - 1)}{2t^{2H_0 - 1}} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(ds).$$

Gathering (3.100) and (3.101) into (3.99) and considering the limit case $b \to \infty$, we have obtained that (3.96) holds true whenever the following relation is met for at least a $h \in \mathcal{G}_0$:

(3.102)
$$\sup_{g \in \mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d} \tilde{h}(s, x) g^2(s, x) \, dx \, ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 \, dx \, ds \right.$$
$$\left. - \frac{q-1}{2t^{2H_0-1}} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(d\xi) \right\} > 0.$$

In order to go from (3.102) to (3.94), we proceed as follows: recalling the definition of $\tilde{\mathcal{F}}$ in (3.79), we will see in the next Lemma 3.23 that if $g \in \mathcal{A}_d$, then $\tilde{\mathcal{F}} g^2 \in \overline{\mathcal{G}_0}$, where $\overline{\mathcal{G}_0}$ is defined by (3.104). With the expression (3.88) for $\tilde{h}(s, x)$ in mind, a direct application of Parseval's identify yields

(3.103) $$\int_0^1 \int_{\mathbb{R}^d} \tilde{h}(s, x) g^2(s, x) \, dx \, ds = \int_{\mathbb{R} \times \mathbb{R}^d} h(\lambda, \xi) \tilde{\mathcal{F}} g^2(\lambda, \xi) \mu_0(d\lambda) \mu(d\xi).$$

Invoking the fact that $q - 1 = 1/(p-1)$, we thus deduce that

$$\int_0^1 \int_{\mathbb{R}^d} \tilde{h}(s, x) g^2(s, x) \, dx \, ds - \frac{q-1}{2t^{2H_0-1}} \int_{\mathbb{R} \times \mathbb{R}^d} |h(\lambda, \xi)|^2 \mu_0(d\lambda) \mu(d\xi) = \phi_t(h, g),$$

where $\phi_t(h, g)$ is defined by (3.95). Hence, it is readily checked that condition (3.102) is equivalent to (3.94) which finishes our proof. $\square$

We now turn to the technical result used in the proof of Lemma 3.22.

LEMMA 3.23. *Recall that the space $\mathcal{A}_d$ is given in Definition 3.10. Let $\overline{\mathcal{G}_0}$ be the space defined by*

(3.104) $$\overline{\mathcal{G}_0} = \{ h \in L^2(\mathbb{R} \times \mathbb{R}^d; \mu_0 \otimes \mu); h(-\lambda, -\xi) = \bar{h}(\lambda, \xi) \mu_0 \otimes \mu\text{-a.e.} \}$$

Then, for any $g \in \mathcal{A}_d$, we have $\tilde{\mathcal{F}}(g^2) \in \overline{\mathcal{G}_0}$.

PROOF. This is a direct consequence of inequality (3.21) and definition of $\tilde{\mathcal{F}}$ in (3.79). $\square$

Now we are ready to prove the main result of this section.

PROOF OF THEOREM 3.14-ITEM (iii). We have shown that $\|u_t^{\epsilon, \diamond}(x)\|_p$ diverges as $\epsilon \to 0$ as long as relation (3.94) is satisfied. Note that, since $\tilde{\mathcal{F}}(g^2) \in \overline{\mathcal{G}_0}$, the functional $\phi_t(h, g)$, defined in (3.95), is continuous in h with respect to the $L^2(\mathbb{R}^{d+1}, \mu_0 \otimes \mu)$-norm. Therefore,

$$\sup_{h \in \mathcal{G}_0} \phi_t(h, g) = \sup_{h \in \overline{\mathcal{G}_0}} \phi_t(h, g).$$

Hence, we aim to show that (3.94) is satisfied for a suitable family of functions $h \in \overline{\mathcal{G}_0}$. Namely, we want to find some functions $h \in \overline{\mathcal{G}_0}$ such that

(3.105) $$\sup_{g \in \mathcal{A}_d} \left\{ \phi_t(h, g) - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 \, dx \, ds \right\} > 0.$$

To this end, we will consider a family of functions $\{h^{\theta, g}; \theta \in \mathbb{R}, g \in \mathcal{A}_d\}$. Each $h^{\theta, g}$ is defined by

$$h^{\theta, g}(\lambda, \xi) = \theta [\tilde{\mathcal{F}}(g^2)](-\lambda, -\xi) = \theta \overline{[\tilde{\mathcal{F}}(g^2)]}(\lambda, \xi),$$

where we recall that $\tilde{\mathcal{F}}(g^2) \in \overline{\mathcal{G}_0}$, according to Lemma 3.23. For such a function $h^{\theta,g}$, we have

$$\phi_t(h^{\theta,g}, g) = \left(\theta - \frac{\theta^2}{2t^{2H_0-1}(p-1)}\right) \int_{\mathbb{R} \times \mathbb{R}^d} |\tilde{\mathcal{F}}(g^2)(\lambda, \xi)|^2 \mu_0(d\lambda)\mu(d\xi).$$

Thus, an elementary computation reveals that

$$\sup_{\theta \in \mathbb{R}} \phi_t(h^{\theta,g}, g) = \frac{(p-1)t^{2H_0-1}}{2} \int_{\mathbb{R} \times \mathbb{R}^d} |\tilde{\mathcal{F}}(g^2)(\lambda, \xi)|^2 \mu(d\lambda)\mu(d\xi).$$

Plugging this information into (3.105), we get

$$\sup_{h \in \overline{\mathcal{G}_0}} \sup_{g \in \mathcal{A}_d} \left\{ \phi_t(h, g) - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 \, dx \, ds \right\}$$

$$\geq \sup_{g \in \mathcal{A}_d} \left\{ \frac{(p-1)t^{2H_0-1}}{2} \int_{\mathbb{R} \times \mathbb{R}^d} |\tilde{\mathcal{F}}(g^2)(\lambda, \xi)|^2 \mu(d\lambda)\mu(d\xi) \right.$$

$$\left. - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 \, dx \, ds \right\}.$$

According to Notation 3.12 and to the definition of $t_0(p)$ in Theorem 3.14, it is clear that the above quantity is strictly positive as soon as $t > t_0(p)$. This concludes the proof. $\quad\square$

4. Stratonovich case. In this section we analyze the moments of equation (1.1), interpreted in the Stratonovich sense, for a wide class of noises. We first briefly recall, in Section 4.1, the main well posedness result of [8] regarding the Stratonovich equation. The moment analysis will then take place in Section 4.2.

4.1. *Interpretation of the solution.* The basic idea behind the Stratonovich interpretation of equation (1.1) can be roughly expressed as follows. We consider a sequence $\{\dot{W}^n; n \geq 1\}$ of smooth approximations of $\dot{W}$ which can be thought of as the mollification considered in (3.6) with $\epsilon = 1/n$. With this approximation in hand, let $\{u^n; n \geq 1\}$ be the sequence of classical solutions associated with $\dot{W}^n$, that is, u^n is the solution of

(4.1) $$\partial_t u_t^n(x) = \frac{1}{2}\Delta u_t^n(x) + u_t^n(x)\dot{W}_t^n(x), \quad t \in \mathbb{R}_+, x \in \mathbb{R}^d,$$

understood in the classical Lebesgue sense. Then, the Stratonovich solution of (1.1) is morally defined as the limit (if it exists) of the sequence u^n.

In some situations, where $\dot{W}$ is not too rough, the above heuristic idea can be rigorously formulated within the so-called Young setting (see, e.g., [21], Section 5). In order to extend these considerations to rougher noises (which overall corresponds to our objective in this study), more intricate machineries must come into the picture as well as renormalization procedures.

Thus, in the companion paper [8], we have relied on the sophisticated *theory of regularity structures* (as introduced by Hairer in [18]) to provide a complete treatment of the Stratonovich model, that is, to show existence and uniqueness of a global solution in a rough regime. For the sake of conciseness, we will here skip the details of this analysis (which involve the exhibition of an abstract solution map and the construction of a related K-rough path) and will directly provide the resulting convergence statement for the approximated equation (4.1).

To this end, we shall need the following piece of notation.

NOTATION 4.1.   Let ρ be the weight given by $\rho(s, x) := p_1(s) p_1(x)$, as considered in (3.6), and define the approximated noise $\dot{W}^n$ of $\dot{W}$ by $\dot{W}^0 := 0$ and, for $n \geq 1$,

$$(4.2) \qquad\qquad \dot{W}^n := \partial_t \partial_{x_1} \cdots \partial_{x_d} W^n,$$

where $W^n := \rho_n * W$ and $\rho_n(s, x) := 2^{n(d+2)} \rho(2^{2n} s, 2^n x)$.

For any vector $(H_0, \mathbf{H}) \in (0, 1)^{d+1}$, we set from now on

$$(4.3) \qquad\qquad \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi) := \frac{1}{|\lambda|^{2H_0 - 1}} \prod_{i=1}^{d} \frac{1}{|\xi_i|^{2H_i - 1}},$$

namely, $c_0 c_{\mathbf{H}} \mathcal{N}_{H_0, \mathbf{H}}$ is the density of the measure $\mu_0 \otimes \mu$ introduced in (1.5). Let us also set

$$(4.4) \qquad\qquad c_{H_0, \mathbf{H}} = \left(\prod_{i=0}^{d} \alpha_{H_i} \right)^{-1/2} \quad \text{with } \alpha_{H_i} := \int_{\mathbb{R}} d\xi \, \frac{|e^{\iota \xi} - 1|^2}{|\xi|^{2H_i + 1}}.$$

Finally, note that, when $2H_0 + H < d + 1$ (where the notation H has been introduced in (1.6)), it can be shown that the integral below is finite (see [8]):

$$(4.5) \qquad\qquad \mathcal{J}_{\rho, H_0, \mathbf{H}} := \int_{\mathbb{R}^{d+1}} |\mathcal{F} \rho(\lambda, \xi)|^2 \mathcal{F} p(\lambda, \xi) \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi) \, d\lambda \, d\xi.$$

With this notation in hand, we are ready to recall the main result of [8] about the approximated equation.

DEFINITION 4.2.   Let $(H_0, \mathbf{H}) \in (0, 1)^{d+1}$ be a vector of Hurst parameters such that

$$(4.6) \qquad\qquad d + \frac{2}{3} < 2H_0 + H \leq d + 1.$$

Fix $\alpha \in \mathbb{R}$ such that

$$-\frac{4}{3} < \alpha < -(d + 2) + 2H_0 + H$$

as well as an arbitrary time horizon $T > 0$ and an initial condition $\psi \in L^\infty(\mathbb{R}^d)$. Finally, let $(\mathfrak{c}^{(n)}_{\rho, H_0, \mathbf{H}})_{n \geq 1}$ be the sequence defined for every $n \geq 1$ as

$$(4.7) \quad \begin{aligned} &\mathfrak{c}^{(n)}_{\rho, H_0, \mathbf{H}} \\ &:= \begin{cases} c^2_{H_0, \mathbf{H}} 2^{2n(d+1-(2H_0+H))} \mathcal{J}_{\rho, H_0, \mathbf{H}} & \text{if } 2H_0 + H < d + 1 \\ c^2_{H_0, \mathbf{H}} \displaystyle\int_{|\lambda| + |\xi|^2 \geq 2^{-2n}} |\mathcal{F} \rho(\lambda, \xi)|^2 \\ \qquad \times \mathcal{F} p(\lambda, \xi) \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi) \, d\lambda \, d\xi & \text{if } 2H_0 + H = d + 1 \end{cases} \end{aligned}$$

and consider the sequence $(u^n)_{n \geq 1}$ of classical solutions of the equation

$$(4.8) \qquad \begin{cases} \partial_t u^n = \dfrac{1}{2} \Delta u^n + u^n \dot{W}^n - \mathfrak{c}^{(n)}_{\rho, H_0, \mathbf{H}} u^n, & t \in [0, T], x \in \mathbb{R}^d, \\ u_0^n(x) = \psi(x). \end{cases}$$

Then, the sequence u^n converges almost surely in $L^\infty([0, T] \times \mathbb{R}^d)$. The limit $u = \lim u^n$ is said to be the Stratonovich solution to the renormalized equation of (1.1).

4.2. *Moments estimates for the Stratonovich equation.* Similarly to the Skorohod situation (see Section 3), we now would like to show that the moments of the renormalized Stratonovich solution u (as introduced in Definition 4.2) are all finite in the subcritical regime (1.7). With this objective in mind, observe first that Definition 4.2 only guarantees *almost sure* convergence of u^n to u and does not provide any estimate on the moments of u. In order to go further, our strategy will somehow consist in bounding the moments of the Stratonovich solution u in terms of those of the Skorohod solution $u^\diamond$ (at the level of their respective approximations) and then exploiting the estimates of Section 3 for the moments of $u^\diamond$.

For more clarity and although both Proposition 3.8 and Definition 4.2 are valid for $H_0 \leq \frac{1}{2}$ (see [7] for the Skorohod case), we will restrict our analysis to the case where $H_0 > \frac{1}{2}$ in the sequel. Besides, for our comparison strategy to be possible, we, of course, need to focus on situations where both u and $u^\diamond$ are well defined. *Therefore, for the remainder of the section we will assume that both conditions (1.7) and (4.6) are met.*

As mentioned above, the idea will be to compare $u^\diamond$ and u through the Feynman–Kac representations of their respective approximations. Just as in Section 3 (resp., Definition 4.2), we denote by $u^{n,\diamond}$ (resp., u^n) the approximation of $u^\diamond$ (resp., u). Namely, $u^{n,\diamond}$ is the solution of (3.12), while u^n solves (4.8), for every fixed $n \geq 1$. Let us assume that both solutions start from an initial condition $\psi \in L^\infty(\mathbb{R}^d)$. At this point, let us recall that, according to [22] (and just as in Proposition 3.6, Proposition 5.2), the Feynman–Kac representation of $u^{n,\diamond}$ is given by

$$(4.9) \qquad u_t^{n,\diamond}(x) = \mathbb{E}\left[\psi(B_t^x) \exp\left(V_t^{n,W}(B^x) - \frac{1}{2}\beta_t^{n,B} \right) \right],$$

where B^x is a standard d-dimensional Wiener process starting at x, independent of W and where $V^{n,W}, \beta_t^n$ are given by

$$V_t^{n,W}(B^x) := \int_0^t du\, \dot{W}^n(u, B_{t-u}^x), \quad \text{and} \quad \beta_t^{n,B} := \mathbf{E}[|V_t^{n,W}(B^x)|^2].$$

On the other hand, it is clear that, for every fixed $n \geq 1$, equation (4.8) is a standard linear parabolic equation, for which a classical Feynman–Kac formula can be applied. This yields the expression

$$(4.10) \qquad u_t^n(x) = \mathbb{E}[\psi(B_t^x) \exp(V_t^{n,W}(B^x) - \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)} t)],$$

where $\mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}$ stands for the renormalization constant in equation (4.8).

Based on the two representations (4.9) and (4.10), we deduce that the desired comparison between $u^{n,\diamond}$ and u^n morally reduces to a comparison between $\frac{1}{2}\beta_t^{n,B}$ and $\mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)} t$. The following uniform estimate will thus be an important step in the procedure.

PROPOSITION 4.3. *Assume that $H_0 > \frac{1}{2}$ and that the condition (4.6) is satisfied. Then, for every fixed $t \geq 0$, it holds that*

$$(4.11) \qquad \sup_{n \geq 1} \left| \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)} t - \frac{1}{2}\mathbb{E}[\beta_t^{n,B}] \right| < \infty.$$

PROOF. Following the definition (4.7) of $\mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}$, we have to treat the cases $2H_0 + H < d + 1$ and $2H_0 + H = d + 1$ separately.

First case: $2H_0 + H < d + 1$. Let us recall that in this situation, $\mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}$ is defined as

$$(4.12) \qquad \begin{aligned} \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)} &:= \mathfrak{c}_{\rho,H_0,\mathbf{H}} \cdot 2^{2n(d+1-(2H_0+H))} \\ &= c_{H_0,\mathbf{H}}^2 \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi\, |\mathcal{F}\rho_n(\lambda,\xi)|^2 \mathcal{F}p(\lambda,\xi)\mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi), \end{aligned}$$

where $\mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi)$ is the quantity defined in (4.3). Keeping this expression in mind and using the covariance formula (2.3), we can recast the expression (3.10) for $\beta_t^{n,B}$ as

$$\beta_t^{n,B} = \mathbf{E}[|V_t^{n,W}(B^x)|^2]$$
$$= c_{H_0,\mathbf{H}}^2 \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi |\mathcal{F}\rho_n(\lambda,\xi)|^2 \mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi) \int_{[0,t]^2} du\, dv\, e^{\iota(\lambda(v-u)+\xi\cdot(B_u-B_v))}.$$

Therefore, invoking elementary symmetry and integration arguments, we get

$$\frac{1}{2}\mathbb{E}[\beta_t^{n,B}]$$

(4.13)
$$= c_{H_0,\mathbf{H}}^2 \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi |\mathcal{F}\rho_n(\lambda,\xi)|^2 \mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi) \int_0^t du \int_0^u dv\, e^{-\iota\lambda(u-v)} e^{-\frac{|\xi|^2}{2}(u-v)}$$

$$= c_{H_0,\mathbf{H}}^2 \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi |\mathcal{F}\rho_n(\lambda,\xi)|^2 \mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi) \int_0^t dv(t-v) e^{-v(\frac{|\xi|^2}{2}+\iota\lambda)}$$

$$= c_{H_0,\mathbf{H}}^2 \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi |\mathcal{F}\rho_n(\lambda,\xi)|^2 \mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi)$$

(4.14)
$$\times \left[\frac{t}{\frac{|\xi|^2}{2}+\iota\lambda} - \frac{1}{\frac{|\xi|^2}{2}+\iota\lambda} \int_0^t dv\, e^{-v(\frac{|\xi|^2}{2}+\iota\lambda)} \right].$$

Hence, owing to the fact that the Fourier transform of the heat kernel p satisfies $\mathcal{F}p(\lambda,\xi) = (|\xi|^2/2 + \iota\lambda)^{-1}$, together with (4.12), we get

(4.15)
$$\frac{1}{2}\mathbb{E}[\beta_t^{n,B}] = \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)} t - r_t^n$$

with a constant r_t^n defined by

(4.16)
$$r_t^n := 2c_{H_0,\mathbf{H}}^2 \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi |\mathcal{F}\rho_n(\lambda,\xi)|^2 \frac{\mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi)}{|\xi|^2+2\iota\lambda} \int_0^t dv\, e^{-v(\frac{|\xi|^2}{2}+\iota\lambda)}.$$

Since $\sup_{n\geq 1} |\mathcal{F}\rho_n(\lambda,\xi)|^2 \lesssim 1$, our claim (4.11) amounts to prove that

(4.17)
$$\mathcal{A}_t \equiv \int_{\mathbb{R}^{d+1}} d\lambda\, d\xi \frac{\mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi)}{||\xi|^2+2\iota\lambda|} \left| \int_0^t dv\, e^{-v(\frac{|\xi|^2}{2}+\iota\lambda)} \right| < \infty.$$

In order to bound $\mathcal{A}_t$, one must decompose the domain $\mathbb{R}^{d+1}$ into a centered ball and some unbounded domains.

Regarding integration over the ball $\{|\lambda| \leq 1, |\xi_1| \leq 1, \ldots, |\xi_d| \leq 1\}$, one has, for all $\tau_0, \tau_1, \ldots, \tau_d \in [0,1]$ such that $\sum_{i=0}^d \tau_i = 1$,

(4.18)
$$\int_{\{|\lambda|\leq 1, |\xi_1|\leq 1, \ldots, |\xi_d|\leq 1\}} d\lambda\, d\xi \frac{\mathcal{N}_{H_0,\mathbf{H}}(\lambda,\xi)}{||\xi|^2+2\iota\lambda|} \left| \int_0^t dv\, e^{-v(\frac{|\xi|^2}{2}+\iota\lambda)} \right|$$

$$\leq t \int_{\{|\lambda|\leq 1, |\xi_1|\leq 1, \ldots, |\xi_d|\leq 1\}} \frac{d\lambda\, d\xi}{||\xi|^2+2\iota\lambda|} \frac{1}{|\lambda|^{2H_0-1}} \prod_{i=1}^d \frac{1}{|\xi_i|^{2H_i-1}}$$

(4.19)
$$\leq t \left(\int_{|\lambda|\leq 1} \frac{d\lambda}{|\lambda|^{2H_0+\tau_0-1}} \right) \left(\prod_{i=1}^d \int_{|\xi_i|\leq 1} \frac{d\xi_i}{|\xi_i|^{2H_i+2\tau_i-1}} \right).$$

At this point, note that, due to the condition $2H_0 + H < d + 1$, we can actually pick the τ_i's such that $2H_0 + \tau_0 - 1 < 1$ and $2H_i + 2\tau_i - 1 < 1$ for $i = 1, \ldots, d$. Indeed, summing the latter constraints, we get $4H_0 + 2H + 2\sum_{i=0}^d \tau_i < 4 + 2d$, which, combined with the

relation $\sum_{i=0}^d \tau_i = 1$, brings us back to the condition $2H_0 + H < d + 1$. Choosing this set of parameters in (4.19), we can conclude that the integral in (4.18) is finite.

As for the integral on the remaining unbounded domains, we will only focus here on the region $\{|\lambda| \geq 1, |\xi_1| \geq 1, \ldots, |\xi_d| \geq 1\}$ in the right-hand side of (4.17). In this case, for all $\tau_0, \tau_1, \ldots, \tau_d \in [0, 1]$ such that $\sum_{i=0}^d \tau_i = 1$ and recalling again the expression (4.3) for $\mathcal{N}_{H_0, \mathbf{H}}$, we have

$$(4.20) \qquad \int_{|\lambda| \geq 1, |\xi_1| \geq 1, \ldots, |\xi_d| \geq 1} d\lambda \, d\xi \frac{\mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi)}{||\xi|^2 + 2\imath\lambda|} \left| \int_0^t dv \, e^{-v(\frac{|\xi|^2}{2} + \imath\lambda)} \right|$$

$$\lesssim \int_{|\lambda| \geq 1, |\xi_1| \geq 1, \ldots, |\xi_d| \geq 1} d\lambda \, d\xi \frac{\mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi)}{\lambda^2 + \xi_1^4 + \cdots + \xi_d^4}$$

$$(4.21) \qquad \lesssim \left(\int_{|\lambda| \geq 1} \frac{d\lambda}{|\lambda|^{2H_0 + 2\tau_0 - 1}} \right) \prod_{i=1}^d \int_{|\xi_i| \geq 1} \frac{d\xi_i}{|\xi_i|^{2H_i + 4\tau_i - 1}}.$$

We can now pick the τ_i's such that $2H_0 + 2\tau_0 > 2$ and $2H_i + 4\tau_i > 2$ for $i = 1, \ldots, d$, due to the fact that we have

$$2(2H_0 + 2\tau_0) + \sum_{i=1}^d (2H_i + 4\tau_i) = 2(2H_0 + H) + 4 > 4 + 2d,$$

where the last inequality is ensured by (4.6). Choosing this set of parameters in (4.21), we obtain that the integral in (4.20) is finite.

The estimate of the integral on the other domains can then clearly be done along similar arguments which achieves the proof of (4.17).

Second case: $2H_0 + H = d + 1$. In this situation, $\mathfrak{c}_{\rho, H_0, \mathbf{H}}^{(n)}$ is defined as

$$\mathfrak{c}_{\rho, H_0, \mathbf{H}}^{(n)} := c_{H_0, \mathbf{H}}^2 \int_{|\lambda| + |\xi|^2 \geq 2^{-2n}} |\mathcal{F}\rho(\lambda, \xi)|^2 \mathcal{F}p(\lambda, \xi) \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi) \, d\lambda \, d\xi$$

which, thanks to the relation $2H_0 + H = d + 1$, can also be written as

$$\mathfrak{c}_{\rho, H_0, \mathbf{H}}^{(n)} = c_{H_0, \mathbf{H}}^2 \int_{|\lambda| + |\xi|^2 \geq 1} d\lambda \, d\xi |\mathcal{F}\rho_n(\lambda, \xi)|^2 \mathcal{F}p(\lambda, \xi) \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi).$$

Therefore, going back to the expression (4.13) of $\frac{1}{2}\mathbb{E}[\beta_t^{n, B}]$ and performing the same trick as in (4.14), we get that

$$\mathfrak{c}_{\rho, H_0, \mathbf{H}}^{(n)} t - \frac{1}{2}\mathbb{E}[\beta_t^{n, B}]$$

$$= -c_{H_0, \mathbf{H}}^2 \int_{|\lambda| + |\xi|^2 \leq 1} d\lambda \, d\xi |\mathcal{F}\rho_n(\lambda, \xi)|^2 \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi) \int_0^t du \int_0^u dv \, e^{-\imath\lambda(u-v)} e^{-\frac{|\xi|^2}{2}(u-v)}$$

$$+ c_{H_0, \mathbf{H}}^2 \int_{|\lambda| + |\xi|^2 \geq 1} d\lambda \, d\xi |\mathcal{F}\rho_n(\lambda, \xi)|^2 \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi) \left[\frac{1}{\frac{|\xi|^2}{2} + \imath\lambda} \int_0^t dv \, e^{-v(\frac{|\xi|^2}{2} + \imath\lambda)} \right],$$

and, accordingly,

$$\sup_{n \geq 1} \left| \mathfrak{c}_{\rho, H_0, \mathbf{H}}^{(n)} t - \frac{1}{2}\mathbb{E}[\beta_t^{n, B}] \right| \lesssim t \int_{|\lambda| + |\xi|^2 \leq 1} d\lambda \, d\xi \mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi)$$

$$+ \int_{|\lambda| + |\xi|^2 \geq 1} d\lambda \, d\xi \frac{\mathcal{N}_{H_0, \mathbf{H}}(\lambda, \xi)}{||\xi|^2 + 2\imath\lambda|} \left| \int_0^t dv \, e^{-v(\frac{|\xi|^2}{2} + \imath\lambda)} \right|.$$

By the very definition of $\mathcal{N}_{H_0,\mathbf{H}}$, the first integral in the latter bound is clearly finite. As for the second integral, its finiteness can be easily derived from similar arguments to those used for (4.20) which achieves the proof of our assertion. $\quad\square$

Based on relations (4.9), (4.10) and (4.11), the fact that one can transfer subcritical estimates from the Skorohod to the Stratonovich equation can be explained in the following way: under the subcritical condition (1.7), the fluctuations of the random variable Y_t^n, given by

$$(4.22) \qquad Y_t^n := \beta_t^{n,B} - \mathbb{E}[\beta_t^{n,B}],$$

are much smaller than the fluctuation of the random variable $V_t^{n,W}(B)$ featuring in (4.9) and (4.10). This assertion is quantified in the following lemma.

LEMMA 4.4. *Assume that $H_0 > \frac{1}{2}$ and that the conditions (1.7) and (4.6) are both satisfied. For $n \geq 1$, let Y_t^n be the random variables defined by (4.22). Then, for all $\theta \geq 0$ and $t \geq 0$, we have*

$$(4.23) \qquad \sup_{n \geq 1} \mathbb{E}[e^{\theta Y_t^n}] < \infty.$$

PROOF. We first fix $n \geq 1$. Our proof is inspired by the computations in [21], Theorem 4.6, where the exponential integrability of a random variable similar to Y_t^n is investigated under more restrictive assumptions on $H_1, \ldots, H_d$. As in [21], our analysis will be based on Le Gall's decomposition for β_t^n. More specifically, we start the following construction: for $N \geq 1$ and $k = 1, \ldots, 2^{N-1}$, we set

$$J_{N,k} = \left[\frac{(2k-2)t}{2^N}, \frac{(2k-1)t}{2^N}\right), \qquad I_{N,k} = \left[\frac{(2k-1)t}{2^N}, \frac{2kt}{2^N}\right)$$

and $A_{N,k} = J_{N,k} \times I_{N,k}$. Then, we can decompose $\beta_t^{n,B}$ as

$$\beta_t^{n,B} = \sum_{N=1}^{\infty} \sum_{k=1}^{2^{N-1}} a_{N,k}^n, \quad \text{where } a_{N,k}^n = \int_{A_{N,k}} \gamma_0^n(r-s)\gamma^n(B_r^x - B_s^x)\, dr\, ds,$$

where γ_0^n (resp., γ^n) is the inverse Fourier transform of the measure $\mu_0^n := \mu_0^{2^{-4n}}$ (resp., $\mu^n := \mu^{2^{-2n}}$), defined by (3.7). Furthermore, owing to the shape of $A_{N,k}$'s combined with the independence of the Brownian increments, one can write

$$(4.24) \qquad a_{N,k}^n \overset{(d)}{=} a_N^n, \quad \text{with } a_N^n = \int_{[0,t/2^N]^2} \gamma_0^n(t+s)\gamma^n(B_r - \tilde{B}_s)\, ds\, dr,$$

where B and $\tilde{B}$ are two independent Brownian motions starting from the origin. It can be shown, exactly along the same lines as in [21], equation (4.13), that the exponential integrability of Y_t^n amounts to the following estimate: for any arbitrarily small $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that, for all $m \geq 1$, we have

$$(4.25) \qquad \mathbb{E}[(a_N^n)^m] \leq C_\varepsilon m! \left(\frac{C\varepsilon}{2^N}\right)^{\alpha m},$$

with $\alpha > 1/2$. In fact, we should point out that the exponential integrability in [21] was obtained thanks to (4.25) with $\alpha = 1$. However, by a closer examination of the proof in [21], it is clear that the exponential integrability (4.23) follows as long as $\alpha > 1/2$ in (4.25). The remainder of our proof is thus devoted to justify (4.25).

Recall from (3.7) that μ_0^n and μ^n are dominated by μ_0 and μ, respectively, where μ_0 and μ are introduced in (1.5). Therefore, for any $\tau > 0$, we have

$$(4.26)$$

$$\mathbb{E}\left[\left(\int_{[0,\tau]^2} \gamma_0^n(t+s)\gamma^n(B_r - \tilde{B}_s)\,ds\,dr\right)^m\right]$$

$$= \int_{(\mathbb{R}^{d+1})^m} \mu_0^n(d\lambda)\mu^n(d\xi)\left|\int_{[0,\tau]^m} \mathbb{E}\left[\prod_{k=1}^m e^{\iota(\lambda_k s_k + \xi_k \cdot B_{s_k})}\right]ds\right|^2$$

$$\leq \int_{(\mathbb{R}^{d+1})^m} \mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,\tau]^m} \mathbb{E}\left[\prod_{k=1}^m e^{\iota(\lambda_k s_k + \xi_k \cdot B_{s_k})}\right]ds\right|^2$$

$$= \mathbb{E}\left[\left(\int_{[0,\tau]^2} \gamma_0(t+s)\gamma(B_r - \tilde{B}_s)\,ds\,dr\right)^m\right].$$

On the other hand, by [6], Inequality (3.1), we have

$$(4.27) \qquad \mathbb{E}\left[\left(\int_{[0,\tau]^2} \gamma_0(t-s)\gamma(B_r - \tilde{B}_s)\,ds\,dr\right)^m\right] \leq C^m(m!)^{d-H}\tau^{(2H_0+H-d)m},$$

where $C > 0$ is a constant independent of $\tau > 0$. Now, combining (4.24), (4.26) and (4.27), together with the trivial relation that $\gamma_0(t+s) \leq \gamma_0(t-s)$, we have, for all $m \geq 1$,

$$\mathbb{E}[a_N^n]^m \leq C^m(m!)^{d-H}\left(\frac{t}{2^N}\right)^{(2H_0+H-d)m}.$$

Note that the above control of moments of a_N^n is uniform in n. Moreover, under our assumption $d - H < 1$, we can conclude that, for any $\epsilon > 0$, the above becomes

$$\mathbb{E}[a_N^n]^m \leq C^m(m!)\left(\frac{\epsilon t}{2^N}\right)^{(2H_0+H-d)m},$$

when m is large enough. Finally, observe that we assume $2H_0 + H - d > \frac{1}{2}$ by (4.6). Condition (4.25) is, therefore, satisfied, and the integrability of Y_t^n (uniformly in n) follows. This finishes the proof of (4.23). $\square$

We are finally in a position to prove the main result of this section, that is, the fact that the Stratonovich solution admits finite moments of any order.

PROPOSITION 4.5. *Assume that $H_0 > \frac{1}{2}$ and that the conditions (1.7) and (4.6) are both satisfied. Let u be the solution of the Stratonovich equation, as given by Definition (4.2), with initial condition $\psi \equiv 1$. Then, for all $t \geq 0$ and $p \geq 1$, we have*

$$\mathbf{E}[|u_t(x)|^p] = c_{p,t} < \infty.$$

REMARK 4.6. The extension of Proposition 4.5 to a general initial condition $\psi \in L^\infty(\mathbb{R}^d)$ is straightforward. Details are omitted here for sake of clarity.

PROOF OF PROPOSITION 4.5. Recall from (4.9) and (4.10) that

$$u_t^n(x) = \mathbb{E}[\exp(V_t^{n,W}(B^x) - \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}t)], \quad \text{and} \quad u_t^{n,\diamond}(x) = \mathbb{E}\left[\exp\left(V_t^{n,W}(B^x) - \frac{1}{2}\beta_t^{n,B}\right)\right].$$

Without loss of generality, we assume that p is an integer in the rest of the proof. We also write $\beta_t^n(B^x)$ for $\beta_t^{n,B}$ in order to keep our notation visible enough. Therefore, we get

$$(4.28) \qquad u_t^n(x) = \mathbb{E}\left[e^{V_t^{n,W}(B^x) - \frac{1}{2}\beta_t^n(B^x)} \cdot e^{\frac{1}{2}(\beta_t^n(B^x) - \mathbb{E}[\beta_t^n(B^x)])}\right] \cdot e^{\frac{1}{2}\mathbb{E}[\beta_t^n(B^x)] - \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}t}.$$

Taking expectation in (4.28), we end up with

$$
\mathbf{E}[(u_t^n(x))^p] = \mathbf{E}\mathbb{E}\left[\prod_{k=1}^p e^{V_t^{n,W}(B^{k,x}) - \frac{1}{2}\beta_t^n(B^{k,x})} \cdot \prod_{k=1}^p e^{\frac{1}{2}(\beta_t^n(B^{k,x}) - \mathbb{E}[\beta_t^n(B^{k,x})])}\right]
$$

(4.29)

$$
\times \prod_{k=1}^p e^{\frac{1}{2}\mathbb{E}[\beta_t^n(B^{k,x})] - \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}t},
$$

where $\{B^{k,x}; k = 1, \ldots, p\}$ is a family of independent Brownian motions which is also in-dependent from the noise W. Moreover, a standard application of Fubini's theorem in (4.29) yields

$$
\mathbf{E}[(u_t^n(x))^p] = \mathbb{E}\left[\mathbf{E}\left[\prod_{k=1}^p e^{V_t^{n,W}(B^{k,x}) - \frac{1}{2}\beta_t^n(B^{k,x})}\right]\right.
$$

(4.30)

$$
\left.\times \prod_{k=1}^p e^{\frac{1}{2}(\beta_t^n(B^{k,x}) - \mathbb{E}[\beta_t^n(B^{k,x})])}\right] \prod_{k=1}^p e^{\frac{1}{2}\mathbb{E}[\beta_t^n(B^{k,x})] - \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}t}.
$$

Now notice that

(4.31)
$$
\mathbf{E}\left[\prod_{k=1}^p e^{V_t^{n,W}(B^{k,x}) - \frac{1}{2}\beta_t^n(B^{k,x})}\right] = \exp\left(\sum_{1 \le j_1 < j_2 \le p} \alpha_t^{j_1 j_2; n}\right),
$$

where $\alpha_t^{j_1,j_2;n}$ is defined the same way as in (3.19) but with γ, γ_0 replaced by γ^n, γ_0^n. Plug-ging (4.31) into (4.31), we have

$$
\mathbf{E}[(u_t^n(x))^p] = \mathbb{E}\left[e^{\sum_{1 \le j_1 < j_2 \le p} \alpha_t^{j_1 j_2; n}} \prod_{k=1}^p e^{\frac{1}{2}(\beta_t^n(B^{k,x}) - \mathbb{E}[\beta_t^n(B^{k,x})])}\right] \prod_{k=1}^p e^{\frac{1}{2}\mathbb{E}[\beta_t^n(B^{k,x})] - \mathfrak{c}_{\rho,H_0,\mathbf{H}}^{(n)}t}.
$$

On the other hand, invoking similar considerations as for (4.26), we get that, for any $\lambda > 0$,

$$
\mathbb{E}[\exp(\lambda \alpha_t^{j_1 j_2; n})] \le \mathbb{E}[\exp(\lambda \alpha_t^{j_1 j_2})].
$$

The rest of the proof then follows from Proposition 3.9, Proposition 4.3, Lemma 4.4 and a simple application of Fatou's lemma and Hölder's inequality. □

REMARK 4.7. Based on the preliminary results of [16], we expect the convergence state-ment in Definition 4.2, that is, the statement giving birth to the Stratonovich solution, to remain true in the rougher situation

(4.32)
$$
d + \frac{1}{2} < 2H_0 + H \le d + \frac{2}{3}.
$$

We then expect the above Skorohod/Stratonovich comparison method to be still possible when conditions (1.7)–(4.32) and $H_0 > \frac{1}{2}$ are met (see, for instance, how the restriction $2H_0 + H > \frac{1}{2}$ naturally arises at the end of the proof of Lemma 4.4).

However, according to the regularity-structure machinery, the transition from the domain (4.6) to the rougher case (4.32) involves the construction of additional highly sophisticated processes, to be seen as third-order counterparts of the second-order elements exhibited in [8] (the so-called K-Lévy area). So far, the only such result in this direction is restricted to the one-dimensional situation (see [16], Proposition 2.14 and Theorem 3.2), and we prefer to wait for more mature existence statements before providing a thorough comparative analysis.

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