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Exponential asymptotics for Brownian self-intersection local times under Dalang's condition*

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Abstract

In this paper, we investigate the exponential asymptotics for Brownian self-intersection times under Dalang's condition. Our theorem includes the setting of non-homogeneous interaction functions.

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1 Introduction

Throughout, $\gamma(\cdot) \geq 0$ is an non-negative definite function on $\mathbb{R}^d$. With the generality this paper allowed, $\gamma(\cdot)$ can be a generalized function that is defined as a linear functional on $\mathcal{S}(\mathbb{R}^d)$, the set of all rapidly decreasing functions known as Schwartz space. The non-negative definity is defined as the property

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)f(x)f(y)dxdy \geq 0 \qquad \forall f\in\mathcal{S}(R^d).$$

By Bochner's theorem, there is a unique measure on $\mathbb{R}^d$, known as the spectral measure of $\gamma(\cdot)$, such that

$$\gamma(x)=\int_{\mathbb{R}^d}e^{i\xi\cdot x}\mu(d\xi).$$

Further, $\mu(d\xi)$ is tempered in the sense that

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^p\mu(d\xi)<\infty$$

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for some $p > 0$. In particular, $\mu(d\xi)$ is locally finite. A special example is when $\gamma(\cdot) = \delta_0(\cdot)$ (Dirac function), $\mu(d\xi)$ is the $(2\pi)^{-d}$-multiple of the Lebesgue measure on $\mathbb{R}^d$.

Let B_s be a d-dimensional Brownian motion. We are interested in the random Hamiltonian

$$\int_0^t\int_0^t \gamma(B_s - B_r)dsdr. \tag{1.1}$$

According to its role played here, $\gamma(\cdot)$ is called interaction function. When $\gamma(\cdot) = \delta_0(\cdot)$, the double time integral in (1.1) only exists in $d = 1$, and is called self-intersection local time for the reason that it measures the ability of a Brownian path path intersects itself. An interested reader is referred to Chapter 4, [2] for the discussion on its exponential asymptotics (or large deviations).

From $\delta_0(\cdot)$ to general $\gamma(\cdot)$, the notion of self-intersection local time is extended to the random Hamiltonian in (1.1) with the meaning of "self-intersection" being interpreted by the geometric shape of $\gamma(\cdot)$. When $\gamma(\cdot) = |\cdot|^{-\alpha}$ for some $0 < \alpha < d$ (known as Riesz potential), the ability of self-intersection is measured by the average distance between two points on the Brownian path. Another example is when

$$\gamma(x) = \sum_{z\in\mathbb{Z}} \delta_{az}(x) \qquad x \in \mathbb{R} \tag{1.2}$$

where $a > 0$ is a given constant. In this case, self-intersection means $B_s = B_r \mod a$ for $s \neq r$.

The exponential asymptotic behavior plays a fundamental role in the problem known as intermittency for a class of stochastic partial different equation driven by a Gaussian noise with $\gamma(\cdot)$ as its covariance function. The goal of this work is to investigate the large-t behaviors for the exponential moments of the self-intersection local times in (1.1) under a condition (on $\gamma(\cdot)$) as general as possible. Here is the main result of the paper:

Theorem 1.1. *Under the Dalang's condition*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty \tag{1.3}$$

the self-intersection local time in (1.1) is properly defined. Further,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_0^t\int_0^t \gamma(B_s-B_r)dsdr\right)^{1/2}\right\} \tag{1.4}$$
$$= \sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\right)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\},$$
$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\frac{1}{t}\int_0^t\int_0^t \gamma(B_s-B_r)dsdr\right\} \tag{1.5}$$
$$= \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\},$$

where

$$\mathcal{F}_d = \left\{g \in \mathcal{L}^2(\mathbb{R}^d);\quad \int_{\mathbb{R}^d}|\nabla g(x)|^2dx < \infty \ \text{ and } \ \int_{\mathbb{R}^d}|g(x)|^2dx = 1\right\}.$$

Further, the variations appearing on the right hand sides of (1.4) and (1.5) are finite.

Remark. The condition (1.3) is introduced by Robert Dalang [6] for solving parabolic Anderson equation with a Gaussian noise that takes $\gamma(\cdot)$ as its spatial covariance function.

By (3.1) and (2.5) below,

$$\mathbb{E}\int_0^t\int_0^t \gamma(B_s - B_r)dsdr = 2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_r^t \exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr$$
$$= \int_{\mathbb{R}^d}\frac{4}{|\xi|^2}\Big[t - \frac{2}{|\xi|^2}\Big(1-e^{-|\xi|^2t/2}\Big)\Big]\mu(d\xi).$$

By a routine computation on the right hand side, one can see that Dalang condition is equivalent to

$$\mathbb{E}\int_0^t\int_0^t \gamma(B_s - B_r)dsdr < \infty$$

for some $t > 0$ (or, equivalently, for all $t > 0$). Therefore, Dalang's condition is obviously necessary for the statement in Theorem 1.1.

There have been some investigations (see, e.g., [1] and [2]) on the exponential asymptotics for self-intersection local times that take form of (1.4) or (1.5). For the author's best knowledge, the results exist only under the homogeneity condition

$$\gamma(cx) = c^{-\alpha}\gamma(x) \qquad x\in\mathbb{R}^d, \quad c > 0 \tag{1.6}$$

for some $0 < \alpha < 2$. Under (1.6), the statements (1.4) and (1.5) are equivalent. Morever, they are equivalent to

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{2-\alpha}}\log\mathbb{E}\exp\Big\{\int_0^t\int_0^t \gamma(B_s - B_r)dsdr\Big\} \tag{1.7}$$
$$= \sup_{g\in\mathcal{F}_d}\Big\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\Big\}.$$

Through Feynman-Kac representation, (1.7) determines inttermittency for the prabolic Anderson model with the Gaussian noise that takes $\gamma(\cdot)$ as its co-variance function (see, e.g., [4]). It is not clear at this moment how (1.7) is extended to the setting beyond homogeneity (1.6).

Dalang's condition (1.3) connects existing results with general $\gamma(\cdot)$. A more substantial extension made in this paper is the encompass of the settings with non-homogeneity. Indeed, this work is partially motivated by some practically interesting models where (1.6) does not hold. One of such settings is when $\gamma(\cdot)$ has the periodicity

$$\gamma(x + az) = \gamma(x) \qquad x\in\mathbb{R}^d, \quad z\in\mathbb{Z}^d \tag{1.8}$$

for some $a > 0$. Clearly, (1.6) and (1.8) can not co-exist. By theory of Fourier series, the periodicity in (1.8) allows Fourier expansion

$$\gamma(x) \sim \sum_{z\in\mathbb{Z}^d}\mu_z\exp\Big\{i\frac{2\pi z\cdot x}{a}\Big\} \qquad x\in\mathbb{R}^d$$

in the sense that

$$\mu_z = \frac{1}{a^d}\int_{[-\frac{a}{2},\frac{a}{2}]^d}\gamma(x)\exp\Big\{-i\frac{2\pi z\cdot x}{a}\Big\}dx$$

where, for the sake of non-negativity of $\gamma(\cdot)$, $\mu_{-z} = \mu_z \ge 0$. In this case, the spectral measure is supported on $a\mathbb{Z}^d$ with $\mu(az) = \mu_z$ ($z\in\mathbb{Z}^d$). The Dalang's condition (1.3) becomes

$$\sum_{z\in\mathbb{Z}^d}\frac{\mu_z}{1+|z|^2} < \infty. \tag{1.9}$$

The case given in (1.2) satisfies (1.9) with $\mu_z = a^{-1}$ ($z \in \mathbb{Z}$).

The non-triviality of Theorem 1.1 can be observed from different aspects: It is the first time that Dalang's condition, which was introduced for a very different reason, becomes the right condition for some precise forms of large deviations. It is rather surprising to have the exponential integrabilities (especially the one needed for (1.5)) merely under (1.3). Even at the deterministic level, it is not obvious at all why variations appearing in Theorem 1.1 should be finite. Under the homogeneity (1.6), the finiteness of the variations are essentially the consequences of Gagliardo-Nirenberg and Hard-Littlewood-Sobolev inequalities. Here we list a deterministic consequence of Dalang's condition (1.3).

Corollary 1.2. *Under the Dalang's condition (1.3) there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y)f^2(x)f^2(y)dxdy \le \frac{1}{2}\|f\|_2^2\|\nabla f\|_2^2 + C\|f\|_2^4 \qquad \forall f \in W^{1,2}(\mathbb{R}^d) \tag{1.10}$$

Proof. Set

$$Q(f) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y)f^2(x)f^2(y)dxdy \qquad f \in W^{1,2}(\mathbb{R}^d).$$

Then

$$\begin{aligned} Q(f) &= \|f\|_2^4 Q\big(\|f\|_2^{-1}f\big) = \frac{1}{2}\|f\|_2^2\|\nabla f\|_2^2 + \|f\|_2^4\Big\{Q\big(\|f\|_2^{-1}f\big) - \frac{1}{2}\big\|\nabla(\|f\|_2^{-1}f)\big\|_2^2\Big\} \\ &\le \frac{1}{2}\|f\|_2^2\|\nabla f\|_2^2 + \|f\|_2^4 \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}. \end{aligned}$$

So the inequality (1.10) holds with

$$C = \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\} < \infty. \tag{1.11}$$

□

The proof of Theorem 1.1 is distributed in three sections: Section 2 is for the construction of self-intersection local time. In this section, we pave a way for the later development where the self-intersection local times is analyzed in terms of its Fourier transform. In Section 3, we establish the exponential inegrabilities of the self-intersection local time and the lower bounds for the exponential asymptotics stated in Theorem 1.1. The main tools used here are sub-additivity and large deviations by Feynman-Kac formula. The upper bounds are proved in Section 4. In addition to some techniques developed along the line of infinite dimensional probability, we adopt a moment comparison (first introduced by Donsker and Varadhan [8]) through Girsanov's theorem. With such comparison, the Brownian self-intersection local time is dominated by the self-intersection local time run by a Ornstein-Uhlenbeck process which has much better properties than Brownian motion as far as ergodicity and tightness are concerned.

2 Defining the self-intersection local times

In literature, the self intersection local time has been constructed in different settings and by different (but equivalent) approaches. An interested reader is referred to [9], [10] and [11] for historic account. For the later development of the paper, and also for reader's convenience, we use this section for the definition of self-intersection local time under Dalang's condition (1.3).

Here, the self-intersection local time is defined as

$$\int_0^t\int_0^t \gamma(B_s - B_r)dsdr \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_0^t \gamma_\epsilon(B_s - B_r)dsdr \qquad \text{in } \mathcal{L}(\Omega, \mathcal{A}, \mathbb{P}) \tag{2.1}$$

where

$$\gamma_\epsilon(x) = \int_{\mathbb{R}^d} \gamma(y-x)p_\epsilon(y)dy$$

and $p_\epsilon(\cdot)$ is the density of the normal distribution $N(0,\epsilon)$. To make this definition work, one has to establish the $\mathcal{L}^1$-convergence requested by (2.1). To this end, all we need is to show that

$$\lim_{\epsilon,\delta\to 0^+} \mathbb{E}\bigg| \int_0^t\int_0^t \gamma_\epsilon(B_s-B_r)dsdr - \int_0^t\int_0^t \gamma_\delta(B_s-B_r)dsdr\bigg| = 0 \tag{2.2}$$

for all $t>0$.

By the fact that the spectral measure of $\gamma_\epsilon(\cdot)$ is $e^{-\epsilon|\xi|^2/2}\mu(d\xi)$, and by Fourier transform

$$\int_0^t\int_0^t \gamma_\epsilon(B_s-B_r)dsdr = \int_{\mathbb{R}^d}\mu(d\xi)e^{-\epsilon|\xi|^2/2}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2. \tag{2.3}$$

Therefore,

$$\begin{aligned}&\bigg|\int_0^t\int_0^t \gamma_\epsilon(B_s-B_r)dsdr - \int_0^t\int_0^t \gamma_\delta(B_s-B_r)dsdr\bigg|\\ &\le \int_{\mathbb{R}^d}\Big|e^{-\epsilon|\xi|^2/2}-e^{-\delta|\xi|^2/2}\Big|\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu(d\xi).\end{aligned}$$

By Dominated convergence theorem, all we need is to show

$$\mathbb{E}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu(d\xi)<\infty \qquad \forall t>0. \tag{2.4}$$

Notice

$$\mathbb{E}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2 = 2\int_0^t\int_r^t \mathbb{E}e^{i\xi\cdot(B_s-B_r)}dsdr = 2\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr \tag{2.5}$$

for any $t>0$. Therefore, by Fubini's theorem

$$\begin{aligned}&\int_0^\infty dte^{-t}\mathbb{E}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2 = 2\int_0^\infty dte^{-t}\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr\\ &= 2\int_0^\infty dre^{-r}\int_r^\infty dse^{-(s-r)}\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}\int_s^\infty dte^{-(t-s)}\\ &= 2\Big(\int_0^\infty e^{-r}dr\Big)\Big(\int_0^\infty dse^{-s}\exp\Big\{-\frac{1}{2}|\xi|^2 s\Big\}\Big)\Big(\int_0^\infty dte^{-t}\Big)\\ &= \frac{2}{1+2^{-1}|\xi|^2}\end{aligned}$$

and therefore

$$\int_0^\infty dte^{-t}\mathbb{E}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu(d\xi) = \int_{\mathbb{R}^d}\frac{2}{1+2^{-1}|\xi|^2}\mu(d\xi).$$

By Dalang's condition (1.3), the right hand side is finite. By the fact that the expectation in (2.5) is monotonic in t, this implies (2.4). □

We end this section with the following lemma.

Lemma 2.1. *Under the Dalang's condition (1.3),*

$$\lim_{t\to 0^+}\frac{1}{t}\mathbb{E}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu(d\xi) = 0. \tag{2.6}$$

Proof. By (2.5),

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d}\Big|\int_0^t e^{i\xi\cdot B_s}ds\Big|^2\mu(d\xi)\\
&=2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr\\
&=2\int_{\{|\xi|\le R\}}\mu(d\xi)\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr\\
&\quad+2\int_{\{|\xi|>R\}}\mu(d\xi)\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr.
\end{aligned}$$

For the first term, we use the simple bound

$$\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr\le\int_0^t\int_r^t dsdr=\frac{1}{2}t^2.$$

So we have

$$2\int_{\{|\xi|\le R\}}\mu(d\xi)\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr\le t^2\mu\big(B(0,R)\big).$$

As for the second term, a straightforward computation gives

$$2\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr=\frac{4}{|\xi|^2}\Big[t-\frac{2}{|\xi|^2}\Big(1-e^{-|\xi|^2t/2}\Big)\Big]\le\frac{4}{|\xi|^2}t.$$

Hence,

$$2\int_{\{|\xi|>R\}}\mu(d\xi)\int_0^t\int_r^t\exp\Big\{-\frac{1}{2}|\xi|^2(s-r)\Big\}dsdr\le t\int_{\{|\xi|>R\}}\frac{4}{|\xi|^2}\mu(d\xi).$$

In summary,

$$\limsup_{t\to 0^+}\frac{1}{t}\mathbb{E}\int_{\mathbb{R}^d}\Big|\int_0^t e^{i\xi\cdot B_s}ds\Big|^2\mu(d\xi)\le\int_{\{|\xi|>R\}}\frac{4}{|\xi|^2}\mu(d\xi).$$

Letting $R\to\infty$ on the right hand side completes the proof. □

3 Exponential integrability, lower bounds and finiteness of the variations

Letting $\epsilon\to 0^+$ in (2.3)

$$\int_0^t\int_0^t\gamma(B_s-B_r)dsdr=\int_{\mathbb{R}^d}\Big|\int_0^t e^{i\xi\cdot B_s}ds\Big|^2\mu(d\xi).\tag{3.1}$$

Viewed from left hand side, the intersection local time is monotonic in t, while from the right hand side, the intersection local time has continuous sample path.

The subject of the discussion in this and next sections are the stochastic processes

$$Z_t=\Big(\int_{\mathbb{R}^d}\Big|\int_0^t e^{i\xi\cdot B_s}ds\Big|^2\mu(d\xi)\Big)^{1/2}\qquad t\ge 0$$

and

$$A_t=\frac{1}{t}Z_t^2=\frac{1}{t}\int_{\mathbb{R}^d}\Big|\int_0^t e^{i\xi\cdot B_s}ds\Big|^2\mu(d\xi)\qquad t>0.$$

Definition 3.1. *A stochastic process X_t ($t>0$) is said to be sub-additive if for any $t_1, t_2>0$, there is a random variable X'_{t_2} such that*

$$X_{t_1+t_2}\le X_{t_1}+X'_{t_2}$$

and that $X'_{t_2}\overset{d}{=}X_{t_2}$ and X'_{t_2} is independent of $\{X_s; s\le t_1\}$.

Using triangle inequality and Jensen inequality, one can exam that Z_t and A_t are sub-additive with

$$Z'_{t_2}=\bigg(\int_{\mathbb{R}^d}\bigg\vert\int_0^{t_2}e^{i\xi\cdot(B_{t_1+s}-B_{t_1})}ds\bigg\vert^2\mu(d\xi)\bigg)^{1/2}$$

and

$$A'_{t_2}=\frac{1}{t_2}\int_{\mathbb{R}^d}\bigg\vert\int_0^{t_2}e^{i\xi\cdot(B_{t_1+s}-B_{t_1})}ds\bigg\vert^2\mu(d\xi),$$

respectively.

With sub-additivity, and with the fact that Z_t is non-negative, non-decreasing, sample-path continuous with $Z_0=0$, by (1.3.7), p.21, [2],

$$\mathbb{P}\big\{Z_t\ge a+b\big\}\le\mathbb{P}\big\{Z_t\ge a\big\}\mathbb{P}\big\{Z_t\ge b\big\}$$

for any $a,b,t>0$. Repeat this inequality we get

$$\mathbb{P}\big\{Z_t\ge m\sqrt{t}\big\}\le\Big(\mathbb{P}\big\{Z_t\ge\sqrt{t}\big\}\Big)^m\qquad m=1,2,\ldots.$$

For any $\theta>0$, by Lemma 2.1 there is a $t_0>0$ such that

$$\sup_{0<t\le t_0}\mathbb{P}\big\{Z_t\ge\theta^{-1}\sqrt{t}\big\}\le\exp\big\{-2\big\}.$$

Consequently,

$$\mathbb{E}_0\exp\big\{\theta Z_t/\sqrt{t}\big\}=1+\int_0^\infty e^b\mathbb{P}_0\big\{Z_t\ge b\theta^{-1}\sqrt{t}\big\}db \tag{3.2}$$
$$\le1+e+\sum_{m=1}^\infty e^{m+1}\mathbb{P}_0\big\{Z_t\ge m\theta^{-1}\sqrt{t}\big\}\le1+e+\sum_{m=0}^\infty e^{m+1}e^{-2m}=\frac{2e^2-1}{e-1}<\infty$$

for all $0<t\le t_0$. It should be pointed out that sample path continuity is essential in above "integrability by sub-additivity" game. A quick reminder is the subordinator, which starts at 0, is non-decreasing and sub-additive (actually additive) but non-integrable.

From (3.2) one can see that for any $\theta>0$,

$$\mathbb{E}\exp\big\{\theta Z_t\big\}<\infty \tag{3.3}$$

as $t>0$ is sufficiently small.

By sub-additivity, for any $t_1,t_2>0$

$$\mathbb{E}\exp\big\{\theta Z_{t_1+t_2}\big\}\le\mathbb{E}\exp\big\{\theta Z_{t_1}\big\}\mathbb{E}\exp\big\{\theta Z_{t_2}\big\} \tag{3.4}$$

whenever the exponential moments on the right hand side are finite. In particular, (3.2) is extended to all $t>0$. Further, the sub-additivity argument shows that the limit

$$\mathcal{M}(\theta)\triangleq\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\big\{\theta Z_t\big\} \tag{3.5}$$

exists and finite for all $\theta>0$.

We now extend (3.3) and (3.5) from Z_t to A_t. Unfortunately, we can not follow the same procedure, as A_t is not monotonic and is not (continuously) defined at 0.

To establish exponential integrability for A_t, we first estimate $\mathbb{E}Z_t^n$ for $0 \le t \le 1$ and $n = 1, 2, \dots$. By (3.2) and Taylor expansion, for any $\theta > 0$, there is a small $t_0 > 0$ such that

$$\mathbb{E}_0 Z_t^n \le \left(\frac{2e^2-1}{e-1}\right)\theta^{-n} n! t^{n/2} \qquad 0 < t < t_0, \quad n = 1, 2, \dots$$

Unfortunately, this bound is not strong enough for exponential integrability of A_t. In the following, we tight it up by replacing "$n!$" by "$\sqrt{n!}$".

By sub-additivity,

$$\mathbb{E}Z_{t_1+t_2}^n \le \mathbb{E}(Z_{t_1} + Z'_{t_2})^n = \sum_{l=0}^n \binom{n}{l} \left\{\mathbb{E}Z_{t_1}^l\right\}\left\{\mathbb{E}Z_{t_2}^{n-l}\right\}.$$

Repeating the above bound,

$$\mathbb{E}Z_t^n \le \sum_{l_1+\cdots+l_m=n} \frac{n!}{l_1!\cdots l_m!} \prod_{k=1}^m \mathbb{E}Z_{t/m}^{l_k} = \sum_{l_1+\cdots+l_m=n} \frac{n!}{l_1!\cdots l_m!} \prod_{k=1}^m \mathbb{E}Z_{t/m}^{l_k}$$

for any integers $n, m \ge 1$ and $t > 0$.

We now let $t \le t_0$ and take $m = n$. By the weaker bound for $\mathbb{E}Z_t^n$,

$$\begin{aligned} \mathbb{E}Z_t^n &\le \sum_{l_1+\cdots+l_n=n} \frac{n!}{l_1!\cdots l_n!} \prod_{k=1}^n \left(\frac{2e^2-1}{e-1}\right)\theta^{-l_j} l_j! \Big(\frac{t}{n}\Big)^{l_j/2} \\ &= \left(\frac{2e^2-1}{(e-1)\theta}\right)^n n! n^{-n/2} t^{n/2} \sum_{l_1+\cdots+l_n=n} 1. \end{aligned}$$

A simple comibinatorial argument gives

$$\sum_{l_1+\cdots+l_n=n} 1 = \binom{2n-1}{n} \le 4^n.$$

In this way, we have the improved bound

$$\mathbb{E}Z_t^n \le \left(\frac{4(2e^2-1)}{(e-1)\theta}\right)^n \sqrt{n!} t^{n/2} \qquad \text{uniformly for } 0 \le t \le t_0 \text{ and} \quad n = 1, 2, \dots.$$

By the definition of A_t, it can be re-written as

$$\begin{aligned} \mathbb{E}A_t^n = \frac{1}{t^n}\mathbb{E}Z_t^{2n} &\le \frac{1}{t^n}\left(\frac{4(2e^2-1)}{(e-1)\theta}\right)^{2n} \sqrt{(2n)!} t^n \\ &\le \left(\frac{4\sqrt{2}e(2e-1)}{(e-1)\theta}\right)^{2n} n! \qquad 0 < t < t_0, \qquad n = 1, 2, \dots. \end{aligned}$$

Since $\theta > 0$ is arbitrary, by Taylor expansion we have proved that for any $\theta > 0$ there is a $t_0 > 0$ such that

$$\mathbb{E}\exp\left\{\theta A_t\right\} < \infty \tag{3.6}$$

for all $0 < t < t_0$. Further, by sub-additivity, A_t satisfies (3.4) (with Z_t being replaced by A_t). Consequently, (3.6) is extended to all $t > 0$ and the limit

$$\mathcal{E}(\theta) \triangleq \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\theta A_t\right\} \tag{3.7}$$

exists and finite for any $\theta > 0$.

The main part of this work is about the evaluation of the limits $\mathcal{M}(\theta)$ and $\mathcal{E}(\theta)$.[1]

In the next step, we establish the lower bounds for (1.4) and (1.5). Consider the Hilbert space

$$\mathcal{H} = \big\{ f \in \mathcal{L}^2(\mathbb{R}^d, \mu(d\xi)); f(-\xi) = \overline{f(\xi)}\ a.e. - \mu \big\}. \tag{3.8}$$

For any $f \in \mathcal{H}$ with $\|f\|_\mu = 1$ and

$$\int_{\mathbb{R}^d} |f(\xi)| \mu(d\xi) < \infty. \tag{3.9}$$

By Cauchy-Schwartz inequality

$$Z_t \geq \int_{\mathbb{R}^d} f(\xi) \bigg(\int_0^t e^{i\xi\cdot B_s} ds \bigg) \mu(d\xi) = \int_0^t \bar{f}(B_s) ds$$

where the function

$$\bar{f}(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} \mu(d\xi) \tag{3.10}$$

is continuous, bounded and real. By Theorem 4.1.6, [2],

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \bigg\{ \int_0^t \bar{f}(B_s) ds \bigg\} = \sup_{g\in\mathcal{F}_d} \bigg\{ \int_{\mathbb{R}^d} \bar{f}(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \bigg\}.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^d} \bar{f}(x) g^2(x) dx = \int_{\mathbb{R}^d} f(\xi) \bigg[\int_{\mathbb{R}^d} e^{i\xi\cdot x} g^2(x) dx \bigg] \mu(d\xi).$$

So we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \bigg\{ \int_0^t \bar{f}(B_s) ds \bigg\} \\ &= \sup_{g\in\mathcal{F}_d} \bigg\{ \int_{\mathbb{R}^d} f(\xi) \bigg[\int_{\mathbb{R}^d} e^{i\xi\cdot x} g^2(x) dx \bigg] \mu(d\xi) - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \bigg\}. \end{aligned} \tag{3.11}$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \big\{ Z_t \big\} \geq \sup_{g\in\mathcal{F}_d} \bigg\{ \int_{\mathbb{R}^d} f(\xi) \bigg[\int_{\mathbb{R}^d} e^{i\xi\cdot x} g^2(x) dx \bigg] \mu(d\xi) - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \bigg\}.$$

Take supremum over $\|f\| = 1$ with (3.9) and notice that the functions satisfying (3.9) are dense in $\mathcal{H}$. So the supremum on the right hand side is equal to

$$\sup_{g\in\mathcal{F}_d} \bigg\{ \bigg(\int_{\mathbb{R}^d} \bigg| \int_{\mathbb{R}^d} e^{i\xi\cdot x} g^2(x) dx \bigg|^2 \mu(d\xi) \bigg)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \bigg\}.$$

By Parseval identity

$$\int_{\mathbb{R}^d} \bigg| \int_{\mathbb{R}^d} e^{i\xi\cdot x} g^2(x) dx \bigg|^2 \mu(d\xi) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dx dy.$$

[1] The identifications of $\mathcal{M}(1)$ and $\mathcal{E}(1)$ are given directly by (1.4) and (1.5), respectively. We can also get $\mathcal{M}(\theta)$ and $\mathcal{E}(\theta)$ for general θ by replacing $\gamma(\cdot)$ by $\theta\gamma(\cdot)$ in Theorem 1.1.

In summary

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\big\{Z_t\big\} \tag{3.12}$$
$$\ge \sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

In view of (3.1), this is the lower bound for (1.4).

The proof of the lower bound for (1.5) is similar: Write

$$A_t = t\int_{\mathbb{R}^d}\left|\frac{1}{t}\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)$$

and let $f\in\mathcal{H}$ satisfy (3.9). Notice that

$$\|f\|^2+\int_{\mathbb{R}^d}\left|\frac{1}{t}\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\ge 2\int_{\mathbb{R}^d}f(\xi)\left[\frac{1}{t}\int_0^t e^{i\xi\cdot B_s}ds\right]\mu(d\xi)=\frac{2}{t}\int_0^t\bar{f}(B_s)ds$$

where $\bar{f}$ is given by (3.10). Hence,

$$\mathbb{E}\exp\big\{A_t\big\}\ge\exp\big\{-t\|f\|_2\big\}\mathbb{E}\exp\left\{2\int_0^t\bar{f}(B_s)ds\right\}.$$

By (3.11) (with $\bar{f}$ being replaced by $2\bar{f}$),

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\exp\big\{A_t\big\}$$
$$\ge -\|f\|^2+\sup_{g\in\mathcal{F}_d}\left\{2\int_{\mathbb{R}^d}f(\xi)\left[\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right]\mu(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$
$$=\sup_{g\in\mathcal{F}_d}\left\{-\|f\|^2+2\int_{\mathbb{R}^d}f(\xi)\left[\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right]\mu(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

Take supremum over f on the right hand side. By the fact that

$$\|h\|^2=\sup_{f\in\mathcal{H}}\Big\{-\|f\|^2+2\langle f,h\rangle\Big\}\qquad\forall h\in\mathcal{H} \tag{3.13}$$

the right hand side becomes

$$\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$
$$=\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

So we have the lower bound for (1.4):

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\exp\big\{A_t\big\} \tag{3.14}$$
$$\ge\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

Finally, the finiteness of the variations appearing in Theorem 1.1 follows from the proved lower bounds (3.12), (3.14) and the existence of the limits in (3.5) and (3.7). □

4 Proof of upper bounds

Let Z_t and A_t be defined as in the last section. By (3.1), all we need is to show

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\exp\big\{Z_t\big\}\\&\le\sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\exp\big\{A_t\big\}\\&\le\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}\tag{4.2}$$

Consider the decomposition

$$\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)=\int_{[-R,R]^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)+\int_{([-R,R]^d)^c}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi).$$

By Hölder inequality,

$$\begin{aligned}\mathbb{E}\exp\big\{Z_t\big\}\le&\left(\mathbb{E}\exp\left\{p\left(\int_{[-R,R]^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}\right\}\right)^{1/p}\\&\times\left(\mathbb{E}\exp\left\{q\left(\int_{([-R,R]^d)^c}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}\right\}\right)^{1/q}\end{aligned}$$

for any conjugate numbers $p,q>1$ (in the following discussion p is close to 1 and q is large). Hence

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\big\{Z_t\big\}\\&\le\frac{1}{p}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{p\left(\int_{[-R,R]^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}\right\}\\&\quad+\frac{1}{q}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{q\left(\int_{([-R,R]^d)^c}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}\right\}.\end{aligned}$$

Notice that the process

$$\widetilde{Z}_t\triangleq q\left(\int_{([-R,R]^d)^c}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}$$

is sub-additive and therefore satisfies (3.4). Consequently,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{q\left(\int_{([-R,R]^d)^c}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}\right\}\\&\le\log\mathbb{E}\exp\left\{q\left(\int_{([-R,R]^d)^c}\left|\int_0^1 e^{i\xi\cdot B_s}ds\right|^2\mu(d\xi)\right)^{1/2}\right\}.\end{aligned}$$

For give $p>1$ that is close to 1, by (3.3) there is $R=R_p>0$ that makes the right hand side arbitrarily small.

In summary, the above argument reduces the upper bound (4.1) to the proof of

$$\lim_{p\to 1^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\bigg(\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg)^{1/2}\bigg\} \tag{4.3}$$
$$\le\sup_{g\in\mathcal{F}_d}\bigg\{\bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}$$

where

$$\mu_p(d\xi)=p^2 1_{[-R_p.R_p]^d}(\xi)\mu(d\xi)$$

and R_p is a properly chosen sequence according to our discussion.

In a parallel argument, the upper bound (4.2) is reduced to

$$\lim_{p\to 1^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\} \tag{4.4}$$
$$\le\sup_{g\in\mathcal{F}_d}\bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}.$$

We prove (4.4) first, as the generating function

$$\mathbb{E}\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\}=\sum_{n=0}^\infty\frac{1}{n!}\frac{1}{t^n}\mathbb{E}\bigg[\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg]^n \tag{4.5}$$

is the sum of integer moments. Indeed, the integer moment enjoins the following nice representation

$$\mathbb{E}\bigg[\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg]^n=\mathbb{E}\int_{(\mathbb{R}^d)^n}\mu_p^{\otimes n}(d\xi)\int_{[0,t]^{2n}}d\mathbf{r}d\mathbf{s}\prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k}-B_{r_k})} \tag{4.6}$$
$$=\int_{(\mathbb{R}^d)^n}\mu_p^{\otimes n}(d\xi)\int_{[0,t]^{2n}}d\mathbf{r}d\mathbf{s}\exp\bigg\{-\frac{1}{2}Var\bigg(\sum_{k=1}^n\xi_k\cdot(B_{s_k}-B_{r_k})\bigg)\bigg\}$$

that allows a moment comparison between the Brownian regime and Ornstein-Uhlenbeck regime performed as following.

Given a small constant $\kappa>0$. let $\mathbb{P}^\kappa$ and $\mathbb{E}^\kappa$ be the law and expectation, respectively, of a d-dimensional Ornstein-Uhlenbeck process starting from 0 with the infinitesimal generator $2^{-1}\Delta-\kappa x\cdot\nabla$. In our following discussion, B_s represents a Brownian motion under $\mathbb{P}$, and an Ornstein-Uhlenbeck process under $\mathbb{P}^\kappa$. By Girsanov's theorem, for any $t>0$,

$$\frac{d\mathbb{P}^\kappa}{d\mathbb{P}}\Big|_{[0,t]}=\exp\bigg\{-\kappa\int_0^t B_s\cdot dB_s-\frac{\kappa^2}{2}\int_0^t|B_s|^2ds\bigg\} \tag{4.7}$$
$$=\exp\bigg\{-\kappa|B_t|^2+\frac{\kappa d}{2}t-\frac{\kappa^2}{2}\int_0^t|B_s|^2ds\bigg\}$$

where the second equality follows from a simple use of Ito formula. In particular,

$$\frac{d\mathbb{P}^\kappa}{d\mathbb{P}}\Big|_{[0,t]}\le\exp\bigg\{\frac{\kappa d}{2}t\bigg\}. \tag{4.8}$$

Applying (4.8) and Lemma 3.9, [8] to the Gaussian laws $\mathbb{P}$ and $\mathbb{P}^\kappa$,

$$Var\bigg(\sum_{k=1}^n\xi_k\cdot(B_{s_k}-B_{r_k})\bigg)\ge\mathrm{Var}^\kappa\bigg(\sum_{k=1}^n\xi_k\cdot(B_{s_k}-B_{r_k})\bigg)$$

where "$\mathrm{Var}^\kappa(\cdot)$ is the variance under the Ornstein-Uhlenbeck law $\mathbb{P}^\kappa$. Notice the moment representation (4.6) holds also under the law $\mathbb{P}^\kappa$. Thus,

$$\mathbb{E}\bigg[\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg]^n \le \mathbb{E}^\kappa\bigg[\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg]^n \tag{4.9}$$

for all inters $n\ge 1$. From (4.5),

$$\mathbb{E}\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\} \le \mathbb{E}^\kappa\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\}. \tag{4.10}$$

We now prove that for fixed $p>1$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\} \\ &\le \frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\bigg\{\int_{[-R_p,R_p]^d}\bigg|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\bigg|^2\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\bigg\}.\end{aligned} \tag{4.11}$$

Let $\mathcal{H}_p$ be a Hilbert space of all complex valued function $f(\xi)$ on $[-R_p,R_p]^d$ with $f(-\xi)=\overline{f}(\xi)$ a.e. μ_p and

$$\|f\|^2\triangleq\int_{[-R_p,R_p]^d}|f(\xi)|^2\mu_p(d\xi)<\infty$$

By Arzelá-Ascoli theorem, for each $L>0$, the class

$$\mathcal{C}_L=\Big\{f\in\mathcal{H}_p;\ \sup_{\xi\in[-R_p,R_p]^d}|f(\xi)|\le 1 \text{ and } |f(\xi)-f(\eta)|\le L|\xi-\eta| \text{ for } \forall\xi,\eta\in[-R_p,R_p]^d\Big\}$$

is relatively compact under the uniform topology and maintains so under the topology of Hilbert norm. Therefore, the closure $\mathcal{K}_L$ of $\mathcal{C}_L$ in $\mathcal{H}_p$ is compact in $\mathcal{H}_p$.

In the discussion below, we view the family

$$X_t(\xi)\triangleq\frac{1}{t}\int_0^t e^{i\xi\cdot B_s}ds\qquad \xi\in[-R_p,R_p]^d,\quad t\ge 1$$

as the stochastic process taking values in $\mathcal{H}_p$. Since $\sup_\xi|X_t(\xi)|\le 1$ and

$$|X_t(\xi)-X_t(\eta)|\le\frac{1}{t}\int_0^t 2\Big|\sin\frac{(\xi-\eta)\cdot B_s}{2}\Big|ds\le|\xi-\eta|\frac{1}{t}\int_0^t|B_s|ds$$

we have that

$$\Big\{X_t(\cdot)\in\mathcal{K}_L\Big\}\supset\bigg\{\int_0^t|B_s|ds\le Lt\bigg\}\qquad\forall L>0.$$

Write

$$A_t=\bigg\{\int_0^t|B_s|ds>Lt\bigg\}.$$

We have the decomposition

$$\begin{aligned}&\mathbb{E}^\kappa\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\} \\ &\le\mathbb{E}^\kappa\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\}1_{\{\{X_t(\cdot)\in\mathcal{K}_L\}} \\ &\quad+\mathbb{E}^\kappa\exp\bigg\{\frac{1}{t}\int_{\mathbb{R}^d}\bigg|\int_0^t e^{i\xi\cdot B_s}ds\bigg|^2\mu_p(d\xi)\bigg\}1_{A_t}.\end{aligned} \tag{4.12}$$

For the second term on the right hand side, we use Cauchy-Schwartz inequality:

$$\begin{aligned}
&\mathbb{E}^\kappa \exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}1_{A_t}\\
&\le \left(\mathbb{E}^\kappa \exp\left\{\frac{2}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}\right)^{1/2}\Big(\mathbb{P}^\kappa(A_t)\Big)^{1/2}\\
&\le \left(\exp\left\{\frac{\kappa d}{2}t\right\}\mathbb{E}\exp\left\{\frac{2}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}\right)^{1/2}\\
&\quad\times\left(\mathbb{E}\exp\left\{-\frac{\kappa}{2}|B_t|^2+\frac{\kappa d}{2}t-\frac{\kappa^2}{2}\int_0^t|B(s)|^2ds\right\}1_{A_t}\right)^{1/2}
\end{aligned}$$

where the last step follows from (4.7) and (4.8). By (3.7),

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\frac{2}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}\le\mathcal{E}(2p^2).$$

Also notice that

$$\begin{aligned}
&\mathbb{E}\exp\left\{-\frac{\kappa}{2}|B_t|^2+\frac{\kappa d}{2}t-\frac{\kappa^2}{2}\int_0^t|B_s|^2ds\right\}1_{A_t}\\
&\le e^{-Lt}\mathbb{E}\exp\left\{\int_0^t\Big(|B_s|-\frac{\kappa^2}{2}|B_s|^2\Big)ds+\frac{\kappa d}{2}t\right\}\\
&\le \exp\left\{\left(-L+\frac{1}{2\kappa^2}+\frac{\kappa d}{2}\right)t\right\}.
\end{aligned}$$

In summary,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}1_{A_t}\\
&\le -\frac{L}{2}+\frac{\kappa d}{2}+\frac{1}{4\kappa^2}+\frac{1}{2}\mathcal{E}(2p^2).
\end{aligned}\tag{4.13}$$

We now bound the first term in the decomposition (4.12). The treatment is based on a simple and universal relation in Hilbert space that is given in (3.13), from which the family

$$G_f=\Big\{h\in\mathcal{H}_p;\quad \|h\|_2^2<-\|f\|^2+2\langle f,h\rangle+\epsilon\Big\}\qquad f\in\mathcal{K}_L$$

form open covers of the compact set $\mathcal{K}_L$, where $\epsilon>0$ is a given small number. Therefore, this family contains a finite sub-family $G_{f_1},\dots,G_{f_m}$ that cover $\mathcal{K}_L$. Consequently, on $\{X\in\mathcal{K}_L\}$,

$$\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)=t\|X_t(\cdot)\|^2\le t\left(\epsilon+\max_{1\le j\le m}\Big\{-\|f_j\|^2+2\langle f_j,X_t\rangle\Big\}\right).$$

Therefore,

$$\begin{aligned}
&\mathbb{E}^\kappa\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}1_{\{\{X_t(\cdot)\in\mathcal{K}_L\}}\\
&\le e^{\epsilon t}\mathbb{E}^\kappa\exp\left\{t\max_{1\le j\le m}\Big\{-\|f_j\|^2+2\langle f_j,X_t\rangle\Big\}\right\}\\
&\le e^{\epsilon t}\sum_{j=1}^m\exp\Big\{-\|f_j\|^2t\Big\}\mathbb{E}^\kappa\exp\Big\{2t\langle f_j,X_t\rangle\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}1_{\{\{X_t(\cdot)\in\mathcal{K}_L\}}\\&\le\epsilon+\max_{1\le j\le m}\left\{-\|f_j\|^2+\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\Big\{2t\langle f_j,X_t\rangle\Big\}\right\}.\end{aligned}$$

From (4.8),

$$\begin{aligned}&\mathbb{E}^\kappa\exp\Big\{2t\langle f_j,X_t\rangle\Big\}\le\exp\Big\{\frac{\kappa d}{2}t\Big\}\mathbb{E}\exp\Big\{2t\langle f_j,X_t\rangle\Big\}\\&=\exp\Big\{\frac{\kappa d}{2}t\Big\}\mathbb{E}\exp\Big\{2\int_0^t\bar{f}_j(B_s)ds\Big\}\end{aligned}$$

Here we keep using the notation $\bar{f}$ for the expression

$$\bar{f}(x)=\int_{[-R_p,R_p]^d}f(\xi)e^{i\xi\cdot x}\mu_p(d\xi).$$

It should be pointed out that for any $f\in\mathcal{H}$, $\bar{f}(x)$ is real, bounded and continuous on $\mathbb{R}^d$.

Applying (3.11) to $2\bar{f}(\cdot)$,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\Big\{2t\langle f_j,X_t\rangle\Big\}\\&=\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{2\int_{\mathbb{R}^d}\bar{f}_j(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\&=\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{2\int_{\mathbb{R}^d}f_j(\xi)\left[\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right]\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&-\|f_j\|+\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\Big\{2t\langle f_j,X_t\rangle\Big\}\\&=\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{-\|f_j\|+2\int_{\mathbb{R}^d}f_j(x)\left[\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right]\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\&\le\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\end{aligned}$$

where the last step follows from the universal fact that $-\|f\|^2+2\langle f,h\rangle\le\|h\|^2$ for any $f,h\in\mathcal{H}_p$.

In summary,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}1_{\{\{X_t(\cdot)\in\mathcal{K}_L\}}\\&\le\epsilon+\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{\int_{[-R_p,R_p]^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Take $\epsilon\to 0^+$ on the right hand side. Together with (4.12) and (4.13),

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}\\&\le\max\left\{-\frac{L}{2}+\frac{\kappa d}{2}+\frac{1}{4\kappa^2}+\frac{1}{2}\mathcal{E}(2p^2),\right.\\&\left.\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{\int_{[-R_p,R_p]^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\right\}.\end{aligned}$$

Letting $L\to\infty$ on the right hand side leads to (4.11).

By (4.10) and (4.11)

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}\\
&\le\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{\int_{[-R_p,R_p]^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\
&\le\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{p^2\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu(d\xi)-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\
&=\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{p^2\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Let $\kappa\to 0^+$ on the right hand side, and then let $p\to 1^+$ on the both sides. We have

$$\begin{aligned}&\lim_{p\to 1^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\frac{1}{t}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right\}\\
&\le\lim_{p\to 1^+}\sup_{g\in\mathcal{F}_d}\left\{p^2\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

By (3.7) and the lower bound (3.14) (applied to $\theta\gamma(\cdot)$), the function

$$\Lambda(\theta)\overset{\Delta}{=}\sup_{g\in\mathcal{F}_d}\left\{\theta\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

is finite on $\mathbb{R}^+$. It is easy to see that $\Lambda(\theta)$ is convex on $\mathbb{R}^+$. Consequently, $\Lambda(\theta)$ is continuous on $\mathbb{R}^+$. In particular, $\Lambda(p^2)\to\Lambda(1)$ as $p\to 1^+$.

In summary, we have proved (4.4).

It remains to prove (4.3). By an obvious modification of the treatment for (4.11), we can prove that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\left\{\theta\left(\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right)^{1/2}\right\}\\
&\le\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{\theta\left(\int_{[-R_p,R_p]^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\end{aligned}\tag{4.14}$$

for all $\theta>0$.

The missing part is a comparison that can play a role as (4.10) in the proof of (4.3). We provide the following replacement: Write

$$\Psi_\kappa(\theta)=\frac{\kappa d}{2}+\sup_{g\in\mathcal{F}_d}\left\{\theta\left(\int_{[-R_p,R_p]^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

By Lemma 1.2.6-(2), p13, [2] (with $p=2$), (4.14) is equivalent to

$$\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n=0}^\infty\frac{\theta^n}{n!}\left\{\mathbb{E}^\kappa\left[\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right]^n\right\}^{1/2}\le 2\Psi_\kappa(2\theta)\qquad\theta>0.$$

By the moment comparison (4.9), we therefore have

$$\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n=0}^\infty\frac{\theta^n}{n!}\left\{\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\right]^n\right\}^{1/2}\le 2\Psi_\kappa(2\theta)\qquad\theta>0.$$

Using Lemma 1.2.6-(2), p13, [2] again,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\bigg(\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\bigg)^{1/2}\right\}\le\Psi_\kappa(\theta).$$

Taking $\theta=1$ and letting $\kappa\to 0^+$ on the right hand side.

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\bigg(\int_{\mathbb{R}^d}\left|\int_0^t e^{i\xi\cdot B_s}ds\right|^2\mu_p(d\xi)\bigg)^{1/2}\right\}\\&\le\sup_{g\in\mathcal{F}_d}\left\{\bigg(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}e^{i\xi\cdot x}g^2(x)dx\right|^2\mu_p(d\xi)\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\&\le\sup_{g\in\mathcal{F}_d}\left\{p\bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\bigg)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Finally, letting $p\to 1^+$ on the both sides leads to (4.3). $\square$

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