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Exponential asymptotics for time–space Hamiltonians

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Abstract. In this paper, we investigate the long time asymptotics of the exponential moment for the following time–space Hamiltonian

$$\int_0^t \int_0^t \frac{1}{|r-s|^{\alpha_0}} \gamma(B_r - B_s)\,\mathrm{d}s\,\mathrm{d}r, \quad t \ge 0,$$

where $(B_s, s \ge 0)$ is a d-dimensional Brownian motion, the kernel $\gamma(\cdot):\mathbb{R}^d \to [0,\infty)$ is a homogeneous function with singularity at zero; and $\alpha_0 \in (0,1)$ together with the scaling parameter of γ satisfies certain conditions. Our work is partially motivated by the studies of the short-range sample-path intersection, the strong coupling polaron, and the parabolic Anderson models with a time–space fractional white noise potential.

Résumé. Dans ce papier, nous étudions le comportement en temps long du moment exponentiel du Hamiltonien dépendant du temps

$$\int_0^t \int_0^t \frac{1}{|r-s|^{\alpha_0}} \gamma(B_r - B_s)\,\mathrm{d}s\,\mathrm{d}r, \quad t \ge 0,$$

où $(B_s, s \ge 0)$ est un mouvement brownien de dimension d, le noyau $\gamma(\cdot):\mathbb{R}^d \to [0,\infty)$ est une fonction homogène avec une singularité en zéro, $\alpha_0 \in (0,1)$ et le paramètre de scaling γ satisfont certaines conditions. Notre travail est partiellement motivé par l'étude des intersections à courte portée de trajectoires, le polaron avec couplage fort et le modèle parabolique d'Anderson avec un potentiel donné par un bruit blanc fractionnaire en espace–temps.

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1. Introduction

Given a d-dimensional Brownian motion $(B_s, s \ge 0)$ starting at 0, the asymptotics (as $t \to \infty$) of the exponential moment

$$\mathbb{E}\exp\left\{\int_0^t \int_0^t \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s\right\} \tag{1.1}$$

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have been well-understood. See Theorem 4.2.1 of [1] for the case $\gamma(x) = \delta_0(x)$ (Dirac delta function) and [4] for the case $\gamma(x) = |x|^{-\alpha}$. This subject is largely motivated by the investigation on sample-path intersection as the integral in (1.1) measures the intensity of self-intersection of the Brownian paths when $\gamma(\cdot) = \delta_0(\cdot)$, or of quasi-self-intersection when $\gamma(\cdot) = |\cdot|^{-\alpha}$. In contrary, the success in the time dependent setting

$$\mathbb{E}\exp\left\{\int_0^t \int_0^t \gamma_0(r-s)\gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s\right\} \tag{1.2}$$

is limited. To our best knowledge, the only successful story is the famous work [8] by Donsker and Varadhan on the asymptotics for polaron together with the follow-up paper [16] by Mansmann in the setting of Dirac polaron. When $d = 3$, Donsker and Varadhan establish the existence of the limit

$$\Lambda(\theta) \equiv \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\exp\left\{\theta\int_0^t \int_0^t \frac{\mathrm{e}^{-|r-s|}}{|B_r - B_s|}\,\mathrm{d}r\,\mathrm{d}s\right\}$$

for any $\theta > 0$. Further, they point out that

$$\lim_{\theta\to 0^+} \theta^{-2}\Lambda(\theta) = \sup_{g\in\mathcal{F}_3}\left\{2\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{g^2(x)g^2(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla g(x)\right|^2\mathrm{d}x\right\},$$

where the class $\mathcal{F}_3$ (or $\mathcal{F}_d$, in general) is defined in (2.9) below.

Following this work and for the case $d = 1$, Mansmann [16] proves that for any $\theta > 0$, the limit

$$\Lambda_0(\theta) \equiv \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\exp\left\{\theta\int_0^t \int_0^t \mathrm{e}^{-|r-s|}\delta_0(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}$$

exists and

$$\lim_{\theta\to 0^+} \theta^{-2}\Lambda_0(\theta) = \sup_{g\in\mathcal{F}_1}\left\{2\int_{-\infty}^{\infty} g^4(x)\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{\infty}\left|g'(x)\right|^2\mathrm{d}x\right\}.$$

To obtain their results Donsker and Varadhan ([8]) adopt the following homogenization procedure

$$\int_0^t \int_0^t \frac{\mathrm{e}^{-|r-s|}}{|B_r - B_s|}\,\mathrm{d}r\,\mathrm{d}s = 2\int_0^t\left[\int_0^{t-s} \frac{\mathrm{e}^{-r}}{|B_{s+r} - B_s|}\,\mathrm{d}r\right]\mathrm{d}s \approx 2\int_0^t\left[\int_0^{\infty} \frac{\mathrm{e}^{-r}}{|B_{s+r} - B_s|}\,\mathrm{d}r\right]\mathrm{d}s$$

and then link the right hand side to their general theory ([5], [6]) on the large deviations for empirical measures.

In this work we shall study the asymptotic behavior of (1.2) when $\gamma_0(t) = |t|^{-\alpha_0}$ with $0 < \alpha_0 < 1$. With few exceptions such as the models of polaron listed above, the general theory ([5], [6]) of Donsker–Varadhan on large deviations for empirical processes provides no solution (even at heuristic level) to the setting of time–dependence. In particular, the method of homogenization used by Donsker and Varadhan [8] in their study of polarons is not applicable to the problems investigated in this paper, simply because

$$\int_0^{\infty} r^{-\alpha_0}\gamma(B_{s+r} - B_s)\,\mathrm{d}r = \infty \quad \text{a.s. } \forall s \geq 0$$

under our set-up (1.6) and our assumption (1.7) listed below.

The motivation to our study of the exponential moment of time–space Hamiltonian comes from the polymer physics. The quantity (1.2) frequently appears as the ground state energy in the model of strongly coupled polarons, where $\gamma(x) = |x|^{-1}$ or $\delta_0(x)$, and the quantity $\gamma_0(r-s)$ appears as the dumping force which decreases as $|r-s|$ increases. We refer to the paper [15] for the physicists' view on this problem. In [11], Section 2.4 the following model

$$H_n(S) = \sum_{\substack{j,k=1\\ j\neq k}}^{n} \frac{1_{\{S_j = S_k\}}}{|j-k|^{\alpha_0}}$$

is proposed for the random polymers of short range interactions, where $\{S_k\}$ is an 1-dimensional simple random walk. Our investigation for the case $\gamma(x) = \delta_0(x)$ is closely relevant to this model in light of invariance principle.

Another motivation of our study is the recent progress in the parabolic Anderson model

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \frac{\partial^{d+1} W^H}{\partial t\, \partial x_1 \cdots \partial x_d}(t,x) u(t,x), \\ u(0,x) = 1, \end{cases} \tag{1.3}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $W^H(t,x)$ is a fractional Brownian sheet with Hurst parameter $H = (H_0, H_1, \dots, H_d)$ satisfying the assumption

$$\frac{1}{2} < H_j < 1 \quad (j = 0, 1, \dots, d) \quad \text{and} \quad 2H_0 + \sum_{j=1}^d H_j > d + 1. \tag{1.4}$$

It is proved in [13] that the equation (1.3) has a weak solution $u(t,x)$ with finite moments of all orders and for any positive integer p, it holds

$$\mathbb{E}\big(u(t,x)^p\big) = \mathbb{E}\left(\exp\left[\frac{c_H}{2} \sum_{j,k=1}^p \int_0^t \int_0^t |s-r|^{-\alpha_0} \prod_{i=1}^d \big|B_s^{j,i} - B_r^{k,i}\big|^{-\alpha_i} \,\mathrm{d}s\,\mathrm{d}r\right]\right),$$

where $B_s^j = (B_s^{j,1}, \dots, B_s^{j,d})$, $j = 1, \dots, p$, are independent d-dimensional standard Brownian motions, $\alpha_0 = 2 - 2H_0$, $\alpha_i = 2 - 2H_i$, and

$$c_H = \prod_{i=0}^d H_i(2H_i - 1).$$

One of our goals is to achieve precise asymptotics for the integer moments of $u(t,x)$. We shall focus on the case $p = 1$ until Section 6, where the case $p \geq 2$ will be considered. Therefore, this problem is relevant to the main subject of the present work with the choice of

$$\gamma(x) = \prod_{j=1}^d |x_j|^{-\alpha_j}, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Motivated by these problems, we study the long time asymptotics for the exponential moments

$$\mathbb{E}\exp\left\{\theta \int_0^t \int_0^t \frac{1}{|r-s|^{\alpha_0}} \gamma(B_r - B_s) \,\mathrm{d}r\,\mathrm{d}s\right\}, \tag{1.5}$$

where $\alpha_0 \in (0,1)$ and the space function $\gamma(x)$ takes one of the following three forms:

$$\gamma(x) = \prod_{j=1}^d |x_j|^{-\alpha_j}, \quad \gamma(x) = |x|^{-\alpha} \quad \text{and} \quad \gamma(x) = \delta_0(x) \tag{1.6}$$

which are referred as, respectively, *the first*, *the second* and *the third forms* of $\gamma(\cdot)$ in our discussion. Throughout the paper, we make the following assumptions on the parameters appearing in our main theorems:

$$\begin{cases} 0 < \alpha_0, \dots, \alpha_d < 1, 2\alpha_0 + \sum_{i=1}^d \alpha_i < 2, & \text{if } \gamma(x) = \prod_{i=1}^d |x_i|^{-\alpha_i}, \\ 0 < \alpha_0 < 1, 0 < \alpha < d, 2\alpha_0 + \alpha < 2, & \text{if } \gamma(x) = |x|^{-\alpha}, \\ d = 1 \text{ and } 0 < \alpha_0 < \frac{1}{2}, & \text{if } \gamma(x) = \delta_0(x). \end{cases} \tag{1.7}$$

To see the connection among all these three cases, we define $\alpha = \sum_{i=1}^d \alpha_i$ when $\gamma(x) = \prod_{i=1}^d |x_i|^{-\alpha_i}$ is of the first form and $\alpha = 1$ when $\gamma(x) = \delta_0(x)$ is of the third form throughout the paper. With this notation we see that α plays the role of the spatial scaling exponent: $\gamma(cx) = |c|^{-\alpha}\gamma(x)$. It is well-known that under the condition (1.7), the double time–integral in (1.5) is well defined and its exponential moment given in (1.5) is finite for any $\theta > 0$ and $t > 0$. For this claim we cite Theorem 3.1 and Theorem 3.4, [13] in the setting of the first and the second form with an easy observation that the function $\gamma(x)$ in the second form is dominated by the one in the first form; and the second half of Proposition 3.3, [12] or Theorem 6.1 of [13] in the setting of the third form.

Let $W^{1,2}(\mathbb{R}^d)$ be the Sobolev space of all functions g on $\mathbb{R}^d$ such that $g, \nabla g \in L^2(\mathbb{R}^d)$. Denote

$$\mathcal{A}_d = \bigg\{ g(s,x); g(s,\cdot) \in W^{1,2}(\mathbb{R}^d), \int_{\mathbb{R}^d} g^2(s,x)\,\mathrm{d}x = 1$$
$$\forall 0 \le s \le 1 \text{ and } \int_0^1 \int_{\mathbb{R}^d} \big|\nabla_x g(s,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}s < \infty \bigg\}, \tag{1.8}$$

$$\mathcal{E}(\alpha_0, d, \gamma) = \sup_{g \in \mathcal{A}_d} \bigg\{ \int_0^1 \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\gamma(x-y)}{|r-s|^{\alpha_0}} g^2(s,x) g^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s$$
$$- \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} \big|\nabla_x g(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s \bigg\}. \tag{1.9}$$

The finiteness of $\mathcal{E}(\alpha_0, d, \gamma)$ and its relationship to other quantities will be established in the Appendix.

Theorem 1.1. *Under the assumption* (1.7),

$$\lim_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)} \log \mathbb{E} \exp\bigg\{ \theta \int_0^t \int_0^t |r-s|^{-\alpha_0} \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s \bigg\}$$
$$= \mathcal{E}(\alpha_0, d, \gamma) \theta^{2/(2-\alpha)} \tag{1.10}$$

for every $\theta > 0$. *Here we recall that* $\alpha = 1$ *in the case when* $\gamma(x) = \delta_0(x)$.

Remark 1.2.

1. *If we formally let* $\alpha = 0$, *then this is a deterministic problem. All quantities are easy to compute. In this case it is easy to verify the result.*
2. *In* [17], *Theorem* 4.1 *the author obtained the following result for the first case by using moment method*,

$$\limsup_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)} \log \mathbb{E} \exp\bigg\{ \int_0^t \int_0^t |r-s|^{-\alpha_0} \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s \bigg\} < \infty.$$

By integral substitution and by scaling the Brownian motion, one can easily establish the following self-similarity property:

$$\int_0^{at} \int_0^{at} |r-s|^{-\alpha_0} \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s \stackrel{d}{=} a^{(4-\alpha-2\alpha_0)/2} \int_0^t \int_0^t |r-s|^{-\alpha_0} \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s. \tag{1.11}$$

With it Theorem 1.1 can be reduced to

Theorem 1.3. *For any* $\theta > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\bigg\{ \theta t^{\alpha_0/2} \bigg(\int_0^t \int_0^t \frac{1}{|r-s|^{\alpha_0}} \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s \bigg)^{1/2} \bigg\}$$
$$= \theta^{4/(4-\alpha)} M(\alpha_0, d, \gamma), \tag{1.12}$$

where

$$M(\alpha_0,d,\gamma)=\sup_{g\in\mathcal{A}_d}\left\{\left(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(s,x)g^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2\mathrm{d}x\,\mathrm{d}s\right\}. \tag{1.13}$$

Proof of Theorem 1.1 based on Theorem 1.3. The scaling property given in (1.11) implies that

$$\int_0^t\int_0^t|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\overset{d}{=}t^{(4-\alpha-2\alpha_0)/2}\int_0^1\int_0^1|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s.$$

Then by Gärtner–Ellis theorem for non-negative random variable (see, e.g., Corollary 1.2.5 in [1]), (1.12) implies that for any $\lambda>0$,

$$\begin{aligned}&\lim_{t\to\infty}t^{-1}\log\mathbb{P}\left\{\int_0^1\int_0^1|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\ge\lambda t^{\alpha/2}\right\}\\&\quad=-\sup_{\theta>0}\left\{\sqrt{\lambda}\theta-M(\alpha_0,d,\gamma)\theta^{4/(4-\alpha)}\right\}=-\frac{\alpha}{4}\left(\frac{4-\alpha}{4}\frac{1}{M(\alpha_0,d,\gamma)}\right)^{(4-\alpha)/\alpha}\lambda^{2/\alpha}.\end{aligned}$$

By the Varadhan's integral lemma (see [1], Theorem 1.1.6)

$$\begin{aligned}&\lim_{t\to\infty}t^{-1}\log\mathbb{E}\exp\left\{\theta t^{(2-\alpha)/2}\int_0^1\int_0^1|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\&\quad=\sup_{\lambda>0}\left\{\lambda\theta-\frac{\alpha}{4}\left(\frac{4-\alpha}{4}\frac{1}{M(\alpha_0,d,\gamma)}\right)^{(4-\alpha)/\alpha}\lambda^{2/\alpha}\right\}\\&\quad=\frac{2-\alpha}{2}2^{\alpha/(2-\alpha)}\left(\frac{4-\alpha}{4}\right)^{-(4-\alpha)/(2-\alpha)}M(\alpha_0,d,\gamma)^{(4-\alpha)/(2-\alpha)}\theta^{2/(2-\alpha)}\\&\quad=\mathcal{E}(\alpha_0,d,\gamma)\theta^{2/(2-\alpha)},\end{aligned}\tag{1.14}$$

where the last equality follows from (A.4) in Lemma A.2 of the Appendix.

Finally, let $a^{(4-\alpha-2\alpha_0)/2}=t^{(2-\alpha)/2}$ and $t=1$. Applying scaling property (1.11), we have

$$\begin{aligned}&t^{(2-\alpha)/2}\int_0^1\int_0^1|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\\&\quad\overset{d}{=}\int_0^{t^{(2-\alpha)/(4-\alpha-2\alpha_0)}}\int_0^{t^{(2-\alpha)/(4-\alpha-2\alpha_0)}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}\tag{1.15}$$

Theorem 1.1 now follows from (1.14), (1.15) and a (time) variable substitution. □

As pointed out before, the general theory of Donsker–Varadhan large deviations does not apply to our setting mainly because of time dependency. Our proof of Theorem 1.3 contains the following ingredients. The comparison of exponential asymptotics between the time–space and space Hamiltonians. The representation (4.3) below of the Hamiltonian as L^2-norm, a time–space Feynman–Kac large deviation principle, and some technology developed in the area of probability in Banach spaces.

The rest of the paper is organized as following. In Section 2 we establish some asymptotic rough bounds for our main theorems by some more direct and elementary method. In Section 3, we develop a time–space version of Feynman–Kac large deviation which may be important for its own sake. The precise upper and lower bounds are established in Section 4 and Section 5, respectively, based on the Feynman–Kac large deviation given in Section 3.

As an application of the main result, we obtain an intermittency effect for the parabolic model (1.3) in Section 6.1. A local version of Theorem 1.3 is given in Section 6.2. Finally, the well-posedness of the variations appearing in our theorems, and their relations are discussed in the Appendix.

2. Asymptotic bounds by comparison

The goal of this section is to prove:

Proposition 2.1. *There is a constant $C > 0$ such that for any $\theta > 0$,*

$$\liminf_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)} \log \mathbb{E}\exp\left\{\theta\int_0^t\int_0^t |r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\} \ge C^{-1}\theta^{2/(2-\alpha)}, \tag{2.1}$$

$$\limsup_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)} \log \mathbb{E}\exp\left\{\theta\int_0^t\int_0^t |r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\} \le C\theta^{2/(2-\alpha)}. \tag{2.2}$$

Compared to Theorem 1.1, the above bounds are less precise. On the other hand, (2.2) is needed in our way to establish Theorem 1.3. In addition, (2.1) and (2.2) can be achieved by some simple observation. Therefore, the proof of them may provide some insight in methodology. Our idea is to compare our setting to the setting of time independence given in (1.1). To this end we first prove:

Lemma 2.2. *Let $\gamma(\cdot)$ be given in (1.6) and assume (1.7) with the exception $\alpha_0 = 0$. There is $C > 0$ such that for each $\theta > 0$,*

$$\lim_{t\to\infty} t^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}\exp\left\{\theta\int_0^t\int_0^t \gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\} = C\theta^{2/(2-\alpha)}. \tag{2.3}$$

Here we specially mention that $\alpha = 1$ for the third form of $\gamma(\cdot)$.

Proof. This result is known for the second form ([4]) and the third form ([16]) with the constant C being identified. Here we give a simpler proof for all three cases.

Our first observation is that (2.3) is equivalent to

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\theta\left(\int_0^t\int_0^t \gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\right\} = C\theta^{4/(4-\alpha)}, \quad \forall\theta > 0. \tag{2.4}$$

Here and elsewhere in the proof, the constant C can be different from place to place. Indeed, by the scaling fact

$$\int_0^{at}\int_0^{at} \gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s \stackrel{d}{=} a^{(4-\alpha)/2}\int_0^t\int_0^t \gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s \tag{2.5}$$

and by a Gärtner–Ellis type result for non-negative random variables (see [1], Corollary 1.2.5), both (2.3) and (2.4) are equivalent to the tail asymptotics

$$\lim_{t\to\infty} t^{-1}\log \mathbb{P}\left\{\int_0^1\int_0^1 \gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s \ge \lambda t^{\alpha/2}\right\} = -C\lambda^{2/\alpha}, \quad \lambda > 0.$$

The proof of (2.4) relies on the argument by sub-additivity. First notice that

$$\gamma(x) = C(\gamma)\int_{\mathbb{R}^d} K(y-x)K(y)\,\mathrm{d}y, \quad x\in\mathbb{R}^d, \tag{2.6}$$

where $C(\gamma) > 0$ is a constant and

$$K(x) = \begin{cases} \prod_{j=1}^{d} |x_j|^{-(1+\alpha_j)/2}, & \text{if } \gamma(x) = \prod_{j=1}^{d} |x_j|^{-\alpha_j}, \\ |x|^{-(d+\alpha)/2}, & \text{if } \gamma(x) = |x|^{-\alpha}, \\ \delta_0(x), & \text{if } \gamma(x) = \delta_0(x). \end{cases} \tag{2.7}$$

Consequently, we have the following representation

$$\int_0^t \int_0^t \gamma(B_r - B_s)\,\mathrm{d}r\,\mathrm{d}s = C(\gamma) \int_{\mathbb{R}^d} \left[\int_0^t K(B_s - x)\,\mathrm{d}s\right]^2 \mathrm{d}x.$$

Notice that the stochastic process

$$Z_t = \left\{\int_{\mathbb{R}^d} \left[\int_0^t K(B_s - x)\,\mathrm{d}s\right]^2 \mathrm{d}x\right\}^{1/2}, \quad t \ge 0$$

is continuous with probability 1, and by the triangle inequality, $Z_{s+t} \le Z_s + Z_t'$ for any $s, t > 0$, where

$$Z_t' = \left\{\int_{\mathbb{R}^d} \left[\int_s^{s+t} K(B_r - x)\,\mathrm{d}r\right]^2 \mathrm{d}x\right\}^{1/2}$$

is equal in law to Z_t and is independent of $\{Z_u; 0 \le u \le s\}$. By [1], Theorem 1.3.5, we have

$$\mathbb{E}\exp\{\theta Z_t\} < \infty \quad (\theta, t > 0) \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\{\sqrt{C(\gamma)}Z_t\} \equiv C \quad \text{exists.}$$

Further, $0 \le C < \infty$. The fact that $C > 0$ follows from the Cauchy–Schwarz inequality

$$\left\{\int_{\mathbb{R}^d} \left[\int_0^t K(B_s - x)\,\mathrm{d}s\right]^2 \mathrm{d}x\right\}^{1/2} \ge \int_{\mathbb{R}^d} f(x) \left[\int_0^t K(B_s - x)\,\mathrm{d}s\right] \mathrm{d}x = \int_0^t f_K(B_s)\,\mathrm{d}s$$

for any measurable function with $\|f\|_2 = 1$, where

$$f_K(x) = \int_{\mathbb{R}^d} f(y) K(x - y)\,\mathrm{d}y.$$

Now we require that f is continuous with compact support. It can be verified that $f_K(x)$ is bounded and continuous in all three cases. By [1], Theorem 4.1.6,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\sqrt{C(\gamma)} \int_0^t f_K(B_s)\,\mathrm{d}s\right\} \\ &\quad = \sup_{g\in\mathcal{F}_d} \left\{\sqrt{C(\gamma)} \int_{\mathbb{R}^d} f_K(x) g^2(x)\,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^d} \left|\nabla g(x)\right|^2 \mathrm{d}x\right\}, \end{aligned} \tag{2.8}$$

where

$$\mathcal{F}_d = \left\{g \in \mathcal{L}^2(\mathbb{R}^d); \int_{\mathbb{R}^d} \left|g(x)\right|^2 \mathrm{d}x = 1 \text{ and } \int_{\mathbb{R}^d} \left|\nabla g(x)\right|^2 \mathrm{d}x < \infty\right\}. \tag{2.9}$$

By Fubini theorem,

$$\int_{\mathbb{R}^d} f_K(x) g^2(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x) \left[\int_{\mathbb{R}^d} K(y - x) g^2(y)\,\mathrm{d}y\right] \mathrm{d}x.$$

Taking supremum over f,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_0^t\int_0^t\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\right\}$$
$$\ge\sup_{g\in\mathcal{F}_d}\left\{\sqrt{C(\gamma)}\sup_f\int_{\mathbb{R}^d}f(x)\left[\int_{\mathbb{R}^d}K(y-x)g^2(y)\,\mathrm{d}y\right]\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}$$
$$=\sup_{g\in\mathcal{F}_d}\left\{\sqrt{C(\gamma)}\left(\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}K(y-x)g^2(y)\,\mathrm{d}y\right]^2\mathrm{d}x\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}.$$

By (2.6),

$$\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}K(y-x)g^2(y)\,\mathrm{d}y\right]^2\mathrm{d}x=C(\gamma)^{-1}\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,\mathrm{d}x\,\mathrm{d}y$$

we reach the conclusion that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\left(\int_0^t\int_0^t\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\right\}$$
$$\ge\sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\} \tag{2.10}$$

and the right hand side is positive.

Summarizing our steps, we have established (2.4) with $0<C<\infty$ in the case $\theta=1$. Replacing t by $\theta^{4/(4-\alpha)}t$ and by the scaling property (2.5) we have proved (2.4) for all $\theta>0$ with the same constant C. □

As a side remark, we point out that the lower bound (2.10) is sharp in the sense that the correspondent upper bound holds. That is, the constant C in (2.4) can be represented as

$$C=\sup_{g\in\mathcal{F}_d}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2\mathrm{d}x\right\}. \tag{2.11}$$

This can be achieved by a simple extention of Theorem 1.3 to the setting of $\alpha_0=0$.

A careful reader may wonder why we do not apply the sub-additivity to Theorem 1.3. Indeed, applying the sub-additivity to time–space case would establish the existence of the limit on the left-hand side (1.12) with the part "$t^{\alpha_0/2}$" being removed. For Theorem 1.3 to be true, of course, the limit value has to be 0. This means that the sub-additivity does not lead to the correct rate in the time–space case. On the other hand, some ideas used here, such as the kernel representation in (2.6) and the argument for the lower bound (2.10), will be adopted to the time–space setting.

Proof of Proposition 2.1. The lower bound (2.1) follows immediately from the fact that

$$\int_0^t\int_0^t|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\ge t^{-\alpha_0}\int_0^t\int_0^t\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s$$
$$\overset{d}{=}\int_0^{t^{(4-\alpha-2\alpha_0)/(4-\alpha)}}\int_0^{t^{(4-\alpha-2\alpha_0)/(4-\alpha)}}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s$$

and Lemma 2.2 with t being replaced by $t^{(4-\alpha-2\alpha_0)/(4-\alpha)}$, where the equality in law comes from the scaling property (2.5) with $a=t^{-(2\alpha_0)/(4-\alpha)}$.

As for the upper bound (2.2), the challenge is to reverse the inequality $|r-s|^{-\alpha_0}\ge t^{-\alpha_0}$ used in the proof of the lower bound. First we notice that

$$\int_0^t\int_0^t|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s=2\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s.$$

The upper bound (2.2) is then equivalent to

$$\limsup_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)}\log \mathbb{E}\exp\left\{\theta\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\le C\theta^{2/(2-\alpha)}. \tag{2.12}$$

We decompose the above integral into three parts

$$\int\int_{\{0\le r<s\le t\}}=\int\int_{\{0\le r<s\le t/2\}}+\int\int_{\{t/2\le r<s\le t\}}+\int_0^{t/2}\int_{t/2}^{t}.$$

Notice the fact that the first, the second terms on the right hand side are mutually independent and identically distributed. It follows from the Hölder inequality that

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\
&\quad\le\left(\mathbb{E}\exp\left\{\theta p\int\int_{\{0\le r<s\le t/2\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{2/p}\\
&\qquad\times\left(\mathbb{E}\exp\left\{\theta q\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{1/q},
\end{aligned}$$

where $p,q>1$ are conjugate numbers ($p^{-1}+q^{-1}=1$). By the scaling property (1.11) we have

$$\begin{aligned}
&\int\int_{\{0\le r<s\le t/2\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\\
&\quad\stackrel{d}{=}\left(\frac{1}{2}\right)^{(4-\alpha-2\alpha_0)/2}\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}$$

Taking $p=2^{(4-\alpha-2\alpha_0)/2}$, then we have

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\
&\quad\le\left(\mathbb{E}\exp\left\{\theta\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{2/p}\\
&\qquad\times\left(\mathbb{E}\exp\left\{\theta q\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{1/q}.
\end{aligned}$$

By the fact that $2/p<1$ under (1.7),

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\int\int_{\{0\le r<s\le t\}}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\
&\quad\le\left(\mathbb{E}\exp\left\{\theta q\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{(1/q)p/(p-2)}.
\end{aligned}$$

In this way, the problem is reduced to the proof of

$$\limsup_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)}\log \mathbb{E}\exp\left\{\theta\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\le C\theta^{2/(2-\alpha)}. \tag{2.13}$$

Denote $A=[t/2,3t/4]\times[t/4,t/2]$ and $B=[t/2,t]\times[0,t/2]\setminus A$. By the Hölder inequality

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\&\quad\le\left(\mathbb{E}\exp\left\{\theta p\int\int_A|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{1/p}\\&\qquad\times\left(\mathbb{E}\exp\left\{\theta q\int\int_B|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\right)^{1/q}.\end{aligned}$$

Taking $p=2^{(4-\alpha-2\alpha_0)/2}$, by the fact that

$$\begin{aligned}&\int\int_A|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\\&\quad\stackrel{d}{=}\int_{t/4}^{t/2}\int_0^{t/4}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\\&\quad\stackrel{d}{=}\left(\frac{1}{2}\right)^{(4-\alpha-2\alpha_0)/2}\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s,\end{aligned}$$

we have

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\int_{t/2}^{t}\int_0^{t/2}|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\&\quad\le\mathbb{E}\exp\left\{\theta q\int\int_B|r-s|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\&\quad\le\mathbb{E}\exp\left\{\theta q\left(\frac{4}{t}\right)^{\alpha_0}\int_0^t\int_0^t\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\}\\&\quad=\mathbb{E}\exp\left\{\theta q4^{\alpha_0}\int_0^{t^{(4-\alpha-2\alpha_0)/(4-\alpha)}}\int_0^{t^{(4-\alpha-2\alpha_0)/(4-\alpha)}}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right\},\end{aligned}$$

where the second step follows partially from the fact that $|r-s|\ge t/4$ on B.

Therefore, the upper bound (2.13) follows from Lemma 2.2 with t being replaced by $t^{(4-\alpha-2\alpha_0)/(4-\alpha)}$. □

3. Time–space large deviations via Feynman–Kac formula

We have seen the critical role played by a Feynman–Kac type large deviation (2.8) in the proof of the lower bound (2.10). We shall see that it is also essential for establishing the precise upper bound. Our goal in this section is to establish a time–space version of such result.

Let $D\subset\mathbb{R}^d$ be an open domain that contains 0. Define the exit time

$$\tau_D=\inf\{t\ge 0;\ B_t\notin D\}. \tag{3.1}$$

In consistent with $\mathcal{F}_d$ defined in (2.9), let $\mathcal{F}_d(D)$ be the set of the smooth functions g on D with $\|g\|_{\mathcal{L}^2(D)}=1$ and $g(\partial D)=0$ and denote for any function f on $\mathbb{R}^d$

$$\lambda_D(f)=\sup_{g\in\mathcal{F}_d(D)}\left\{\int_D f(x)g^2(x)\,\mathrm{d}x-\frac{1}{2}\int_D\left|\nabla g(x)\right|^2\mathrm{d}x\right\} \tag{3.2}$$

and write $\lambda(f)=\lambda_{\mathbb{R}^d}(f)$.

Proposition 3.1. *Let $f(t,x)$ be a measurable function defined on $[0,1]\times\mathbb{R}^d$. Assume that for each $0\le s\le 1$, $f(s,\cdot)$ is bounded and continuous on $\mathbb{R}^d$, and that the family of functions $\{f(\cdot,x);x\in\mathbb{R}^d\}$ is equicontinuous on $[0,1]$. For any bounded open domain $D\subset\mathbb{R}^d$ that contains 0*,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left\{\int_0^t f\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]=\int_0^1\lambda_D\big(f(s,\cdot)\big)\,\mathrm{d}s. \tag{3.3}$$

In addition,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\int_0^t f\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\}=\int_0^1\lambda\big(f(s,\cdot)\big)\,\mathrm{d}s. \tag{3.4}$$

Proof. Let the integer $n\ge 1$ be fixed but arbitrary and let $0=s_0<s_1<\cdots<s_n=1$ be the uniform partition of $[0,1]$. Set $f_j(x)=f(s_{j-1},x)$ $(j=1,\dots,n)$ and define $f^*(s,x)$ on $[0,1]\times\mathbb{R}^d$ as $f^*(s,x)=f_j(x)$ whenever $s\in[s_{j-1},s_j)$. In addition, put $f^*(1,x)\equiv f^*(1^-,x)$. As the first step we establish (3.3) and (3.4) for f^*. By the Markov property,

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]\\&\quad=\mathbb{E}\left[\exp\left\{\int_0^{s_{n-1}t} f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\}E\big(n,n^{-1}t,B_{s_{n-1}t}\big);\tau_D\ge s_{n-1}t\right],\end{aligned}$$

where

$$E(j,t,x)=\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t\right]$$

and the notation "$\mathbb{E}_x$" denotes the expectation with respect to the Brownian motion starting at x. Hence,

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]\\&\quad\le\sup_{x\in D}E\big(n,n^{-1}t,x\big)\mathbb{E}\left[\exp\left\{\int_0^{s_{n-1}t} f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge s_{n-1}t\right].\end{aligned}$$

Repeating this procedure,

$$\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]\le\prod_{j=1}^n\sup_{x\in D}E\big(j,n^{-1}t,x\big).$$

Or,

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]\\&\quad\le\prod_{j=1}^n\sup_{x\in D}\mathbb{E}_x\left[\exp\left\{\int_0^{t/n} f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t/n\right].\end{aligned} \tag{3.5}$$

With a slight modification, one can show that for any $\delta>0$

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]\\&\quad\ge\prod_{j=1}^n\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left\{\int_0^{t/n} f_j(B_s)\,\mathrm{d}s\right\};|B_{s_{j-1}t}|<\delta,\tau_D\ge t/n\right].\end{aligned} \tag{3.6}$$

We now claim that for each j,

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in D}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t\right]\le\lambda_D(f_j). \tag{3.7}$$

Indeed, define $\tau'_D=\inf\{t\ge 1;\, B_t\notin D\}$. Then for any $x\in D$,

$$\begin{aligned}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t\right]&\le C\mathbb{E}_x\left[\exp\left\{\int_1^t f_j(B_s)\,\mathrm{d}s\right\};\tau'_D\ge t\right]\\&=C\int_D p(y-x)\mathbb{E}_y\left[\exp\left\{\int_0^{t-1} f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t-1\right]\mathrm{d}y,\end{aligned}$$

where $p(x)$ is the density function of B_1 and the last step follows from the Markov property. By the fact that $p(x)$ is uniformly bounded on $\mathbb{R}^d$, and by the inequality (see [2], Lemma 4.1)

$$\int_D\mathbb{E}_y\left[\exp\left\{\int_0^{t-1} f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t-1\right]\mathrm{d}y\le|D|\exp\bigl\{(t-1)\lambda_D(f_j)\bigr\}$$

we obtain the following bound

$$\sup_{x\in D}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};\tau_D\ge t\right]\le C|D|\exp\bigl\{(t-1)\lambda_D(f_j)\bigr\} \tag{3.8}$$

which leads to (3.7).

Replacing t by t/n in (3.7), by (3.5) we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_D\ge t\right]\le\frac{1}{n}\sum_{j=1}^n\lambda_D(f_j). \tag{3.9}$$

We now claim that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\}\le\frac{1}{n}\sum_{j=1}^n\lambda(f_j). \tag{3.10}$$

First, the factorization bound (3.5) remains true if D is replaced by $D_t=\{x\in\mathbb{R}^d;\,|x|<t^2\}$. Second, for any $1\le j\le n$, by an argument similar to the one used for (3.7) and noticing that $\lambda_{D_t}(f_j)\le\lambda(f_j)$ we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in D_t}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};\tau_{D_t}\ge t\right]\le\lambda(f_j).$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_{D_t}\ge t\right]\le\frac{1}{n}\sum_{j=1}^n\lambda(f_j).$$

Third, noticing that in the decomposition,

$$\begin{aligned}&\mathbb{E}\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\}\\&\quad=\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_{D_t}\ge t\right]+\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t},B_s\right)\mathrm{d}s\right\};\tau_{D_t}<t\right]\end{aligned}$$

the second term is negligible as it yields the bound

$$\exp\{Ct\}\mathbb{P}\Big\{\max_{s\le t}|B_s|\ge t^2\Big\}\le \exp\{Ct\}\exp\{-ct^3\}$$

we have (3.10).

Given $\delta>0$, define the δ-interior D^o_δ of D as

$$D^o_\delta=\{x\in D;\,|x-y|>\delta \text{ for any } y\in\partial D\}.$$

Take δ sufficiently small so that $0\in D^o_{2\delta}$. For the lower bound, we claim that for any $j=1,\dots,n$,

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};|B_t|<\delta,\tau_D\ge t\right]\ge\lambda_{D^o_{2\delta}}(f_j). \tag{3.11}$$

Indeed, by the boundedness of f_j, for any $x\in\mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};|B_t|<\delta,\tau_D\ge t\right]\\
&\quad\ge C^{-1}\mathbb{E}_x\left[\exp\left\{\int_1^{t-1} f_j(B_s)\,\mathrm{d}s\right\};|B_t|<\delta,\tau_D\ge t\right]\\
&\quad=C^{-1}\int_D p_D(y-x)\mathbb{E}_y\left[\exp\left\{\int_0^{t-2} f_j(B_s)\,\mathrm{d}s\right\}\mathbb{P}_{B_{t-2}}\{|B_1|<\delta,\tau_D\ge 1\};\tau_D\ge t-2\right]\mathrm{d}y,
\end{aligned}$$

where $p_D(x)$ is the density of the measure

$$\mu(A)=\mathbb{P}\{B_1\in A,\tau_D\ge 1\},\quad A\subset D,$$

and the last step follows from the Markov property.

It is well-known (see [14], Theorem 11.3) that there is a $\varepsilon>0$ such that $p_D(x)\ge\varepsilon$ for all $x\in D^0_\delta$. In particular, for any $z\in D^o_{2\delta}$,

$$\mathbb{P}_z\{|B_1|<1,\tau\ge 1\}=\int_{\{|y|<\delta\}}p_D(y-z)\,\mathrm{d}y\ge c\varepsilon\delta^d.$$

Consequently, the integral appearing on the right hand side of the previous estimate is bounded from below by

$$\begin{aligned}
&\int_{D^o_{2\delta}}p_D(y-x)\mathbb{E}_y\left[\exp\left\{\int_0^{t-2} f_j(B_s)\,\mathrm{d}s\right\}\mathbb{P}_{B_{t-2}}\{|B_1|<\delta,\tau_D\ge 1\};\tau_{D^o_{2\delta}}\ge t-2\right]\mathrm{d}y\\
&\quad\ge c\varepsilon^2\delta^d\int_{D^o_{2\delta}}\mathbb{E}_y\left[\exp\left\{\int_0^{t-2} f_j(B_s)\,\mathrm{d}s\right\};\tau_{D^o_{2\delta}}\ge t-2\right]\mathrm{d}y.
\end{aligned}$$

Summarizing our computation, we have

$$\begin{aligned}
&\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\};|B_t|<\delta,\tau_D\ge t\right]\\
&\quad\ge c\varepsilon^2\delta^d C^{-1}\int_{D^o_{2\delta}}\mathbb{E}_y\left[\exp\left\{\int_0^{t-2} f_j(B_s)\,\mathrm{d}s\right\};\tau_{D^o_{2\delta}}\ge t-2\right]\mathrm{d}y.
\end{aligned} \tag{3.12}$$

For any $g \in \mathcal{F}_d(D^0_{2\delta})$,

$$\begin{aligned}&\int_{D^o_{2\delta}} \mathbb{E}_y\left[\exp\left\{\int_0^{t-2} f_j(B_s)\,\mathrm{d}s\right\}; \tau_{D^o_{2\delta}} \ge t-2\right]\mathrm{d}y\\ &\quad\ge \|g\|_\infty^{-2}\int_{D^o_{2\delta}} g(y)\mathbb{E}_y\left[\exp\left\{\int_0^{t-2} f_j(B_s)\,\mathrm{d}s\right\}g(B_{t-2}); \tau_{D^o_{2\delta}} \ge t-2\right]\mathrm{d}y\\ &\quad= \|g\|_\infty^{-2}\big\langle g, \mathrm{e}^{(t-2)A}g\big\rangle,\end{aligned}$$

where A is the linear operator $A = 2^{-1}\Delta + f_j$. Here we point out the relevance of the semi-group $\{T_t; t \ge 0\}$ of self-adjoint operators on $\mathcal{L}^2(D^o_{2\delta})$ defined as

$$T_t h(x) = \mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\}h(B_t); \tau_{D^o_{2\delta}} \ge t\right], \quad h \in \mathcal{L}^2\big(D^o_{2\delta}\big)$$

and the relation $T_t = \mathrm{e}^{tA}$ which is a consequence of Feynman–Kac formula. See, e.g., [1], Section 4.1, for detail.

By spectral representation, associated to g there is a probability measure $\mu_g(\mathrm{d}\lambda)$ on $(-\infty,\infty)$ such that

$$\int_{-\infty}^{\infty}\lambda\mu_g(\mathrm{d}\lambda) = \langle g, Ag\rangle = \int_{D^o_{2\delta}} f_j(x)g^2(x)\,\mathrm{d}x - \frac{1}{2}\int_{D^o_{2\delta}}\big|\nabla g(x)\big|^2\,\mathrm{d}x$$

and

$$\big\langle g, \mathrm{e}^{(t-2)A}g\big\rangle = \int_{-\infty}^{\infty}\mathrm{e}^{(t-2)\lambda}\mu_g(\mathrm{d}\lambda) \ge \exp\left\{(t-2)\int_{-\infty}^{\infty}\lambda\mu_g(\mathrm{d}\lambda)\right\},$$

where the second step follows from Jensen's inequality.

Summarizing the steps from (3.12), we obtain

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left\{\int_0^t f_j(B_s)\,\mathrm{d}s\right\}; |B_t| < \delta, \tau_D \ge t\right]\\ &\quad\ge \int_{D^o_{2\delta}} f_j(x)g^2(x)\,\mathrm{d}x - \frac{1}{2}\int_{D^o_{2\delta}}\big|\nabla g(x)\big|^2\,\mathrm{d}x.\end{aligned}$$

Taking supremum over $g \in \mathcal{F}_d(D^0_{2\delta})$ leads to (3.11).

Combining (3.6) and (3.11), we get

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t}, B_s\right)\mathrm{d}s\right\}; \tau_D \ge t\right] \ge \frac{1}{n}\sum_{j=1}^n \lambda_{D^0_{2\delta}}(f_j).$$

Letting $\delta \to 0^+$ on the right hand side gives

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left\{\int_0^t f^*\left(\frac{s}{t}, B_s\right)\mathrm{d}s\right\}; \tau_D \ge t\right] \ge \frac{1}{n}\sum_{j=1}^n \lambda_D(f_j). \tag{3.13}$$

As a direct consequence of (3.13), we have

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\int_0^t f^*\left(\frac{s}{t}, B_s\right)\mathrm{d}s\right\} \ge \frac{1}{n}\sum_{j=1}^n \lambda_D(f_j)$$

for any bounded open domain $D \subset \mathbb{R}^d$. Letting $D \uparrow \mathbb{R}^d$ on the right hand side yields

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \int_0^t f^*\left(\frac{s}{t}, B_s\right) \mathrm{d}s \right\} \ge \frac{1}{n} \sum_{j=1}^n \lambda(f_j). \tag{3.14}$$

Combining (3.7) with (3.13), and (3.10) with (3.10),

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[\exp\left\{ \int_0^t f^*\left(\frac{s}{t}, B_s\right) \mathrm{d}s \right\}; \tau_D \ge t \right] = \frac{1}{n} \sum_{j=1}^n \lambda_D(f_j) \tag{3.15}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \int_0^t f^*\left(\frac{s}{t}, B_s\right) \mathrm{d}s \right\} = \frac{1}{n} \sum_{j=1}^n \lambda(f_j). \tag{3.16}$$

Notice that $\lambda_D(f_j) = \lambda_D(f(s_{j-1}, \cdot))$, and that $\lambda_D(f(s, \cdot))$ is continuous in s. Consequently, the average on the right hand side of (3.15) converges to the integral on the right hand side of (3.3) as $n \to \infty$. In addition,

$$\max_{0\le s\le t} \left| f^*\left(\frac{s}{t}, B_s\right) - f\left(\frac{s}{t}, B_s\right) \right| \le \max_{x\in\mathbb{R}^d} \max_{0\le s\le 1} \left| f^*(s,x) - f(s,x) \right|.$$

By the equicontinuity assumption the right hand side tends to 0 as $n \to \infty$. Consequently, (3.3) follows from (3.15). Similarly, (3.4) follows from (3.16). □

Recall that the class $\mathcal{A}_d$ is given in (1.8). More generally, let

$$\mathcal{A}_d(D) = \left\{ g \in \mathcal{A}_d;\ g(s, \cdot) \text{ is supported in } D \text{ for every } 0 \le s \le 1 \right\}. \tag{3.17}$$

We end this section with the following remark on the right hand sides of (3.3) and (3.4): It is not hard to see that for any bounded open domain $D \subset \mathbb{R}^d$,

$$\int_0^1 \lambda_D\bigl(f(s,\cdot)\bigr) \mathrm{d}s = \sup_{g\in\mathcal{A}_d(D)} \left\{ \int_0^1 \int_D f(s,x) g^2(s,x) \,\mathrm{d}x\,\mathrm{d}s - \frac{1}{2} \int_0^1 \int_D \left|\nabla_x g(s,x)\right|^2 \mathrm{d}x\,\mathrm{d}s \right\}. \tag{3.18}$$

In addition,

$$\int_0^1 \lambda\bigl(f(s,\cdot)\bigr) \mathrm{d}s = \sup_{g\in\mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(s,x) g^2(s,x) \,\mathrm{d}x\,\mathrm{d}s - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} \left|\nabla_x g(s,x)\right|^2 \mathrm{d}x\,\mathrm{d}s \right\}. \tag{3.19}$$

4. Lower bounds

In this section, we establish the lower bound of Theorem 1.3:

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \theta t^{\alpha_0/2} \left(\int_0^t \int_0^t \frac{1}{|r-s|^{\alpha_0}} \gamma(B_r - B_s) \,\mathrm{d}r\,\mathrm{d}s \right)^{1/2} \right\} \ge \theta^{4/(4-\alpha)} M(\alpha_0, d, \gamma). \tag{4.1}$$

Similar to (2.6), there is a constant $C(\alpha_0) > 0$ such that

$$|s|^{-\alpha_0} = C_0(\alpha_0) \int_{\mathbb{R}} |u-s|^{-(1+\alpha_0)/2} |u|^{-(1+\alpha_0)/2} \,\mathrm{d}u, \quad s \in \mathbb{R}. \tag{4.2}$$

Together with (2.6), this gives

$$
\begin{aligned}
&t^{\alpha_0}\int_0^t\int_0^t \frac{1}{|r-s|^{\alpha_0}}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\\
&\quad=\int_0^t\int_0^t\left|\frac{r-s}{t}\right|^{-\alpha_0}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\\
&\quad=C_0(\alpha_0)C(\gamma)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K(x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x. \qquad (4.3)
\end{aligned}
$$

Let $f(u,x)$ be a bounded, continuous and locally supported function on $\mathbb{R}\times\mathbb{R}^d$ with $\|f\|_2=1$. By the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K(x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}\\
&\quad\ge\int_{\mathbb{R}\times\mathbb{R}^d}f(u,x)\left[\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K(x-B_s)\,\mathrm{d}s\right]\mathrm{d}u\,\mathrm{d}x\\
&\quad=\int_0^t\left[\int_{\mathbb{R}\times\mathbb{R}^d}f(u,x)\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K(x-B_s)\,\mathrm{d}u\,\mathrm{d}x\right]\mathrm{d}s\\
&\quad=\int_0^t\bar{f}\left(\frac{s}{t},B_s\right)\mathrm{d}s,
\end{aligned}
$$

where, one can easily check, that the function

$$
\bar{f}(s,x)=\int_{\mathbb{R}\times\mathbb{R}^d}f(u,y)|u-s|^{-(1+\alpha_0)/2}K(y-x)\,\mathrm{d}u\,\mathrm{d}y,\qquad (s,x)\in[0,1]\times\mathbb{R}^d
$$

satisfies all restriction given in Proposition 3.1. Hence, by (3.4) in Proposition 3.1 and (3.19) we have

$$
\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta t^{\alpha_0/2}\left(\int_0^t\int_0^t\frac{1}{|r-s|^{\alpha_0}}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\right\}\\
&\quad\ge\sup_{g\in\mathcal{A}_d}\left\{\theta\sqrt{C_0(\alpha_0)C(\gamma)}\int_0^1\int_{\mathbb{R}^d}\bar{f}(s,x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2\mathrm{d}x\,\mathrm{d}s\right\}.
\end{aligned}
$$

Taking supremum over f on the right hand side and noticing that

$$
\int_0^1\int_{\mathbb{R}^d}\bar{f}(s,x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s=\int_{\mathbb{R}\times\mathbb{R}^d}f(u,y)\left[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}K(x-y)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right]\mathrm{d}u\,\mathrm{d}y,
$$

we obtain that

$$
\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta t^{\alpha_0/2}\left(\int_0^t\int_0^t\frac{1}{|r-s|^{\alpha_0}}\gamma(B_r-B_s)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\right\}\\
&\quad\ge\sup_{g\in\mathcal{A}_d}\left\{\theta\left(C_0(\alpha_0)C(\gamma)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}K(x-y)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}y\right)^{1/2}\right.\\
&\qquad\left.-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2\mathrm{d}x\,\mathrm{d}s\right\}.
\end{aligned}
$$

By (2.6) and (4.2)

$$C_0(\alpha_0)C(\gamma)\int_{\mathbb{R}\times\mathbb{R}^d}\bigg[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}K(x-y)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}y$$
$$=\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(s,x)g^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r.$$

Finally, the lower bound (4.1) follows from Lemma 4.1 below.

Lemma 4.1. *For any* $\theta>0$,

$$\sup_{g\in\mathcal{A}_d}\bigg\{\theta\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(s,x)g^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}r\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}=\theta^{4/(4-\alpha)}M(\alpha_0,d,\gamma).$$

Proof. Replace the function $g(s,x)$ in the variation on the left hand side by

$$\theta^{d/(4-\alpha)}g\big(s,\theta^{2/(4-\alpha)}x\big).$$

With integration substitution $w=\theta^{2/(4-\alpha)}x$ and $z=\theta^{2/(4-\alpha)}y$, the variation becomes

$$\sup_{g\in\mathcal{A}_d}\bigg\{\theta\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(\theta^{-2/(4-\alpha)}(w-z))}{|r-s|^{\alpha_0}}g^2(s,w)g^2(r,z)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r\bigg)^{1/2}$$
$$-\frac{1}{2}\theta^{4/(4-\alpha)}\int_0^1\int_{\mathbb{R}^d}\big|\nabla_w g(s,w)\big|^2\,\mathrm{d}w\,\mathrm{d}s\bigg\}$$
$$=\sup_{g\in\mathcal{A}_d}\bigg\{\theta^{4/(4-\alpha)}\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(w-z)}{|r-s|^{\alpha_0}}g^2(s,w)g^2(r,z)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}r\bigg)^{1/2}$$
$$-\frac{1}{2}\theta^{4/(4-\alpha)}\int_0^1\int_{\mathbb{R}^d}\big|\nabla_w g(s,w)\big|^2\,\mathrm{d}w\,\mathrm{d}s\bigg\}$$
$$=\theta^{4/(4-\alpha)}M(\alpha_0,d,\gamma),$$

where the first equality comes from the scaling $\gamma(cx)=c^{-\alpha}\gamma(x)$ for any $c>0$. □

5. Upper bound

Recall that $K(x)$ is defined in (2.7). By the scaling property (1.11) and the representation in (4.3), the bound (2.2) in Proposition 2.1 leads to

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigg\{\theta\bigg(\int_{\mathbb{R}\times\mathbb{R}^d}\bigg[\int_0^t\big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}K(x-B_s)\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}x\bigg)^{1/2}\bigg\}$$
$$\le C\theta^{4/(4-\alpha)},\quad \theta>0. \tag{5.1}$$

To establish the upper bound for Theorem 1.3, all we need is to tight up the constant $C>0$. More precisely, we need to show that (5.1) holds with

$$C=\big(C_0(\alpha_0)C(\gamma)\big)^{-2/(4-\alpha)}M(\alpha_0,d,\gamma).$$

What we did in Section 4 was essentially to bound a L^2-norm by the linear functionals from below and then apply Proposition 3.1 to the linear functionals. The opposite direction of this approach requires some exponential tightness in L^2-space which does not hold directly in our setting, due to non-compactness of the space $\mathbb{R}\times\mathbb{R}^d$. The treatment is compactification by folding. For this purpose we need to localize the kernels $|\cdot|^{-(1+\alpha_0)/2}$ and $K(\cdot)$ and to remove their singularities at 0.

In the following discussion, let $l:\mathbb{R}^+\longrightarrow[0,1]$ be a smooth function satisfying the following properties: $l(u)=1$ for $u\in[0,1]$, $l(u)=0$ for $u\geq 3$ and $-1\leq l'(u)\leq 0$ for all $u>0$. Let $R>0$ be a large number and write

$$\psi_R(u)=|u|^{-(1+\alpha_0)/2}l\big(R^{-1}|u|\big).$$

In connection to (2.7), we write

$$K_R(x)=\begin{cases}\prod_{j=1}^d|x_j|^{-(1+\alpha_j)/2}l(R^{-1}|x_j|), & \text{when } \gamma(x)=\prod_{i=1}^d|x_i|^{-\alpha_i},\\ |x|^{-(d+\alpha)/2}l(R^{-1}|x|), & \text{when } \gamma(x)=|x|^{-\alpha},\\ \delta_0(x), & \text{when } \gamma(x)=\delta_0(x).\end{cases}\tag{5.2}$$

Set

$$Q_R(u,x)=|u|^{-(1+\alpha_0)/2}K(x)-\psi_R(u)K_R(x).$$

We claim that

$$\lim_{R\to\infty}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^t Q_R\big(u-t^{-1}s,x-B_s\big)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}=0.\tag{5.3}$$

We first consider the case when $\gamma(x)=\prod_{i=1}^d|x_i|^{-\alpha_i}$. By the triangle inequality we have that

$$\begin{aligned}&\biggl(\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^t Q_R\big(u-t^{-1}s,x-B_s\big)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\\ &\quad\leq\biggl(\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^t\big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}\big(1-l\big(R^{-1}\big|u-t^{-1}s\big|\big)\big)K(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\\ &\qquad+\sum_{j=1}^d\biggl(\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^t\big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}K_j(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\\ &\quad=X_0(t,R)+\sum_{j=1}^d X_j(t,R),\end{aligned}$$

where

$$K_j(x)=\big(1-l\big(R^{-1}|x_j|\big)\big)K(x),\quad j=1,\dots,d.$$

To show (5.3) it suffices to establish that for any $\theta>0$

$$\lim_{R\to\infty}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\big\{\theta X_j(t,R)\big\}=0,\quad j=0,\forall 1,\dots,d.\tag{5.4}$$

For a fixed α_0' satisfying $0<\alpha_0'<\alpha_0$,

$$|u|^{-(1+\alpha_0)/2}\big(1-l\big(R^{-1}|u|\big)\big)\leq|u|^{-(1+\alpha_0)/2}1_{\{|u|\geq R\}}\leq R^{-(\alpha_0-\alpha_0')/2}|u|^{-(1+\alpha_0')/2}.$$

Consequently, we have

$$X_0(t,R) \le R^{-(\alpha_0-\alpha_0')/2}\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t \left|u-t^{-1}s\right|^{-(1+\alpha_0')/2} K(x-B_s)\,\mathrm{d}s\right]^2\right)^{1/2}.$$

Applying (5.1) with α_0 being replaced by α_0', and θ being replaced by $\theta R^{-(\alpha_0-\alpha_0')/2}$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\{\theta X_0(t,R)\} \le C_0 R^{-(2(\alpha_0-\alpha_0'))/(4-\alpha)}\theta^{4/(4-\alpha)},$$

where $C_0 = C_0(\alpha_0', \alpha_1, \ldots, \alpha_d)$ is independent of R. This leads to (5.4) in the case $j = 0$.

Similarly, for all $1 \le j \le d$ we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\{\theta X_j(t,R)\} \le C_j R^{-(2(\alpha_j-\alpha_j'))/(4-\alpha')}\theta^{4/(4-\alpha')},$$

where $\alpha' = \alpha_1 + \cdots + \alpha_{j-1} + \alpha_j' + \alpha_{j+1} + \cdots + \alpha_d$. This leads to (5.4) with $j = 1, \ldots, d$.

The proofs of (5.3) for the other forms of $\gamma(\cdot)$ are similar.

By the triangle inequality and Hölder inequality,

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t \left|u-t^{-1}s\right|^{-(1+\alpha_0)/2} K(x-B_s)\,\mathrm{d}s\right]^2 \mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right\}\\
&\quad\le \left(\mathbb{E}\exp\left\{p\theta\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t \psi_R\big(u-t^{-1}s\big)K_R(x-B_s)\,\mathrm{d}s\right]^2 \mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right\}\right)^{1/p}\\
&\qquad\times\left(\mathbb{E}\exp\left\{q\theta\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t Q_R\big(u-t^{-1}s, x-B_s\big)\,\mathrm{d}s\right]^2 \mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right\}\right)^{1/q}
\end{aligned}$$

for any numbers $p, q > 1$ with $p^{-1} + q^{-1} = 1$. Applying (5.3) (with θ being replaced by $q\theta$) to the right hand side, and by the fact that p can be arbitrarily close to 1, we reduced the proof of (5.1) to the proof of

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t \psi_R\big(u-t^{-1}s\big)K_R(x-B_s)\,\mathrm{d}s\right]^2 \mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right\}\\
&\quad\le \theta^{4/(4-\alpha)}\big(C_0(\alpha_0)C(\gamma)\big)^{-2/(4-\alpha)} M(\alpha_0, d, \gamma) \quad \forall R > 0.
\end{aligned} \tag{5.5}$$

To remove the singularity of the functions $\psi_R(u)$ and $K_R(x)$ at 0 in the case when $\gamma(x) = \prod_{i=1}^d |x_i|^{-\alpha_i}$ or when $\gamma(x) = |x|^{-\alpha}$, let $b > 0$ be a small number and we use the following smooth truncations

$$\psi_{R,b}(u) = \psi_R(u)\big(1 - l\big(b^{-1}|u|\big)\big) = |u|^{-(1+\alpha_0)/2} l\big(R^{-1}|u|\big)\big(1 - l\big(b^{-1}|u|\big)\big) \tag{5.6}$$

and

$$K_{R,b}(x) = \begin{cases} \prod_{j=1}^d |x_j|^{-(1+\alpha_j)/2} l(R^{-1}|x_j|)(1 - l(b^{-1}|x_j|)), & \text{when } \gamma(x) = \prod_{i=1}^d |x_i|^{-\alpha_i},\\ |x|^{-(d+\alpha)/2} l(R^{-1}|x|)(1 - l(b^{-1}|x|)), & \text{when } \gamma(x) = |x|^{-\alpha}. \end{cases} \tag{5.7}$$

We claim that

$$\lim_{b\to0^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t Q_{R,b}\big(u-t^{-1}s, x-B_s\big)\,\mathrm{d}s\right]^2 \mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right\} = 0, \tag{5.8}$$

where $Q_{R,b}(u,x) = \psi_R(u)K_R(x) - \psi_{R,b}(x)K_{R,b}(x)$.

We only consider the case when $\gamma(x)=\prod_{j=1}^d |x_j|^{-(1+\alpha_j)/2}$. The other one can be dealt with similarly. The proof of (5.8) is similar to the proof of (5.3) with the observation that for any $\bar\alpha_j>\alpha_j$ $(j=0,\dots,d)$

$$\psi_R(u)-\psi_{R,b}(u)\le |u|^{-(1+\alpha_0)/2}1_{\{|u|<3b\}}\le (3b)^{(\bar\alpha_0-\alpha_0)/2}|u|^{-(1+\bar\alpha_0)/2}$$

and

$$|x_j|^{-(1+\alpha_j)/2}l\bigl(R^{-1}|x_j|\bigr)l\bigl(b^{-1}|x_j|\bigr)\le (3b)^{(\bar\alpha_j-\alpha_j)/2}|x_j|^{-(1+\bar\alpha_j)/2}\quad (j=1,\dots,d)$$

and that we can make $\bar\alpha_j$ arbitrarily close to α_j so (1.7) remains true when α_j is replaced by $\bar\alpha_j$ for any $j=0,\dots,d$.

In the settings of $\gamma(x)=\prod_{i=1}^d |x_i|^{-\alpha_i}$ and $\gamma(x)=|x|^{-\alpha}$, therefore, the problem is further reduced from the proof of (5.5) to

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^t \psi_{R,b}\bigl(u-t^{-1}s\bigr)K_{R,b}(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}\\&\quad\le\theta^{4/(4-\alpha)}\bigl(C_0(\alpha_0)C(\gamma)\bigr)^{-2/(4-\alpha)}M(\alpha_0,d,\gamma)\end{aligned}\tag{5.9}$$

for any fixed large number $R>0$ and small number $b>0$.

As for the case when $\gamma(x)=\delta_0(x)$, where $K_R=K=\delta_0$, by the similar argument, the problem is reduced to

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^t \psi_{R,b}\bigl(u-t^{-1}s\bigr)\delta_0(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}\\&\quad\le\theta^{4/3}\bigl(C_0(\alpha_0)C(\delta_0)\bigr)^{-2/3}M(\alpha_0,1,\delta_0)\end{aligned}\tag{5.10}$$

for the same $\psi_{R,b}$.

Unfortunately, the singularity of the Dirac function can not be removed by truncation. A separate treatment is needed here. Let $h(x)$ be a smooth and locally supported probability density on $\mathbb{R}$ and write $h_\varepsilon(x)=\varepsilon^{-1}h(\varepsilon^{-1}x)$. We claim that

$$\lim_{\varepsilon\to0^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^t \psi_{R,b}\bigl(u-t^{-1}s\bigr)(\delta_0-h_\varepsilon)(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}=0.\tag{5.11}$$

Indeed, by sub-additivity

$$\begin{aligned}&\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^t \psi_{R,b}\bigl(u-t^{-1}s\bigr)(\delta_0-h_\varepsilon)(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}\\&\quad\le\biggl(\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^1 \psi_{R,b}\bigl(u-t^{-1}s\bigr)(\delta_0-h_\varepsilon)(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}\biggr)^t.\end{aligned}$$

Thus, it suffices to show that

$$\lim_{\varepsilon\to0^+}\limsup_{t\to\infty}\mathbb{E}\exp\biggl\{\theta\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^1 \psi_{R,b}\bigl(u-t^{-1}s\bigr)(\delta_0-h_\varepsilon)(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\biggr\}=1.\tag{5.12}$$

By the triangle inequality

$$\begin{aligned}&\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^1 \psi_{R,b}\bigl(u-t^{-1}s\bigr)(\delta_0-h_\varepsilon)(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\\&\quad\le\biggl(\int_{\mathbb{R}\times\mathbb{R}}\biggl[\int_0^1 \bigl|\psi_{R,b}\bigl(u-t^{-1}s\bigr)-\psi_{R,b}(u)\bigr|\delta_0(x-B_s)\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}x\biggr)^{1/2}\end{aligned}$$

$$+\left(\int_{\mathbb{R}\times\mathbb{R}}\left[\int_0^1\left|\psi_{R,b}\left(u-t^{-1}s\right)-\psi_{R,b}(u)\right|h_\varepsilon(x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}$$
$$+\left(\int_{\mathbb{R}}\psi_{R,b}^2(u)\,\mathrm{d}u\right)^{1/2}\left(\int_{\mathbb{R}}\left[L(1,x)-L_\varepsilon(1,x)\right]^2\mathrm{d}x\right)^{1/2},$$

where

$$L(1,x)=\int_0^1\delta_0(x-B_s)\,\mathrm{d}s\quad\text{and}\quad L_\varepsilon(1,x)=\int_0^1 h_\varepsilon(x-B_s)\,\mathrm{d}s=\int_{\mathbb{R}}h_\varepsilon(x-y)L(t,y)\,\mathrm{d}y$$

are local time and smoothed local time, respectively.

Notice the fact that

$$\lim_{t\to\infty}\int_{\mathbb{R}}\max_{0\le s\le 1}\left|\psi_{R,b}\left(u-t^{-1}s\right)-\psi_{R,b}(u)\right|^2\mathrm{d}u=0.$$

By Jensen's inequality

$$\int_{\mathbb{R}}L_\varepsilon^2(1,x)\,\mathrm{d}x\le\int_{\mathbb{R}}\left[\int_{\mathbb{R}}h_\varepsilon(x-y)L^2(1,y)\,\mathrm{d}y\right]\mathrm{d}x$$
$$=\int_{\mathbb{R}}\left[\int_{\mathbb{R}}h_\varepsilon(x-y)\,\mathrm{d}x\right]L^2(1,y)\,\mathrm{d}y=\int_{\mathbb{R}}L^2(1,y)\,\mathrm{d}y.$$

With the continuity of the Brownian local time, and the exponential integrability of the self-intersection local time (see, e.g., [1], Chapter 4), (5.12) holds. So does (5.11).

By (5.11), the proof of (5.10) is reduced to the proof of

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\int_{\mathbb{R}\times\mathbb{R}}\left[\int_0^t\psi_{R,b}\left(u-t^{-1}s\right)h_\varepsilon(x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right\}$$
$$\le\theta^{4/3}\left(C_0(\alpha_0)C(\delta_0)\right)^{-2/3}M(\alpha_0,1,\delta_0)\tag{5.13}$$

for fixed R, b and ε. In the remaining part of this section, we prove (5.9) for the first, second forms of $\gamma(\cdot)$, and (5.13) for the third form of $\gamma(\cdot)$.

Let $M>2R$ be fixed but arbitrary. For the first and second forms of $\gamma(\cdot)$, we have

$$\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t\psi_{R,b}\left(u-t^{-1}s\right)K_{R,b}(x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x$$
$$=\sum_{k\in\mathbb{Z}}\sum_{z\in\mathbb{Z}^d}\int_{[0,M]^{d+1}}\left[\int_0^t\psi_{R,b}\left(Mk+u-t^{-1}s\right)K_{R,b}(Mz+x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x$$
$$\le\int_{[0,M]^{d+1}}\left[\sum_{k\in\mathbb{Z}}\sum_{z\in\mathbb{Z}^d}\int_0^t\psi_{R,b}\left(Mk+u-t^{-1}s\right)K_{R,b}(Mz+x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x$$
$$=\int_{[0,M]^{d+1}}\left[\int_0^t\widetilde{\psi}_M\left(u-t^{-1}s\right)\widetilde{K}_M(x-B_s)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x,$$

where

$$\widetilde{\psi}_M(u)=\sum_{k\in\mathbb{Z}}\psi_{R,b}(kM+u)\quad\text{and}\quad\widetilde{K}_M(x)=\sum_{z\in\mathbb{Z}^d}K_{R,b}(Mz+x).\tag{5.14}$$

With $K_{R,b}$ being replaced by h_ε, the above bound remains true for the third form of $\gamma(\cdot)$. The notations $\widetilde{K}_M$ and $\widetilde{\psi}_M$ are also adopted to this case.

For any fixed $t>0$, the following process

$$\eta_t(u,x)=\int_0^t \widetilde{\psi}_M\bigl(u-t^{-1}s\bigr)\widetilde{K}_M(x-B_s)\,\mathrm{d}s,\quad (u,x)\in[0,M]^{d+1}$$

can be considered as a process with values in the Hilbert space $\mathcal{L}^2([0,M]^{d+1})$ (the norm on this Hilbert space will be denoted by $\|\cdot\|$). We claim that there is a fixed compact set $K\subset\mathcal{L}^2([0,M]^{d+1})$ such that for any $t>0$, $t^{-1}\eta_t(\cdot,\cdot)\in K$ a.s.

By the locality of $\psi_{R,b}$, $K_{M,b}$ and h_ε and the fact $M>2R$, $\widetilde{\psi}_M$ and $\widetilde{K}_M$ are actually the periodic extensions of $\psi_{R,b}$ and $K_{R,b}$ (or h_ε), respectively. In particular, $\widetilde{\psi}_M$ and $\widetilde{K}_M$ are bounded, smooth functions with bounded derivatives. Consequently, there is a constant $C>0$, such that

$$t^{-1}\bigl\|\eta_t(\cdot,\cdot)\bigr\|\le C\quad\text{and}\quad t^{-1}\bigl\|\eta_t(\cdot+v_1,\cdot+w_1)-\eta_t(\cdot+v_2,\cdot+w_2)\bigr\|\le C\bigl|(v_1,w_1)-(v_2,w_2)\bigr|$$

for all t and $(v_1,w_1),(v_2,w_2)\in[0,M]^{d+1}$. Thus, our claim follows from the classic fact ([9], 8.21, Theorem IV) that the set

$$\begin{aligned}A=\bigl\{&f\in\mathcal{L}^2\bigl([0,M]^{d+1}\bigr);\ \|f\|\le C\text{ and }\bigl\|f(\cdot+v_1,\cdot+w_1)-f(\cdot+v_2,\cdot+w_2)\bigr\|\\&\le C\bigl|(v_1,w_1)-(v_2,w_2)\bigr|\text{ for }(v_1,w_1),(v_2,w_2)\in[0,M]^{d+1}\bigr\}\end{aligned}$$

is pre-compact in $\mathcal{L}^2([0,M]^{d+1})$, with the choice K as the closure of A.

Let $\delta>0$ be fixed. For any $g\in K$ by the Hahn–Banach theorem ([18], p. 108, Corollary 2) and by the fact that bounded and continuous functions are dense in $\mathcal{L}^2([0,M]^{d+1})$, there is a bounded and continuous function $f\in\mathcal{L}^2([0,M]^{d+1})$ such that $\|f\|=1$ and that $\|g\|<\delta+\langle f,g\rangle$. By the finite-cover theorem, one can pick $f_1,\dots,f_m$ from these functions such that $\|g\|<\delta+\max_{1\le i\le m}\langle f_i,g\rangle$ for any $g\in K$. In particular,

$$\mathbb{E}\exp\bigl\{\theta\|\eta_t\|\bigr\}\le \mathrm{e}^{\delta\theta t}\sum_{i=1}^m\mathbb{E}\exp\bigl\{\theta\langle f_i,\eta_t\rangle\bigr\}.\tag{5.15}$$

Let $1\le i\le m$ be fixed. Notice that

$$\langle f_i,\eta_t\rangle=\int_0^t\biggl[\int_{[0,M]^{d+1}}f_i(u,x)\widetilde{\psi}_M\bigl(u-t^{-1}s\bigr)\widetilde{K}_M(x-B_s)\,\mathrm{d}u\,\mathrm{d}x\biggr]\mathrm{d}s=\int_0^t\bar{f}_i\biggl(\frac{s}{t},B_s\biggr)\mathrm{d}s,$$

where

$$\bar{f}_i(s,x)=\int_{[0,M]^{d+1}}f_i(u,y)\widetilde{\psi}_M(u-s)\widetilde{K}_M(y-x)\,\mathrm{d}u\,\mathrm{d}y,\quad (s,x)\in[0,1]\times\mathbb{R}^d$$

satisfies all assumptions made in Proposition 3.1. Hence, by (3.4) combined with (3.19),

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\bigl\{\theta\langle f_i,\eta_t\rangle\bigr\}=\sup_{g\in\mathcal{A}_d(D)}\biggl\{\int_0^1\int_{\mathbb{R}^d}\bar{f}_i(s,x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(s,x)\bigr|^2\,\mathrm{d}x\,\mathrm{d}s\biggr\}.\tag{5.16}$$

Notice that

$$\begin{aligned}\int_0^1\int_{\mathbb{R}^d}\bar{f}_i(s,x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s&=\int_{[0,M]^{d+1}}f_i(u,y)\biggl[\int_0^1\int_{\mathbb{R}^d}\widetilde{\psi}_M(u-s)\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\biggr]\mathrm{d}u\,\mathrm{d}y\\&\le\biggl(\int_{[0,M]^{d+1}}\biggl[\int_0^1\int_{\mathbb{R}^d}\widetilde{\psi}_M(u-s)\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\biggr]^2\mathrm{d}u\,\mathrm{d}y\biggr)^{1/2}.\end{aligned}$$

The quadratic integral inside $(\cdots)^{1/2}$ on the right hand side is equal to

$$\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d} g^2(r,x_1)g^2(s,x_2)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x_1x_2 \times\left[\int_0^M \widetilde{\psi}_M(u-s)\widetilde{\psi}_M(u-r)\,\mathrm{d}u\right]\left[\int_{[0,M]^d}\widetilde{K}_M(y-x_1)\widetilde{K}_M(y-x_2)\,\mathrm{d}y\right].$$

We now claim that

$$\int_0^M \widetilde{\psi}_M(u-s)\widetilde{\psi}_M(u-r)\,\mathrm{d}u=\int_{\mathbb{R}}\psi_{R,b}(u-s)\psi_{R,b}(u-r)\,\mathrm{d}u. \tag{5.17}$$

Notice that

$$\psi_{R,b}(jM+u-r)\psi_{R,b}(kM+u-s)=0,\quad u\in\mathbb{R}, s,r\in[0,1],\text{ and } j,k\in\mathbb{Z}\text{ with } j\neq k.$$

Consequently,

$$\widetilde{\psi}_M(u-s)\widetilde{\psi}_M(u-r)=\sum_{k\in\mathbb{Z}}\psi_{R,b}(kM+u-s)\psi_{R,b}(kM+u-r),\quad u\in\mathbb{R}, r,s\in[0,1]$$

which leads to (5.17).

Noticing that the right-hand side of (5.17) is no greater than

$$\int_{\mathbb{R}}|u-s|^{-(1+\alpha_0)/2}|u-r|^{-(1+\alpha_0)/2}\,\mathrm{d}u=C_0(\alpha_0)^{-1}|r-s|^{-\alpha_0},$$

we have

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^d}\bar{f}_i(s,x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &\le\big(C_0(\alpha_0)\big)^{-1}\int_{[0,M]^d}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}|r-s|^{-\alpha_0}\widetilde{K}_M(y-x_1)\widetilde{K}_M(y-x_2)g^2(r,x_1)g^2(s,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y\\ &=\left(\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}y\right)^{1/2}.\end{aligned} \tag{5.18}$$

Then by Lemma A.3, we have proved that for each $i=1,2,\dots,m$, the variation on the right hand side of (5.16) is no greater than

$$\begin{aligned}&\sup_{g\in\mathcal{A}_d}\left\{\theta\big(C_0(\alpha_0)C(\gamma)\big)^{-1/2}\left(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(s,x)g^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\right.\\ &\qquad\left.-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla_x g(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\right\}\\ &=\theta^{4/(4-\alpha)}\big(C_0(\alpha_0)C(\gamma)\big)^{-2/(4-\alpha)}M(\alpha_0,d,\gamma),\end{aligned}$$

where the equality follows from Lemma 4.1.

By (5.15), therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\big\{\theta\|\eta_t\|\big\}\le\theta\delta+\theta^{4/(4-\alpha)}\big(C_0(\alpha_0)C(\gamma)\big)^{-2/(4-\alpha)}M(\alpha_0,d,\gamma).$$

Finally, (5.9) and (5.13) follow from the fact that δ can be arbitrarily small.

6. Some related results

6.1. *Intermittency of a parabolic Anderson model*

Recall the parabolic Anderson model (1.3)

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \frac{\partial^{d+1} W^H}{\partial t\, \partial x_1 \cdots \partial x_d}(t,x) u(t,x), \\ u(0,x) = 1. \end{cases}$$

The p-moment of the solution is given by

$$\mathbb{E}\big(u(t,x)^p\big) = \mathbb{E}\left(\exp\left[\frac{c_H}{2}\sum_{j,k=1}^{p}\int_0^t\int_0^t |s-r|^{-\alpha_0}\gamma\big(B_r^j - B_s^k\big)\,\mathrm{d}s\,\mathrm{d}r\right]\right), \tag{6.1}$$

where we recall $\alpha_i = 2-2H_i$, $i=0,1,\ldots,d$ and $\gamma(x) = \prod_{i=1}^d |x_i|^{-\alpha_i}$ is of the first form. We have the following precise asymptotics for the moments of the solution $u(t,x)$.

Theorem 6.1. *Let $\mathcal{E}(\alpha_0,d,\gamma)$ be the quantity defined by (1.9). Then*

$$\lim_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)} \log \mathbb{E} u^p(t,x) = p^{(4-\alpha)/(2-\alpha)}\left(\frac{c_H}{2}\right)^{2/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma).$$

Proof. Notice that

$$\begin{aligned}&\sum_{j,k=1}^{p}\int_0^t\int_0^t |s-r|^{-\alpha_0}\gamma\big(B_r^j - B_s^k\big)\,\mathrm{d}s\,\mathrm{d}r \\ &\quad = C_0(\alpha_0)C(\alpha)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^p\int_0^t \big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}K\big(x-B_s^j\big)\,\mathrm{d}s\right]^2 \mathrm{d}u\,\mathrm{d}x.\end{aligned}$$

By scaling, an argument similar to the proof of Theorem 1.1 based on Theorem 1.3 given in Section 1, it suffices to show

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left[\theta\left(C_0(\alpha_0)C(\alpha)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^p\int_0^t\big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}K\big(x-B_s^j\big)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right] \\ &\quad = p\theta^{4/(4-\alpha)}M(\alpha_0,d,\gamma).\end{aligned}$$

Unlike before, the upper bound is easier to obtain. Observing that

$$\begin{aligned}&\left(C_0(\alpha_0)C(\alpha)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^p\int_0^t\big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}K\big(x-B_s^j\big)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2} \\ &\quad \le \sum_{j=1}^p\left(C_0(\alpha_0)C(\alpha)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\int_0^t\big|u-t^{-1}s\big|^{-(1+\alpha_0)/2}K\big(x-B_s^j\big)\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2} \\ &\quad = \sum_{j=1}^p\left(t^{\alpha_0}\int_0^t\int_0^t |s-r|^{-\alpha_0}\gamma\big(B_r^j-B_s^j\big)\,\mathrm{d}s\,\mathrm{d}r\right)^{1/2}\end{aligned}$$

and the fact that B^j, $j=1,\dots,d$ are independent, and using Theorem 1.3, we get

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left[\theta\left(C_0(\alpha_0)C(\alpha)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^p\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K\left(x-B_s^j\right)\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right]$$
$$\le p\theta^{4/(4-\alpha)}M(\alpha_0,d,\gamma).$$

The lower bound can be established as in Section 4. More precisely, assume $f(u,x)$ to be a bounded, continuous and locally supported function on $\mathbb{R}\times\mathbb{R}^d$ with $\|f\|_2=1$. Then from the Cauchy–Schwarz inequality, we see

$$\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^p\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K\left(x-B_s^j\right)\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}$$
$$\ge\int_{\mathbb{R}\times\mathbb{R}^d}f(u,x)\left[\sum_{j=1}^p\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K\left(x-B_s^j\right)\mathrm{d}s\right]\mathrm{d}u\,\mathrm{d}x$$
$$=\sum_{j=1}^p\int_0^t\bar{f}\left(t^{-1}s,B_s^j\right)\mathrm{d}s,$$

where

$$\bar{f}(s,x)=\int_{\mathbb{R}\times\mathbb{R}^d}f(u,y)|u-s|^{-(1+\alpha_0)/2}K(y-x)\,\mathrm{d}u\,\mathrm{d}y,\quad (s,x)\in[0,1]\times\mathbb{R}^d.$$

By the independence of the Brownian motions $B^1,\dots,B^p$, we have

$$\mathbb{E}\exp\left\{\theta\sqrt{C_0(\alpha_0)C(\alpha)}\sum_{j=1}^p\int_0^t\bar{f}\left(t^{-1}s,B_s^j\right)\mathrm{d}s\right\}=\left(\mathbb{E}\exp\left\{\theta\sqrt{C_0(\alpha_0)C(\alpha)}\int_0^t\bar{f}\left(t^{-1}s,B_s\right)\mathrm{d}s\right\}\right)^p.$$

Applying (3.4) in Proposition 3.1

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left[\theta\left(C_0(\alpha_0)C(\alpha)\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^p\int_0^t\left|u-t^{-1}s\right|^{-(1+\alpha_0)/2}K\left(x-B_s^j\right)\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}x\right)^{1/2}\right]$$
$$\ge p\sup_{g\in\mathcal{A}_d}\left\{\theta\sqrt{C_0(\alpha_0)C(\gamma)}\int_0^1\int_{\mathbb{R}^d}\bar{f}(s,x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2\mathrm{d}x\,\mathrm{d}s\right\}.$$

Taking supremum over f on right-hand side leads to the desired lower bound. □

By Theorem 6.1, it is easy to verify that when $p<q$,

$$\lim_{t\to\infty}\frac{[\mathbb{E}u^p(t,x)]^{1/p}}{[\mathbb{E}u^q(t,x)]^{1/q}}=0.$$

This is a sufficient condition for the model (1.3) to preserve intermittency, which means, roughly speaking,that when the time t is large, the total mass $u(t,x)$ will be mainly concentrated on a small number of remote islands. Refer to [10] and the references therein for more details about intermittency.

6.2. *A local version of Theorem 1.3*

For possible future application, we post a local version of Theorem 1.3. For any $0<b<R<\infty$, recall that $\psi_{R,b}(u)$ and $K_{R,b}(x)$ are the truncated kernels defined in (5.6) and (5.7), respectively. Here we allow $b=0$ or $R=\infty$ in a natural way. For example,

$$\psi_{R,0}(u)=|u|^{-(1+\alpha_0)/2}l\big(R^{-1}|u|\big)\quad\text{and}\quad K_{\infty,0}(x)=K(x).$$

Let $D\subset\mathbb{R}^d$ be an open and bounded domain with $0\in D$ and recall that the exit time $\tau_D=\inf\{t\ge 0; B_t\notin D\}$ and that the class $\mathcal{A}_d(D)$ is defined by (3.17). For any $\theta>0$, define

$$\begin{aligned}&M_{\mathrm{loc}}(\alpha_0,d,\gamma,\theta,R_0,R,b_0,b,D)\\&\quad=\sup_{g\in\mathcal{A}_d(D)}\bigg\{\theta\bigg(\int_{\mathbb{R}\times\mathbb{R}^d}\bigg[\int_0^1\int_D\psi_{R_0,b_0}(u-s)K_{R,b}(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}\\&\qquad-\frac{1}{2}\int_0^1\int_D\big|\nabla g(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}.\end{aligned}\tag{6.2}$$

Theorem 6.2. *Under the assumption* (1.7), *for any* $\theta>0$, $0\le b_0<R_0\le\infty$, $0\le b<R\le\infty$

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\bigg[\exp\bigg\{\theta\bigg(\int_{\mathbb{R}\times\mathbb{R}^d}\bigg[\int_0^t\psi_{R_0,b_0}\big(u-t^{-1}s\big)K_{R,b}(x-B_s)\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}x\bigg)^{1/2}\bigg\};\tau_D\ge t\bigg]\\&\quad=M_{\mathrm{loc}}(\alpha_0,d,\gamma,\theta,R_0,R,b_0,b,D)\quad\forall\theta>0.\end{aligned}\tag{6.3}$$

Proof. The argument is essentially the same as the one used for Theorem 1.3 with the following exceptions: First, we replace (3.4) and (3.19) by (3.3) and (3.18), respectively. In particular, the establishment of the lower bound is now straightforward. Second, folding for the upper bound is no longer needed. Take the first and second forms of $\gamma(\cdot)$ for example. On the event $\{\tau_D\ge t\}$

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}^d}\bigg[\int_0^t\psi_{R_0,b_0}\big(u-t^{-1}s\big)K_{R,b}(x-B_s)\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}x\\&\quad=\int_{(-3R_0,3R_0+1)\times D_R}\bigg[\int_0^t\psi_{R_0,b_0}\big(u-t^{-1}s\big)K_{R,b}(x-B_s)\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}x,\end{aligned}$$

where D_R is $(3R)$-neighborhood of D. As a crucial fact, the process

$$\frac{1}{t}\int_0^t\psi_{R_0,b_0}\big(\cdot-t^{-1}s\big)K_{R,b}(\cdot-B_s)\,\mathrm{d}s,\quad t\ge 0$$

takes values in a fixed compact set $K\subset\mathcal{L}^2([-3R_0,3R_0+1]\times D_R)$ for each $t>0$ when $0<b_0<R_0<\infty$ and $0<b<R<\infty$. In other cases, the proper truncations proposed in Section 5 reduce the problem to this setting. □

Appendix

Lemma A.1. *Under the assumption* (1.7), *there is a constant* $C>0$ *such that*

$$\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\le C\bigg(\int_0^1\int_{\mathbb{R}^d}\big|\nabla_x f(s,x)\big|^2\,\mathrm{d}x\bigg)^{\alpha/2}\tag{A.1}$$

for all $f\in\mathcal{A}_d$.

Proof. Recall the inequality

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)f^2(x)f^2(y)\,\mathrm{d}x\,\mathrm{d}y\le C\|f\|_2^{4-\alpha}\|\nabla f\|_2^{\alpha},\qquad f\in W^{1,2}\big(\mathbb{R}^d\big)$$

for which we cite (5.30), (5.18) and Lemma 5.7 in [3] for the first, second and third forms of $\gamma(\cdot)$, respectively.

For two possibly different functions $f,g\in W^{1,2}(\mathbb{R}^d)$, by the representation (2.6)

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)f^2(x)g^2(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=C(\gamma)\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d}K(z-x)f^2(x)\,\mathrm{d}x\bigg]\bigg[\int_{\mathbb{R}^d}K(z-y)g^2(y)\,\mathrm{d}y\bigg]\mathrm{d}z\\
&\quad\le C(\gamma)\bigg\{\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d}K(z-x)f^2(x)\,\mathrm{d}x\bigg]^2\mathrm{d}z\bigg\}^{1/2}\bigg\{\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d}K(z-y)g^2(y)\,\mathrm{d}y\bigg]^2\mathrm{d}z\bigg\}^{1/2}\\
&\quad=\bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)f^2(x)f^2(y)\,\mathrm{d}x\,\mathrm{d}y\bigg\}^{1/2}\bigg\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,\mathrm{d}x\,\mathrm{d}y\bigg\}^{1/2}\\
&\quad\le\|f\|_2^{(4-\alpha)/2}\|\nabla f\|_2^{\alpha/2}\|g\|_2^{(4-\alpha)/2}\|\nabla g\|_2^{\alpha/2}.
\end{aligned}$$

In particular,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\le C\bigg(\int_{\mathbb{R}^d}\big|\nabla_x f(s,x)\big|^2\,\mathrm{d}x\bigg)^{\alpha/4}\bigg(\int_{\mathbb{R}^d}\big|\nabla_x f(r,x)\big|^2\,\mathrm{d}x\bigg)^{\alpha/4}$$

holds for all $f\in\mathcal{A}_d$ with the same constant $C>0$.

Hence,

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-t|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\\
&\quad\le C\int_0^1\int_0^1|r-s|^{-\alpha_0}\bigg(\int_{\mathbb{R}^d}\big|\nabla_x f(s,x)\big|^2\,\mathrm{d}x\bigg)^{\alpha/4}\bigg(\int_{\mathbb{R}^d}\big|\nabla_x f(r,x)\big|^2\,\mathrm{d}x\bigg)^{\alpha/4}\mathrm{d}r\,\mathrm{d}s\\
&\quad\le C\bigg(\int_0^1\int_0^1|r-s|^{-4\alpha_0/(4-\alpha)}\,\mathrm{d}r\,\mathrm{d}s\bigg)^{(4-\alpha)/4}\bigg(\int_0^1\int_{\mathbb{R}^d}\big|\nabla_x f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^{\alpha/2},
\end{aligned}$$

where the last step follows from Hölder inequality. Thus, the conclusion follows from the fact that $4\alpha_0(4-\alpha)^{-1}<1$ under (1.7). □

Let $\kappa(\alpha_0,d,\gamma)$ be the best constant in (A.1).

Lemma A.2. *Under the assumption* (1.7),

$$M(\alpha_0,d,\gamma)=\frac{4-\alpha}{4}\bigg(\frac{\alpha}{2}\bigg)^{\alpha/(4-\alpha)}\kappa(\alpha_0,d,\gamma)^{2(4-\alpha)},\tag{A.2}$$

$$\mathcal{E}(\alpha_0,d,\gamma)=\frac{2-\alpha}{2}\alpha^{\alpha/(2-\alpha)}\kappa(\alpha_0,d,\gamma)^{2/(2-\alpha)},\tag{A.3}$$

$$\mathcal{E}(\alpha_0,d,\gamma)=\frac{2-\alpha}{2}2^{\alpha/(2-\alpha)}\bigg(\frac{4-\alpha}{4}\bigg)^{-(4-\alpha)/(2-\alpha)}M(\alpha_0,d,\gamma)^{(4-\alpha)/(2-\alpha)}.\tag{A.4}$$

Proof. Clearly, (A.4) is a consequence of (A.2) and (A.3). By definition

$$M(\alpha_0,d,\gamma)\le \sup_{g\in\mathcal{A}_d}\left\{\kappa(\alpha_0,d,\gamma)^{1/2}\left(\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x f(s,x)\right|^2\mathrm{d}x\right)^{\alpha/4}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x f(s,x)\right|^2\mathrm{d}x\right\}$$
$$\le \sup_{\lambda>0}\left\{\kappa(\alpha_0,d,\gamma)^{1/2}\lambda^{\alpha/2}-\frac{1}{2}\lambda^2\right\}$$
$$=\frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\kappa(\alpha_0,d,\gamma)^{2/(4-\alpha)}.$$

Similarly,

$$\mathcal{E}(\alpha_0,d,\gamma)\le \sup_{\lambda>0}\left\{\kappa(\alpha_0,d,\gamma)\lambda^{\alpha}-\frac{1}{2}\lambda^2\right\}=\frac{2-\alpha}{2}\alpha^{\alpha/(2-\alpha)}\kappa(\alpha_0,d,\gamma)^{2/(2-\alpha)}.$$

On the other hand, given $0<\varepsilon<\kappa(\alpha_0,d\gamma)$, let $f\in\mathcal{A}_d$ satisfy

$$\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-t|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s$$
$$>\big(\kappa(\alpha_0,d,\gamma)-\varepsilon\big)\left(\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x f(s,x)\right|^2\mathrm{d}x\right)^{\alpha/2}.$$

For any $\beta>0$, $g_\beta(s,x)=\beta^{d/2}f(s,\beta x)$ is in $\mathcal{A}_d$. Hence,

$$M(\alpha_0,d,\gamma)$$
$$\ge\left(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-t|^{\alpha_0}}g_\beta^2(s,x)g_\beta^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g_\beta(s,x)\right|^2\mathrm{d}x$$
$$=\beta^{\alpha/2}\left(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-t|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}-\frac{\beta^2}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x f(s,x)\right|^2\mathrm{d}x,$$

where the last step follows from integration substitution. Maximizing the right hand side by picking optimal β,

$$M(\alpha_0,d,\gamma)\ge\frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\left(\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x f(s,x)\right|^2\mathrm{d}x\right)^{-\alpha/(4-\alpha)}$$
$$\times\left(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-t|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\right)^{2/(4-\alpha)}$$
$$\ge\frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\big(\kappa(\alpha_0,d,\gamma)-\varepsilon\big)^{2/(4-\alpha)}.$$

Letting $\varepsilon\to 0^+$ leads to

$$M(\alpha_0,d,\gamma)\ge\frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\kappa(\alpha_0,d,\gamma)^{2/(4-\alpha)}.$$

In a similar way, we can prove that

$$\mathcal{E}(\alpha_0,d,\gamma)\ge\frac{2-\alpha}{2}\alpha^{\alpha/(2-\alpha)}\kappa(\alpha_0,d,\gamma)^{2/(2-\alpha)}.$$

□

Lemma A.3. *Let $\widetilde{K}_M(\cdot)$ be defined as in the proof of Theorem 1.3 in Section 5. We have*

$$\limsup_{M\to\infty}\sup_{g\in\mathcal{A}_d}\left\{\left(\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right]^2\mathrm{d}u\,\mathrm{d}y\right)^{1/2}\right.$$
$$\left.-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla g(s,x)\right|^2\mathrm{d}x\,\mathrm{d}s\right\}$$
$$\le \left(C_0(\alpha_0)C(\gamma)\right)^{-2/(4-\alpha)}M(\alpha_0,d,\gamma).$$

Proof. We shall prove the lemma in the settings of the first and second forms of $\gamma(\cdot)$, and it can be proved in the same spirit for the third form of $\gamma(\cdot)$.

Noticing that $K_{R,b}(\cdot)$ is supported on $[-2R,2R]^d$, for any $y\in[0,M]^d$

$$\int_{\mathbb{R}^d}\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x=\int_{\mathbb{R}^d}K_{R,b}(y-x)\widetilde{g}^2(s,x)\,\mathrm{d}x$$
$$=\int_{[-2R,M+2R]}K_{R,b}(y-x)\widetilde{g}^2(s,x)\,\mathrm{d}x,$$

where

$$\widetilde{g}(s,x)=\sqrt{\sum_{z\in\mathbb{Z}^d}g^2(s,Mz+x)}.$$

Clearly, $\widetilde{g}(s,x)$ is periodic in x in the sense that

$$\widetilde{g}(s,zM+x)=\widetilde{g}(s,x),\qquad z\in\mathbb{Z}^d,x\in\mathbb{R}^d.$$

In addition,

$$\int_{[0,M]^d}\widetilde{g}^2(s,x)\,\mathrm{d}x=1\quad\text{and}\quad\left|\nabla\widetilde{g}(s,x)\right|^2\le\sum_{z\in\mathbb{Z}^d}\left|\nabla g(x+Mz)\right|^2\quad\text{for } s\in[0,1].$$

Let $g\in\mathcal{A}_d$. By Lemma 3.4 in [7], for each $s\in[0,1]$, there is an $a(s)\in\mathbb{R}^d$ such that

$$\int_{[0,M]^d\setminus[\sqrt{M},M-\sqrt{M}]^d}\widetilde{g}^2\bigl(s,x+a(s)\bigr)\,\mathrm{d}x\le\frac{2d}{\sqrt{M}}.$$

Notice that there is a finite partition of $[-2R,M+2R]^d\setminus[0,M]^d$ such that each part under this partition can be shifted by zM ($z\in\mathbb{Z}^d$) unit to become a part of $[0,M]^d\setminus[R,M-R]^d\subset[0,M]^d\setminus[\sqrt{M},M-\sqrt{M}]^d$. Let $F\subset[-2R,M+2R]^d\setminus[0,M]^d$ be a member of the partition and $z\in\mathbb{Z}^d$ be such that $F+zM\subset[0,M]^d\setminus[\sqrt{M},M-\sqrt{M}]^d$. By periodicity

$$\int_F\widetilde{g}^2\bigl(s,x+a(s)\bigr)\,\mathrm{d}x=\int_{F+zM}\widetilde{g}^2\bigl(s,x+a(s)\bigr)\,\mathrm{d}x\le\frac{2d}{\sqrt{M}}.$$

Therefore, there is a constant $C>0$ independent of g such that

$$\int_{[-2R,M+2R]^d\setminus[0,M]^d}g^2\bigl(s,x+a(s)\bigr)\,\mathrm{d}x\le\frac{C(2d)}{\sqrt{M}}.$$

Write $E=[-2R,M+2R]^d\setminus[\sqrt{M},M-\sqrt{M}]^d$. Summarizing our computation

$$\int_E\widetilde{g}^2\bigl(s,x+a(s)\bigr)\,\mathrm{d}x\le\frac{C}{\sqrt{M}},$$

where C is a positive generic constant which depends on the dimension d only. We may assume $a \equiv 0$, that is

$$\int_E \widetilde{g}^2(s,x)\,\mathrm{d}x \le \frac{C}{\sqrt{M}}, \tag{A.5}$$

for otherwise we may replace $g(s,\cdot)$ with $g(s,a(s)+\cdot)$.

Define the function ϕ on $\mathbb{R}$ by

$$\phi(\lambda) = \begin{cases} \lambda M^{-1/2}, & 0 \le \lambda \le \sqrt{M}, \\ 1, & \sqrt{M} \le \lambda \le M - \sqrt{M}, \\ M^{1/2} - \lambda M^{-1/2}, & M - \sqrt{M} \le \lambda \le M, \\ 0, & \text{otherwise}, \end{cases}$$

and write

$$\varphi(x) = \phi(x_1)\cdots\phi(x_d), \quad x = (x_1,\dots,x_d) \in \mathbb{R}^d,$$

$$f(s,x) = \widetilde{g}(s,x)\varphi(x) \Big/ \sqrt{\int_{\mathbb{R}^d} \widetilde{g}^2(s,y)\varphi^2(y)\,\mathrm{d}y} = \widetilde{g}(s,x)\varphi(x)/\sqrt{A(s)}.$$

Then $|\varphi| \le 1$, $|\nabla\varphi| \le \sqrt{d/M}$ and $f \in \mathcal{A}_d$. Note that

$$\begin{aligned} 1 &= \int_{[0,M]^d} \widetilde{g}^2(s,y)\,\mathrm{d}y \ge A(s) \\ &= \int_{[0,M]^d} \widetilde{g}^2(s,y)\varphi^2(y)\,\mathrm{d}y \\ &\ge 1 - \int_E \widetilde{g}^2(s,y)\,\mathrm{d}y \ge 1 - \frac{C}{\sqrt{M}}. \end{aligned}$$

Let $A = 1 - \frac{C}{\sqrt{M}}$, and we have

$$A \le A(s) \le 1, \quad s \in [0,1].$$

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^d} \big|\nabla f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le \frac{1}{A}\bigg[\int_0^1 \int_{\mathbb{R}^d} \big|\nabla\widetilde{g}(s,x)\big|^2 \big|\varphi(x)\big|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_0^1 \int_{\mathbb{R}^d} \big|\widetilde{g}(s,x)\big|^2 \big|\nabla\varphi(x)\big|^2\,\mathrm{d}x\,\mathrm{d}s + 2\int_0^1 \int_{\mathbb{R}^d} \widetilde{g}(s,x)\varphi(x)\langle\nabla\widetilde{g},\nabla\varphi\rangle\,\mathrm{d}x\,\mathrm{d}s\bigg] \\ &\quad\le \frac{1}{A}\bigg[\int_0^1 \int_{[0,M]^d} \big|\nabla\widetilde{g}(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \frac{d}{M}\int_0^1 \int_{[0,M]^d} \big|\widetilde{g}(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s + 2\bigg(\int_0^1 \int_{[0,M]^d} \big|\nabla\widetilde{g}(s,x)\big|^2 \big|\nabla\varphi(x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2}\bigg] \\ &\quad\le \frac{1}{A}\bigg[\int_0^1 \int_{[0,M]^d} \big|\nabla\widetilde{g}(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{d}{M} + 2\sqrt[4]{\frac{d}{M}}\bigg(\sqrt{\frac{d}{M}}\int_0^1 \int_{[0,M]^d} \big|\nabla\widetilde{g}(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2}\bigg] \\ &\quad\le \frac{1}{A}\bigg\{\bigg(1+\sqrt{\frac{d}{M}}\bigg)\int_0^1 \int_{[0,M]^d} \big|\nabla\widetilde{g}(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{d}{M} + \sqrt{\frac{d}{M}}\bigg\}, \end{aligned} \tag{A.6}$$

where in the last step we used the fact $2xy \le x^2 + y^2$.

On the other hand, notice that $\widetilde{g}^2(s,x) = A(s) f^2(s,x)$ for all $x \in [-2R, M+2R]\backslash E$:

$$\begin{aligned}
&\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{[-2R,M+2R]^d}|u-s|^{-(1+\alpha_0)/2}K_{R,b}(y-x)\widetilde{g}^2(s,x)\,\mathrm{d}x\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}\\
&\quad\le\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{[-2R,M+2R]^d\backslash E}|u-s|^{-(1+\alpha_0)/2}K_{R,b}(y-x)\widetilde{g}^2(s,x)\,\mathrm{d}x\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}\\
&\qquad+\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{E}|u-s|^{-(1+\alpha_0)/2}K_{R,b}(y-x)\widetilde{g}^2(s,x)\,\mathrm{d}x\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}. \qquad (A.7)
\end{aligned}$$

The first term on the right hand side is no larger than, noticing that $A(s) \le 1$,

$$\begin{aligned}
&\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}K_{R,b}(y-x)f(s,x)\,\mathrm{d}x\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}\\
&\quad\le\bigg(\big(C_0(\alpha_0)C(\gamma)\big)^{-1/2}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\bigg)^{1/2}.
\end{aligned}$$

For the second term, assuming that the function $K_{R,b}(\cdot)$ is uniformly bounded from above by N_0 where N_0 depends on (R,b,d).

$$\begin{aligned}
&\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_E|u-s|^{-(1+\alpha_0)/2}K_{R,b}(y-x)\widetilde{g}^2(s,x)\,\mathrm{d}x\bigg]^2\mathrm{d}u\,\mathrm{d}y\\
&\quad=C(\alpha_0)^{-1}\int_{[0,M]^d}\int_0^1\int_0^1|r-s|^{-\alpha_0}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}y\int_E K_{R,b}(y-x_1)\widetilde{g}^2(s,x_1)\,\mathrm{d}x_1\int_E K_{R,b}(y-x_2)\widetilde{g}^2(r,x_2)\,\mathrm{d}x_2\\
&\quad\le C(\alpha_0)^{-1}\int_{[0,M]^d}\int_0^1\int_0^1|r-s|^{-\alpha_0}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}y\int_E K_{R,b}(y-x_1)\widetilde{g}^2(s,x_1)\,\mathrm{d}x_1\int_{[0,M]^d}N_0\widetilde{g}^2(r,x_2)\,\mathrm{d}x_2\\
&\quad\le N_0C(\alpha_0)^{-1}\int_{\mathbb{R}^d}K_{R,b}(y)\,\mathrm{d}y\int_0^1\int_0^1|r-s|^{-\alpha_0}\,\mathrm{d}r\,\mathrm{d}s\int_E\widetilde{g}^2(s,x_1)\,\mathrm{d}x_1\\
&\quad\le N_0C(\alpha_0)^{-1}\int_{\mathbb{R}^d}K_{R,b}(y)\,\mathrm{d}y(1-\alpha_0)^{-1}\int_0^1\big[s^{1-\alpha_0}+(1-s)^{1-\alpha_0}\big]\int_E\widetilde{g}^2(s,x_1)\,\mathrm{d}x_1\,\mathrm{d}s\\
&\quad=\frac{C}{\sqrt{M}}, \qquad (A.8)
\end{aligned}$$

where C depends on (R,b,d,α_0) only and the last second step follows (A.5).

Combing (A.6), (A.7) and (A.8), we obtain

$$\begin{aligned}
&\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla g(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\le\bigg(\big(C_0(\alpha_0)C(\gamma)\big)^{-1/2}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\bigg)^{1/2}+CM^{-1/4}\\
&\qquad-\frac{1}{2}\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}A\int_0^1\int_{\mathbb{R}^d}\big|\nabla f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{1}{2}\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}\bigg(\frac{d}{M}+\sqrt{\frac{d}{M}}\bigg)\\
&\quad\le\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}A\bigg\{\bigg(1+\sqrt{\frac{d}{M}}\bigg)
\end{aligned}$$

$$\times \frac{1}{A}\big(C_0(\alpha_0)C(\gamma)\big)^{-1/2}\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}+CM^{-1/4}+\frac{1}{2}\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}\bigg(\frac{d}{M}+\sqrt{\frac{d}{M}}\bigg).$$

Therefore,

$$\sup_{g\in\mathcal{A}_d}\bigg\{\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla g(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}$$
$$\le\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}A\sup_{f\in\mathcal{A}_d}\bigg\{\bigg(1+\sqrt{\frac{d}{M}}\bigg)\frac{1}{A}\big(C_0(\alpha_0)C(\gamma)\big)^{-1/2}$$
$$\times\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}+CM^{-1/4}+\frac{1}{2}\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}\bigg(\frac{d}{M}+\sqrt{\frac{d}{M}}\bigg)$$
$$=\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}A\bigg[\bigg(1+\sqrt{\frac{d}{M}}\bigg)\frac{1}{A}\bigg]^{4/(4-\alpha)}$$
$$\times\big(C_0(\alpha_0)C(\gamma)\big)^{-2/(4-\alpha)}\sup_{f\in\mathcal{A}_d}\bigg\{\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}+CM^{-1/4}+\frac{1}{2}\bigg(1+\sqrt{\frac{d}{M}}\bigg)^{-1}\bigg(\frac{d}{M}+\sqrt{\frac{d}{M}}\bigg),$$

where the last step follows Lemma 4.1.

Finally, noticing that $A=1-\frac{C}{\sqrt{M}}$, we have

$$\limsup_{M\to\infty}\sup_{g\in\mathcal{A}_d}\bigg\{\bigg(\int_{[0,M]^d}\int_{\mathbb{R}}\bigg[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}\widetilde{K}_M(y-x)g^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\bigg]^2\mathrm{d}u\,\mathrm{d}y\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla g(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}$$
$$\le\big(C_0(\alpha_0)C(\gamma)\big)^{-2/(4-\alpha)}\sup_{f\in\mathcal{A}_d}\bigg\{\bigg(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}f^2(s,x)f^2(r,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\bigg)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla f(s,x)\big|^2\,\mathrm{d}x\,\mathrm{d}s\bigg\}.$$

Then the proof concludes. □

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