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Parabolic Anderson model with rough or critical Gaussian noise

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Abstract. This paper considers the parabolic Anderson equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^{d+1} W^{\mathbf{H}}}{\partial t\partial x_1\cdots\partial x_d}$$

generated by a $(d+1)$-dimensional fractional noise with the Hurst parameter $\mathbf{H} = (H_0, H_1, \dots, H_d)$. The existence/uniqueness, Feynman–Kac's moment formula and the precise intermittency exponents are formulated in the case when some of $H_1, \dots, H_d$ are less than one half, and in the case when the Dalang's condition

$$d - \sum_{k=1}^{n} H_j < 1 \quad \text{is replaced by } d - \sum_{k=1}^{n} H_j = 1.$$

Some partial result is also achieved for the case when $H_0 < 1/2$ which brings insight on what to expect as the Gaussian noise is rough in time.

Résumé. Cet article s'intéresse à l'équation d'Anderson parabolique

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^{d+1} W^{\mathbf{H}}}{\partial t\partial x_1\cdots\partial x_d}$$

engendrée par un bruit fractionnaire de dimension $(d+1)$ et de paramètre de Hurst $\mathbf{H} = (H_0, H_1, \dots, H_d)$. L'existence et l'unicité, la formule des moments de Feynman–Kac et les exposants précis d'intermittence sont formulés dans le cas où l'un des paramètres $H_1, \dots, H_d$ est inférieur à un demi, et dans le cas où la condition de Dalang

$$d - \sum_{k=1}^{n} H_j < 1 \quad \text{est remplacée par } d - \sum_{k=1}^{n} H_j = 1.$$

Des résultats partiels sont aussi obtenus dans la cas $H_0 < 1/2$, ce qui donne une intuition de ce qui doit être attendu dans le cas où le bruit Gaussien est rugueux en temps.

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1. Introduction

In this paper we consider the parabolic Anderson equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \theta \dot{W}^{\mathbf{H}}(t,x) \diamond u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

with the fractional Gaussian noise

$$\dot{W}^{\mathbf{H}}(t,x) = \frac{\partial^{d+1} W^{\mathbf{H}}}{\partial t \partial x_1 \cdots \partial x_d}(t, x_1, \dots, x_d), \quad \text{where } x = (x_1, \dots, x_d) \tag{1.2}$$

given as the formal derivative of a fractional Brownian sheet $W^{\mathbf{H}}(t,x)$ ($(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$) with the Hurst index $\mathbf{H} = (H_0, H_1, \dots, H_d)$ ($0 < H_0, \dots, H_d < 1$), which is defined as a mean zero Gaussian field with the covariance function

$$\mathbb{E}\big\{W^{\mathbf{H}}(s,x) W^{\mathbf{H}}(t,y)\big\} = R_{H_0}(s,t) \prod_{j=1}^d R_{H_j}(x_j, y_j),$$

where

$$R_{H_j}(u,v) = \frac{1}{2}\big\{|u|^{2H_j} + |v|^{2H_j} - |u-v|^{2H_j}\big\}, \quad u, v \in \mathbb{R}, j = 0, 1, \dots, d.$$

In (1.1), $\theta > 0$ is a given constant and the notation "$\diamond$" represents the Wick product.

Throughout the paper, the equation is interpreted in the sense given in the following definition.

Definition 1.1. An adapted random field $\{u(t,x); (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\}$ is a solution to the equation (1.1), if for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $u(t,x) \in \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, the process

$$\big\{p_{t-s}(x-y) u(s,y) \mathbf{1}_{[0,t]}(s); (s,y) \in \mathbb{R}^+ \times \mathbb{R}^d\big\}$$

is Skorokhod integrable with respect to the Gaussian differential $W^{\mathbf{H}}(\delta s, \delta y)$, and $u(t,x)$ satisfies

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) W^{\mathbf{H}}(\delta s, \delta y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.3}$$

where $p_s(y)$ ($(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$) is the Brownian semi-group and the differential notation "$W(\delta s, \delta y)$" is used for the Skorokhod integral.

$u(t,x)$ is said to be a local solution to the equation (1.1) if there is a $t_0 > 0$ such that $u(t,x)$ is defined and satisfies all requests given above for $(t,x) \in [0, t_0) \times \mathbb{R}^d$.

The definition of the Wick product, the Skorokhod integration and the notion of some other material on Malliavin calculus needed in this paper are briefly recalled in the next section.

The parabolic Anderson equation with Gaussian noise has been studied extensively in literature. For the purpose of comparison, we recall the most notable formulation known as Dalang's condition. Given a mean-zero Gaussian noise $\dot{W}(t,x)$ with the covariance function of the form

$$\mathrm{Cov}\big(\dot{W}(t,x), \dot{W}(s,y)\big) = \gamma_0(t-s)\gamma(x-y), \quad (t,x), (s,y) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{1.4}$$

By the Bochner's theorem, the non-negative definite functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ yield the spectral representation

$$\gamma_0(u) = \int_{\mathbb{R}} e^{i\lambda u} \mu_0(d\lambda) \quad \text{and} \quad \gamma(x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mu(d\xi) \tag{1.5}$$

with the tempered measures μ_0 and μ (known as the spectral measures) on $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

In the case when the noise $\dot{W}(t,x)$ is white in time (i.e., $\gamma_0(\cdot)=\delta_0(\cdot)$) and $\gamma(\cdot)\ge 0$, Dalang [8] points out that the condition (known as Dalang's condition)

$$\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\mu(d\xi)<\infty \tag{1.6}$$

is sufficient and necessary for the solvability of the parabolic Anderson equation with the Gaussian noise $\dot{W}(t,x)$. In a general setting of Gaussian noise, Hu, Huang, Nualart and Tindel [14] shows that the Dalang's condition is also sufficient for the solvability whenever $\gamma_0(\cdot)$ is locally integrable and $\gamma(\cdot)\ge 0$.

Return to the setting of the fractional noise $W^{\mathbf{H}}$ given in (1.2). In connection to (1.5),

$$\mu_0(d\lambda)=\widehat{C}_0|\lambda|^{1-2H_0}\,d\lambda \quad\text{and}\quad \mu(d\xi)=\prod_{j=1}^d\widehat{C}_j|\xi_j|^{1-2H_j}\,d\xi \quad \bigl(\xi=(\xi_1,\dots,\xi_d)\bigr), \tag{1.7}$$

where

$$\widehat{C}_j=\frac{\Gamma(2H_j+1)\sin(\pi H_j)}{2\pi}>0,\quad j=0,1,\dots,d.$$

It is easy to see from (1.7) that the Dalang's condition is equivalent to

$$d-\sum_{j=1}^d H_j<1 \tag{1.8}$$

in the setting of fractional Gaussian noise $\dot{W}=\dot{W}^{\mathbf{H}}$.

Notice that

$$\gamma(x)=\prod_{j=1}^d\widehat{C}_j\int_{-\infty}^{\infty}e^{i\xi x_j}|\xi|^{1-2H_j}\,d\xi=\prod_{j=1}^d\gamma_j(x_j)\quad\text{(say)}. \tag{1.9}$$

For a $0\le j\le d$ with $H_j\ge 1/2$, the correspondent covariance function has the explicit representation

$$\gamma_j(x)=\widehat{C}_j\int_{-\infty}^{\infty}e^{i\xi x_j}|\xi|^{1-2H_j}\,d\xi=\begin{cases}C_{H_j}|x|^{-(2-2H_j)}, & H_j>1/2,\\ \delta_0(x), & H_j=1/2\end{cases} \tag{1.10}$$

for $x\in\mathbb{R}$, where $C_{H_j}=H_j(2H_j-1)$.

Consequently, $\gamma_0(\cdot)$ and $\gamma(\cdot)$ (as the tensor product of $\gamma_1(\cdot),\dots,\gamma_d(\cdot)$) are non-negative and local integrable as $H_0,\dots,H_d\ge 1/2$. By Theorem 3.6 in [14], therefore, the Parabolic Anderson equation (1.1) has an unique solution in the sense of Definition 1.1 under the condition (1.8).

In this work, we are particularly interested in the case when $H_j<1/2$ for some $0\le j\le d$. The fractional Gaussian noise $W^{\mathbf{H}}$ given in (1.2) is said to be rough if $H_j<1/2$ for some $0\le j\le d$, to be rough in time if $H_0<1/2$, to be rough in space if $H_j<1/2$ for some $1\le j\le d$, and to be rough in the jth ($1\le j\le d$) component if $H_j<1/2$. When $W^{\mathbf{H}}$ is rough, either $\gamma_0(\cdot)$ or $\gamma(\cdot)$ formally given in (1.5) is not point-wisely defined, nor is it non-negative in any reasonable sense, despite of being non-negative definite as co-variance function, for its Fourier transform given in (1.7) is no longer non-negative definite.

Despite of extensive literature in Parabolic Anderson models – particularly in the most interesting setting of the fractional noise, very little has been known about how the roughness impacts the system (1.1). Investigation started in recent years. The reader is referred to the references [1,5,9,13,15] and [17] where the model is $(1+1)$-dimensional with $\mathbf{H}=(H_0,H)$ satisfying $H_0\ge 1/2$ and $H<1/2$. In the work [1] and [13], for example, it is shown that when $H_0=1/2$ and $H<1/2$, the system is solvable under the extra condition $H>1/4$ which is very likely to be necessary. This example shows that in the presence of roughness, the Dalang's condition is no longer sufficient as it is automatic in $d=1$.

In a more recent work [17], Huang, Lê and Nualart consider the system rough in space but color in time. Specifically, they solve the parabolic Anderson equation with a time-fractional ($H_0 > 1/2$) and space-rough ($H < 1/2$) fractional noise under the same condition "$H > 1/4$". With respect to [17], we shall see (Theorem 1.2) that the condition "$H > 1/4$" is not sharp as $H_0 > 1/2$ and should be replaced by $H_0 + H > 3/4$.

We are concerned with the system (1.1) in the case when the fractional noise is rough in time or space. More specifically, we are interested in the following problems naturally raised from the study of parabolic Anderson models.

1. Given that Dalang's condition (1.8) (or (1.6) in more general setting) does not contain the information on the time component of the Gaussian noise, a natural question is whether or not the singularity of the time component (measured by H_0 in our model) of the fractional noise impacts the solvability in the presence of space roughness. If so, how?
2. The model of fractional noise that is possibly multi-dimensional and rough in space. A key problem in this regard is how to separate the rough components from the non-rough components as they alternate the system solvability in different ways.
3. Perhaps, the most interesting and challenging case related to the rough noise is when the noise is rough in time (i.e., $H_0 < 1/2$). To the best of our knowledge, the system (1.1) has never been solved for any Hurst parameter $\mathbf{H} = (H_0, H_1, \dots, H_d)$ as far as $H_0 < 1/2$. In other words, it was not clear if the noise is even allowed to be rough in time. Unfortunately, we are not able to solve this problem completely in this paper. See Proposition 1.4 below for a partial result.
4. What can we say when the noise is critical in space in the sense that the Dalang's condition (1.8) is replaced by its critical version

$$d - \sum_{j=1}^{d} H_j = 1? \tag{1.11}$$

   In this case, we say that the fractional noise $\dot{W}^{\mathbf{H}}$ is critical (to the Dalang's condition).

   The motive behind "4" comes partially from the investigation of the parabolic Anderson equation with space-time white noise (i.e., the case when $H_0 = \cdots = H_d = 1/2$) which is of special interest due to its close connection to the KPZ equation [12]. The $(1+1)$-dimension is the only setting that is solvable in the sense of Definition 1.1. The case of $(1+2)$-dimension is critical in the sense of (1.11). A critical setting investigated by Hairer and Labbé [11] is the case of time independent white noise $\dot{W}(x)$ ($x \in \mathbb{R}^2$). Given that the time-independence reduces singularity of the Gaussian noise and therefore increases the solvability of the system, our concern is on the degree of the singularity in time that the equation (1.1) can tolerate in the setting of criticality. Theorem 1.2 below applies to the models with $(1+2)$-dimensional Gaussian noises fractional in time and white in space (i.e., $\mathbf{H} = (H_0, 1/2, 1/2)$ with $H_0 > 1/2$).
5. The precise moment asymptotics such as intermittency and high moment asymtotics (see Theorem 1.5 below). It is worth of mentioning that the roughness in Gaussian noise posts some substantial challenges, as we shall see in the later development.

For the sake of simplicity, we assume throughout the paper that the initial value $u_0(x)$ in (1.1) satisfies the condition

$$0 < \inf_{x\in\mathbb{R}^d} u_0(x) \le \sup_{x\in\mathbb{R}^d} u_0(x) < \infty. \tag{1.12}$$

Set $J_* = \{1 \le j \le d;\, H_j < 1/2\}$, $J^* = \{1 \le j \le d;\, H_j \ge 1/2\}$, $d_* = \#\{J_*\}$, $d^* = \#\{J^*\}$,

$$H_* = \sum_{j\in J_*} H_j, \qquad H^* = \sum_{j\in J^*} H_j, \qquad H = H_* + H^* = \sum_{j=1}^{d} H_j.$$

Theorem 1.2. *Let $H_0 > 1/2$ and write $\alpha_0 = 2 - 2H_0$. Under the assumption*

$$\begin{cases} d - H < 1, \\ 4(1 - H_0) + 2(d - H) + (d_* - 2H_*) < 4 \end{cases} \tag{1.13}$$

the parabolic Anderson equation (1.1) *admits a unique mild global solution* $u(t,x)$ *in the sense of Definition* 1.1. *Further,* $u(t,x)$ *satisfies the Feynman–Kac moment representation*

$$\mathbb{E}u^m(t,x)=\mathbb{E}_x\left[\exp\left\{C_{H_0}\theta^2\sum_{1\le j<k\le m}\int_0^t\int_0^t\frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\,dr\,ds\right\}\prod_{j=1}^m u_0\big(B_j(t)\big)\right] \tag{1.14}$$

for $m=2,3,\dots,$ *where* $C_{H_0}=H_0(2H_0-1)$, $B_1(t),\dots,B_m(t)$ *are independent* d*-dimensional Brownian motions and the notation "*$\mathbb{E}_x$*"* ($x\in\mathbb{R}^d$) *is for the Brownian expectation with* $B_1(0)=\cdots=B_m(0)=x$.

When the assumption (1.13) *is replaced by*

$$\begin{cases} d-H=1, \\ 4(1-H_0)+(d_*-2H_*)<2 \end{cases} \tag{1.15}$$

the parabolic Anderson equation (1.1) *admits a unique mild local solution* $u(t,x)$ *in the sense of Definition* 1.1. *Further,* $u(t,x)$ *satisfies the Feynman–Kac moment representation* (1.14) *for* $m=2$ *and* $0<t<t_0$ *where* $t_0>0$ *is the critical exponent given in Definition* 1.1.

Since $\gamma(\cdot)$ is not defined pointwisely in the presence of roughness, the time integral appearing in the moment formula (1.14) and the time integrals appearing in (1.17), Theorem 1.3 below may not be directly defined. They are defined by a procedure of approximation. See (2.10) and (2.11) below.

When the fractional noise is not rough in space, the assumption (1.13) becomes the Dalang's condition and is H_0-independent. This explains why Dalang's condition accurate even when the noise is colored in time ($H_0>1/2$) as far as $\gamma(\cdot)\ge 0$. When $\dot W^{\mathbf{H}}$ is rough in space, on the other hand, the assumption (1.13) becomes H_0-dependent. For example, the condition for solvability in the $(1+1)$-dimension $\mathbf{H}=(H_0,H)$ with $H_0>1/2$ and $H<1/2$ is now given as $H_0+H>3/4$ instead of $H>1/4$ [17].

Theorem 1.2 claims a local solution for (1.1) as $\mathbf{H}=(H_0,1/2,1/2)$ for any $H_0>1/2$. Can the local solution be extended into a global solution in a space larger than $\mathcal{L}^2(\Omega)$ (such as $\mathcal{L}^1(\Omega)$, for example)? We leave this question to the future study.

Theorem 1.3. *Let* $H_0=1/2$. *Under the assumption*

$$2(d-H)+(d_*-2H_*)<2 \tag{1.16}$$

the parabolic Anderson equation (1.1) *admits a unique global mild solution* $u(t,x)$ *in the sense of Definition* 1.1. *Further, we have the Feynman–Kac moment representation*

$$\mathbb{E}u^m(t,x)=\mathbb{E}_x\left[\exp\left\{\theta^2\sum_{1\le j<k\le m}\int_0^t\gamma\big(B_j(s)-B_k(s)\big)\,ds\right\}\prod_{j=1}^m u_0\big(B_j(t)\big)\right] \tag{1.17}$$

for $m=2,3,\dots.$

Notice that the condition (1.16) does not allow $d-H=1$. In particular, the setting of $(1+2)$ dimensional time-space white noise (i.e., $\mathbf{H}=(1/2,1/2,1/2)$) is excluded by Theorem 1.3. In view of the moment representation (1.17), it seems hopeless to expect the solvability in terms of Definition 1.1, as the Brownian local time

$$\int_0^t\delta_0\big(B(s)-\widetilde B(s)\big)\,ds$$

can not be defined properly for 2-dimensional Brownian motions.

Perhaps, the biggest challenge comes from the setting when the fractional noise $\dot W^{\mathbf{H}}$ is rough in time, i.e., the case when $H_0<1/2$ with the time covariance function $\gamma_0(\cdot)$ being formally defined in (1.5). Unlike the setting $H_0>1/2$ where $\gamma_0(\cdot)$ is a constant multiple of the well-defined and non-negative function $|\cdot|^{-(2-2H_0)}$, the time covariance function $\gamma_0(\cdot)$ can not be defined point-wise as $H_0 < 1/2$. In the recent work [2], Chen, Hu, Kalbasi and Nualart investigate the parabolic Anderson equation with the time-derivative Gaussian noise in the form of

$$\frac{\partial}{\partial t} W^{\mathbf{H}}(t,x),$$

which allows $H_0 < 1/2$. It should be pointed that taking no differential to the space variable eliminates the spatial singularity and makes the situation easier than the setting posted in this paper. Another relevant literature existence is the work by Deya (2016) (Theorem 1.2, [9]) where the equation is $(1+1)$-dimensional with a fractional Gaussian noise that is allowed to be rough in time or in space. Nevertheless, the solution in [9] takes a meaning different from Definition 1.1. Such difference often leads to different conclusions when it comes to the issue of solvability. In the setting $H_0 > 1/2$, for instance, the condition for solvability given in Deya (Theorem 1.2, (i), [9]) is "$2H_0 + H > 2$", versus to "$H_0 + H > 3/4$" obtained in Theorem 1.2 with $d = 1$. To the author's best knowledge, nothing substantial has been discovered by far when the noise is rough in time and when the system (1.1) is interpreted in Definition 1.1. Unfortunately, solving the equation (1.1) for $H_0 < 1/2$ is out of our capability at this time. Instead, we establish a partial result which casts insight on what to expect when $H_0 < 1/2$.

To see the relevance of the next result, let us mention the fact that the solvability of the system (1.1) rests on the exponential integrability of the time integral

$$\int_0^t \int_0^t \gamma_0(s-r)\gamma\big(B(s) - \tilde{B}(r)\big)\,dr\,ds \tag{1.18}$$

defined by approximation (see (2.10) and (2.11) below), where $B(t)$ and $\tilde{B}(t)$ are two independent d-dimensional Brownian motions. By Taylor expansion, a sharp bound for the integral moment of the time integral in (1.18) is needed. To the author's best knowledge, the only result existing in literature is obtained by Hu and Nualart [16] where it is proved that the the time integral in (1.18) has all finite positive moments when $\mathbf{H} = (H_0, 1/2)$ with $3/8 < H_0 < 1/2$. The following proposition fortifies Hu–Nualart's result and provides for the first time a quantified bound to the moments of the time integral in (1.18).

Proposition 1.4. *Let $1/4 < H_0 < 1/2$. Under the assumption*

$$4(1-2H_0) + 2(d-H) + (d_* - 2H_*) < 2 \tag{1.19}$$

we have

$$\mathbb{E}_0\bigg[\int_0^t \int_0^t \gamma_0(s-r)\gamma\big(B(s) - \tilde{B}(r)\big)\,dr\,ds\bigg]^n \le (n!)^{(d-H)+(2-2H_0)} C^n t^{n(H+2H_0-d)} \tag{1.20}$$

for any $t > 0$ and $n = 1, 2, \dots$, where $C > 0$ is a constant independent of n and t.

Unfortunately, the moment bound in (1.20) is not strong enough for the exponential integrability required by the solvability of (1.1), since $(d-H) + (2-2H_0) > 1$ under our condition. On the other hand, it does provide some insight on the condition for solving (1.1) when $H_0 < 1/2$. Further, we shall provide some evidence (Remark 5.2 below) indicating that the bound (1.20) can be improved. Finally, we conjecture that under (1.19), the correct bound should be

$$\mathbb{E}_0\bigg[\int_0^t \int_0^t \gamma_0(s-r)\gamma\big(B(s) - \tilde{B}(r)\big)\,dr\,ds\bigg]^n \le (n!)^{(d-H)+(1-2H_0)} C^n t^{n(H+2H_0-d)} \tag{1.21}$$

instead of (1.20). By the fact that $(d-H) + (1-2H_0) < 1$ under the assumption (1.19), the bound (1.21) is sufficient for the exponential integrability that is required by the solvability of the system (1.1). We leave this problem to the future investigation.

We now move to the intermittency and moment asymptotics for the solution of (1.1). By Lemma 5.1 and Lemma 5.2, [4], the variations

$$\mathcal{E}=\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,dx\,dy-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla g(x)\right|^2dx\right\} \tag{1.22}$$

and

$$\begin{aligned}\mathcal{E}(H_0)=\sup_{g\in\mathcal{A}_d}\bigg\{&\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|s-r|^{2-2H_0}}g^2(s,x)g^2(r,y)\,dx\,dy\,dr\,ds\\&-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2x\,ds\bigg\}\quad(H_0>1/2)\end{aligned} \tag{1.23}$$

are finite under the Dalang's condition "$d-H<1$", where $\mathcal{F}_d$ and $\mathcal{A}_d$ are the function classes defined as

$$\mathcal{F}_d=\left\{g\in W^{1,2}(\mathbb{R}^d);\ \int_{\mathbb{R}^d}\left|g(x)\right|^2dx=1\right\} \tag{1.24}$$

and

$$\mathcal{A}_d=\left\{g;\ g(s,\cdot)\in\mathcal{F}_d\ \forall 0\le s\le 1\right\}. \tag{1.25}$$

Theorem 1.5. *Set $\alpha=2d-2H$ and $\alpha_0=2-2H_0$. In the assumption of Theorem 1.2*

$$\lim_{t\to\infty}t^{-\frac{4-\alpha-2\alpha_0}{2-\alpha}}\log\mathbb{E}u^p(t,x)=p\left(\frac{C_{H_0}\theta^2(p-1)}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0),\quad p\ge 2, \tag{1.26}$$

$$\lim_{m\to\infty}m^{-\frac{4-\alpha}{2-\alpha}}\log\mathbb{E}u^m(t,x)=\left(\frac{C_{H_0}\theta^2}{2}\right)^{\frac{2}{2-\alpha}}t^{\frac{4-\alpha-2\alpha_0}{2-\alpha}}\mathcal{E}(H_0)\quad\forall t>0 \tag{1.27}$$

for every $x\in\mathbb{R}^d$ and $t>0$, where $\mathcal{E}(H_0)$ is given in (1.23) and $C_{H_0}=H_0(2H_0-1)$.

In the assumption of Theorem 1.3,

$$\lim_{m\to\infty}m^{-\frac{4-\alpha}{2-\alpha}}\log\mathbb{E}u^m(t,x)=\left(\frac{\theta^2}{2}\right)^{\frac{2}{2-\alpha}}t\mathcal{E}\quad\forall t>0 \tag{1.28}$$

for every $x\in\mathbb{R}^d$ and $t>0$, where $\mathcal{E}$ is given in (1.22).

In [4], (1.26) and (1.27) are established when the fractional noise is not rough in space. The roughness posts a substantial challenge which will be addressed later. The intermittency with white-time noise are not fully understood even in the non-rough setting, we refer [4] for some related discussion.

Finally, we outline the rest of the paper and highlight some of the key points appearing in our approach. Some mathematical background such as Malliavin calculus and Feynman–Kac formula is briefly reviewed in Section 2. In particular, solving the equation (1.1) is reduced to the problem of the exponential integrability for the Brownian Hamiltonian in (1.18). Theorem 1.2, Theorem 1.3 and Proposition 1.4 are proved in Section 3, Section 4 and Section 5, respectively. The central problem in the proof is the moment estimate for the Brownian Hamiltonian in (1.18). New ideas developed here include the technique of the variable separation which allows us separate the time component from space component, and the rough space components from the non-rough space components. Another tool substantial to our argument is time-exponentiation which requires some innovative treatment (Lemma 5.1) in the presence of roughness. Theorem 1.5 is proved in Section 6. The moment asymptotics in (1.26) and (1.27) is obtained in [4] in the non-rough setting. The treatment of compactification by folding, which is essential to the proof in [4], is no longer working in the presence of roughness. The argument we adopt here is partially inspired by the recent paper of Huang, Lê and Nualart [17] where the compactification is installed by a comparison between Brownian and Ornstein–Uhlenbeck Hamiltonians, an idea goes back at least to Donsker–Varahdan [10]. Finally, the hyper-contractivity inequality by Lê [18] allows us to obtain (1.26) for the possibly non-integer moments.

2. Malliavin calculus and Feynman–Kac representation

In this section we provide a mathematical construction for the parabolic Anderson equation (1.1) in the Itô–Skorokhod sense by briefly recalling some basics in Malliavin calculus that is related to the development of this work. The material in this section is essentially known (see, e.g., [14]) and is collected here for the reader's convenience. For possible future reference, the Gaussian noise $\dot{W}^{\mathbf{H}}(t,x)$ appearing in (1.1) is replaced by a more general time-space Gaussian noise $\dot{W}(t,x)$ with the time and space covariance functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ given as in (1.4).

In connection to (1.4), $\dot{W}$ is viewed as a mean zero Gaussian field $W(\phi)$ ($\phi\in\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d)$) with the covariance function

$$\operatorname{Cov}\bigl(W(\varphi),W(\psi)\bigr)=\int_{(\mathbb{R}^+\times\mathbb{R}^d)^2}\gamma_0(s-t)\gamma(x-y)\varphi(s,x)\psi(t,y)\,dx\,dy\,ds\,dt,$$

where $\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d)$ is the Schwartz space of the infinitely differentiable and rapidly decreasing (at ∞) functions om $\mathbb{R}^+\times\mathbb{R}^d$.

Let $\mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P})$ be the space of square integrable random variables spanned by the Gaussian field $W(\phi)$ ($\phi\in\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d)$). The linear isometry $W(\cdot)$ between the inner product space $\{W(\phi);\varphi\in\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d)\}$ with the inner product defined by the covariance function in (2.1) and Schwartz space $\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d)$ endowed with the inner product

$$\langle\phi,\psi\rangle_{\mathcal{H}}=\int_{(\mathbb{R}^+\times\mathbb{R}^d)^2}\gamma_0(s-t)\gamma(x-y)\varphi(s,x)\psi(t,y)\,dx\,dy\,ds\,dt \tag{2.1}$$

is extended to a linear isometry between $\mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P})$ and the Hilbert space $\mathcal{H}$, the closure of $\mathcal{S}^d(\mathbb{R}^+\times\mathbb{R}^d)$ under the above inner product, and is denoted as

$$W(\varphi)=\int_{\mathbb{R}^+\times\mathbb{R}^d}\varphi(t,x)W(dt\,dx),\quad \varphi\in\mathcal{H}. \tag{2.2}$$

The Malliavin derivative $D(F)$ ($F\in\mathcal{L}^2(\Omega)$) is defined in the way of approximation. First we consider the random variable F of the form $F=f(W(\phi_1),\dots,W(\phi_n))$ ($\phi_1,\dots,\phi_n\in\mathcal{H}$) with sufficiently smooth function f on $\mathbb{R}^n$ and define

$$D(F)=\sum_{j=1}^n\frac{\partial f}{\partial x_j}\bigl(W(\phi_1),\dots,W(\phi_n)\bigr)\phi_j.$$

The operator D is then extended on $\mathbb{D}^{1,2}$, the Sobolev space as the closure of the space of the smooth and cylindrical random variables under the norm

$$\bigl\|D(F)\bigr\|_{1,2}=\sqrt{\mathbb{E}F^2+\mathbb{E}\bigl\|D(F)\bigr\|_{\mathcal{H}}^2}.$$

In this way, the derivative D becomes a linear operator from $\mathbb{D}^{1,2}\subset\mathcal{L}^2(\Omega)$ to the Hilbert space $\mathcal{L}^2(\Omega,\mathcal{H})$, the space of the $\mathcal{H}$-valued random variables η with $\mathbb{E}\|\eta\|_{\mathcal{H}}^2<\infty$.

The Skorokhod integral is defined as the dual operator of the differential operator in the spirit of "integration by parts": We denote by δ: $\mathcal{L}^2(\Omega,\mathcal{H})\longrightarrow\mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P})$ the adjoint operator of the derivative D: For any $h\in\mathcal{L}^2(\Omega,\mathcal{H})$, $\delta(h)$ is uniquely determined by

$$\mathbb{E}\bigl[\delta(h)F\bigr]=\mathbb{E}\langle DF,h\rangle_{\mathcal{H}},\quad F\in\mathbb{D}^{1,2}. \tag{2.3}$$

The operator δ is also called Skorokhod integral in the notation

$$\delta(h)=\int_{\mathbb{R}^+\times\mathbb{R}^d} h(t,x)W(\delta t,\delta x). \tag{2.4}$$

Given a random field $h(t,x)$ in the probability space $(\Omega,\mathcal{A},\mathbb{P})$ generated by the Gaussian noise $\dot W$, by definition h is Skorohod integrable means that $h\in\mathcal{L}^2(\Omega,\mathcal{H})$, or

$$\mathbb{E}\|h\|_{\mathcal{H}}^2=\mathbb{E}\int_{(\mathbb{R}^+\times\mathbb{R}^d)^2}\gamma_0(s-t)\gamma(x-y)h(s,x)h(t,y)\,dx\,dy\,ds\,dt<\infty. \tag{2.5}$$

The notion of Skorokhod integration coincides with the Gaussian integration given in (2.2) when acting on a deterministic field $h\in\mathcal{H}$. Indeed, for any $\varphi\in\mathcal{H}$ take $F=W(\varphi)$ (so $D(F)=\varphi$) in (2.3) we have

$$\mathbb{E}\big[\delta(h)W(\varphi)\big]=\langle\varphi,h\rangle_{\mathcal{H}},\quad \varphi\in\mathcal{H}.$$

This leads to $\delta(h)=W(h)$.

For any $h\in\mathcal{H}$ and $F\in\mathbb{D}^{1,2}$, Fh is Skorokhod integrable. The Wick product between F and $W(h)$ is defined as following:

$$F\diamond W(h)=\delta(Fh). \tag{2.6}$$

With the definition of the Wick product, the use of the notation $u\diamond\dot W$ in the parabolic Anderson equation (1.1) is interpreted as the symbolic limit of the Skorokhod integral

$$u\diamond W\big(h_\epsilon(\cdot-t,\cdot-x)\big)=\int_{\mathbb{R}^+\times\mathbb{R}^d}h_\epsilon(s-t,y-x)u(s,y)W(\delta s,\delta y)$$

as $\epsilon\to 0^+$, where $h_\epsilon(s,y)=\epsilon^{-(d+1)}h(\epsilon^{-1}s,\epsilon^{-1}y)$ with a sufficiently regular probability density function $h(s,y)$ on $\mathbb{R}^+\times\mathbb{R}^d$.

The solution of the Parabolic Anderson equation (1.1) (with $\dot W^{\mathbf{H}}$ being generalized into $\dot W$) in the sense of Definition 1.1 is uniquely given in the form of the following Wiener-chaos expansion

$$u(t,x)=\sum_{n=0}^\infty I_n\big(f_n(\cdot,t,x)\big),\quad (t,x)\in[0,t_0)\times\mathbb{R}^d. \tag{2.7}$$

which comes essentially from the iteration of the integral equation (1.3), where for each n, $I_n(f_n(\cdot,t,x))$ is formally given as a n-multiple Skorokhod integral with the symmetrified integrand

$$\begin{aligned}&f_n(s_1,x_1,\dots,s_n,x_n;t,x)\\&\quad=\frac{1}{n!}p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})\left(\prod_{k=1}^{n-1}p_{s_{\sigma(k+1)}-s_{\sigma(k)}}(x_{\sigma(k+1)}-x_{\sigma(k)})\right)(p_{s_{\sigma(1)}}*u_0)(x_{\sigma(1)})\mathbf{1}_{[0,t]^n}(s),\end{aligned} \tag{2.8}$$

where σ denotes the permutation on $\{1,\dots,n\}$ determined by the order $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$. By the $\mathcal{L}^2$-orthogonality of the expansion, the system (1.1) (or (1.3), more precisely) is uniquely solvable in the sense of Definition 1.1 whenever

$$\mathbb{E}u^2(t,x)=\sum_{n=0}^\infty n!\big\|f_n(\cdot,t,x)\big\|_{\mathcal{H}^{\otimes n}}^2<\infty,\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \tag{2.9}$$

and is locally solvable with uniqueness if (2.9) holds for $(t,x)\in[0,t_0)\times\mathbb{R}^d$ with $t_0>0$ being given in Definition 1.1.

The right hand side of (2.9) can be represented in terms of Feynman–Kac moment of the Brownian motions. To this end, let $B(t)$ and $\widetilde{B}(t)$ be two independent d-dimensional Brownian motions. The time integrals appearing in our main theorems are defined as the moment-limit

$$\int_0^t \int_0^{\tilde{t}} \gamma_0(s-r)\gamma\big(B(s)-\tilde{B}(r)\big)\,dr\,ds$$
$$\equiv \lim_{\epsilon,\delta\to 0^+} \int_0^t \int_0^{\tilde{t}} \gamma_0^\delta(s-r)\gamma^\epsilon\big(B(s)-\tilde{B}(r)\big)\,dr\,ds, \quad t,\tilde{t}>0 \tag{2.10}$$

in connection to the colored-time noise, and

$$\int_0^t \gamma\big(B(s)-\tilde{B}(s)\big)\,ds \equiv \lim_{\epsilon\to 0^+}\int_0^t \gamma^\epsilon\big(B(s)-\tilde{B}(s)\big)\,ds, \quad t>0 \tag{2.11}$$

in connection to the white-time noise, whenever the limits exist, where $\gamma_0^\delta(\cdot)$ and $\gamma^\epsilon(\cdot)$ are the properly smoothized versions of the covariance functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$, respectively.

Let $\mu_0^\delta(\cdot)$ and $\mu^\epsilon(\cdot)$ be the spectral measures of $\gamma_0^\delta(\cdot)$ and $\gamma^\epsilon(\cdot)$, respectively. For each integer $n\ge 1$, by Fourier transform one can show that

$$\mathbb{E}_0\bigg[\int_0^t\int_0^{\tilde{t}} \gamma_0^\delta(s-r)\gamma^\epsilon\big(B(s)-\tilde{B}(r)\big)\,dr\,ds\bigg]^n = \int_{(\mathbb{R}^{d+1})^n}\mu_0^\delta(d\lambda)\mu^\epsilon(d\xi)\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)\,ds\bigg|^2,$$

$$\mathbb{E}_0\bigg[\int_0^t \gamma^\epsilon\big(B(s)-\tilde{B}(s)\big)\,ds\bigg]^n = \int_{(\mathbb{R}^d)^n}\mu^\epsilon(d\xi)\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot(B(s_k)-\widetilde{B}(s_k))}\bigg)\,ds.$$

Here and elsewhere in the remaining of the paper, we use the conventions such that

$$\mu(d\xi)=\prod_{k=1}^n \mu(d\xi_k) \quad \text{and} \quad ds = ds_1\cdots ds_n$$

for the product measures whenever applied to the context of n-multiple integrations.

Let μ_0 and μ be the spectral measures of $\gamma_0(\cdot)$ and $\gamma(\cdot)$ (resp.) given in (1.5). Notice that μ_0^δ and μ^ϵ are dominated by and converge to (as $\delta,\epsilon\to 0$) μ_0 and μ, respectively. Hence, the existences of the $\mathcal{L}^n(\Omega,\mathcal{A},\mathbb{P})$-limits in in (2.10) and (2.11) are the consequences of the moment integrabilities

$$\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_{[0,t]^n}\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\,ds\bigg|^2<\infty \tag{2.12}$$

and

$$\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot(B(s_k)-\widetilde{B}(s_k))}\,ds<\infty, \tag{2.13}$$

respectively. Under (2.12) and (2.13), respectively,

$$\mathbb{E}_0\bigg[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\tilde{B}(r)\big)\,dr\,ds\bigg]^n$$
$$=\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)\,ds\bigg|^2 \tag{2.14}$$

and

$$\mathbb{E}_0\bigg[\int_0^t \gamma\big(B(s)-\tilde{B}(s)\big)\,ds\bigg]^n = \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot(B(s_k)-\widetilde{B}(s_k))}\bigg)\,ds. \tag{2.15}$$

In connection to (2.9),

$$\big\|I_n\big(f_n(\cdot,t,x)\big)\big\|^2_{\mathcal{H}^{\otimes n}} = \frac{1}{(n!)^2}\mathbb{E}_x\bigg\{\bigg[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\tilde{B}(r)\big)\,dr\,ds\bigg]^n u_0\big(B(t)\big)u_0\big(\tilde{B}(t)\big)\bigg\}.$$

By Taylor's expansion, we have the identity

$$\begin{aligned}&\sum_{n=0}^\infty n!\big\|f_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}}\\ &\quad=\mathbb{E}_x\bigg[\exp\bigg\{\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\tilde{B}(r)\big)\,dr\,ds\bigg\}u_0\big(B(t)\big)u_0\big(\tilde{B}(t)\big)\bigg]\end{aligned} \tag{2.16}$$

in the sense that both sides are finite or infinite together.

In particular,

$$\sum_{n=0}^\infty n!\big\|f_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}} = \mathbb{E}_x\bigg[\exp\bigg\{\int_0^t \gamma\big(B(s)-\tilde{B}(s)\big)\,ds\bigg\}u_0\big(B(t)\big)u_0\big(\tilde{B}(t)\big)\bigg], \tag{2.17}$$

when the noise $\dot{W}(t,x)$ is white in time.

Under the initial condition (1.12), therefore, the solvability (i.e., existence and uniqueness) claims made in Theorem 1.2 and Theorem 1.3 are reduced to the verification of the exponential integrabilities

$$\mathbb{E}_0\exp\bigg\{\theta\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\tilde{B}(r)\big)\,dr\,ds\bigg\}<\infty \tag{2.18}$$

and

$$\mathbb{E}_0\exp\bigg\{\theta\int_0^t \gamma\big(B(s)-\tilde{B}(s)\big)\,ds\bigg\}<\infty, \tag{2.19}$$

respectively, for every $t>0$. Similarly, the local solvability defined in Definition 1.1 is implied by the exponential integrability in (2.18) or (2.19) for $t<t_0$.

By a standard procedure known as *replica*, we have the Feynman–Kac moment representations

$$\mathbb{E}u^m(t,x)=\mathbb{E}_x\bigg[\exp\bigg\{\sum_{1\le j<k\le m}\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B_j(s)-B_k(r)\big)\,dr\,ds\bigg\}\prod_{j=1}^m u_0\big(B_j(t)\big)\bigg] \tag{2.20}$$

$(m=2,3,\dots)$ and in particular

$$\mathbb{E}u^m(t,x)=\mathbb{E}_x\bigg[\exp\bigg\{\sum_{1\le j<k\le m}\int_0^t \gamma\big(B_j(s)-B_k(s)\big)\,ds\bigg\}\prod_{j=1}^m u_0\big(B_j(t)\big)\bigg] \tag{2.21}$$

$(m=2,3,\dots)$ when the noise is white in time, as soon as the exponential moments are finite. Consequently, (2.18) and (2.19) for all $t>0$ lead to these representations for all t; and (2.18) and (2.19) for $t<t_0$ imply the representations for $m=2$ and $t<t_0$.

3. Proof of Theorem 1.2

First, we claim that the bound

$$\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2 \le C^n(n!)^{d-H}t^{(H+2H_0-d)n} \tag{3.1}$$

is sufficient for the proof of Theorem 1.2, where the constant $C>0$ is independent of $t>0$, and $n=1,2,\dots$. Here we recall our convention that

$$\mu(d\xi)=\prod_{k=1}^n \mu(d\xi_k), \qquad \mu_0(d\lambda)=\prod_{k=1}^n \mu_0(d\lambda_k) \quad \text{and} \quad ds=ds_1\cdots ds_n.$$

Indeed, (3.1) validates (through (2.12)) the definition of the Brownian time integral in (2.10). By the relation (2.14) and Taylor's expansion, (3.1) leads to the exponential integrability (2.18) for all $t>0$ when $d-H<1$, and for some $t>0$ when $d-H=1$. The remaining of Theorem 1.2 follows as the direct consequences of the exponential integrability (2.18), according to the discussion in Section 2.

We now start to prove (3.1). By the Brownian scaling and the homogeneity of the spectral measures $\mu_0(d\lambda)$ and $\mu(d\xi)$,

$$\begin{aligned}&\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\\ &\quad = t^{(H+2H_0-d)n}\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2.\end{aligned}$$

Hence, we only need to consider the case $t=1$, i.e., to establish the bound

$$\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2 \le C^n(n!)^{d-H}, \quad n=1,2,\dots. \tag{3.2}$$

To simplify our notation, we will use the same "C" for possibly different positive constants independent of $n=1,2,\dots$.

The first step is to separate the time and space components by Hölder inequality. Set $\alpha_0\equiv 2-2H_0$ and recall the relation (1.10)

$$|u|^{-\alpha_0}=C\int_{\mathbb{R}} e^{i\lambda u}\mu_0(d\lambda).$$

So we have

$$\begin{aligned}&\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\\ &\quad = C^n\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,1]^{2n}}\left(\prod_{k=1}^n |s_k-r_k|^{-\alpha_0}\right)\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)\\ &\qquad \times\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)dr\,ds.\end{aligned} \tag{3.3}$$

The rigorous justification of the above identity may need a standard procedure of approximation which is omitted here.

Notice that $\alpha_0 < 1$. Under the assumption (1.13) there is a $\beta > 1$ such that

$$\frac{2(d-H)+(d_*-2H_*)}{2} < 1+\beta^{-1} < 2-\alpha_0.$$

Let $\bar\beta > 1$ be the conjugate of of β. We have that $\bar\beta\alpha_0 < 1$.

Let β be fixed in the following argument. By Hölder's inequality, for any $(s_1,\dots,s_n)\in[0,1]^n$,

$$\begin{aligned}
&\int_{[0,1]^n}\left(\prod_{k=1}^n |s_k-r_k|^{-\alpha_0}\right)\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)dr\\
&\quad\le\left\{\int_{[0,1]^n}\left(\prod_{k=1}^n |s_k-r_k|^{-\bar\beta\alpha_0}\right)dr\right\}^{1/\bar\beta}\left\{\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)^{\beta}dr\right\}^{1/\beta}\\
&\quad\le\left(\prod_{k=1}^n\int_0^1 |s_k-r|^{-\bar\beta\alpha_0}\,dr\right)^{1/\bar\beta}\left\{\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)dr\right\}^{1/\beta}.
\end{aligned}$$

Here we have used the fact that

$$0 < \mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)} \le 1. \tag{3.4}$$

Notice that

$$\sup_{u\in\mathbb{R}}\int_0^1 |u-r|^{-\bar\beta\alpha_0}\,dr < \infty.$$

So we have the bound

$$\begin{aligned}
&\int_{[0,1]^n}\left(\prod_{k=1}^n |s_k-r_k|^{-\alpha_0}\right)\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)dr\\
&\quad\le C^n\left\{\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B(r_k)}\right)dr\right\}^{1/\beta}.
\end{aligned}$$

Summarizing our argument since (3.3),

$$\begin{aligned}
&\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\\
&\quad\le C^n\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left[\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right]^{1+\beta^{-1}}.
\end{aligned}\tag{3.5}$$

Set

$$I_n(t,\tilde t)=\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left[\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right]^{\frac{1+\beta^{-1}}{2}}\left[\int_{[0,\tilde t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right]^{\frac{1+\beta^{-1}}{2}},$$

where $t,\tilde t > 0$. All we need is the bound $I_n(1,1)\le (n!)^{d-H}C^n$. To this end, our second step is time exponentiation.

In view of (3.4), $I_n(t,\tilde{t})$ is non-decreasing in both t and $\tilde{t}$. By the Brownian scaling and space homogeneity,

$$I_n(t,t)=t^{n(1+\beta^{-1}-(d-H))}I_n(1,1),\quad t>0. \tag{3.6}$$

Let τ and $\tilde{\tau}$ be independent exponential times with parameter 1. We have that $\tau\wedge\tilde{\tau}$ is exponential with parameter 2. Therefore,

$$\begin{aligned}2\int_0^\infty e^{-2t}I_n(t,t)\,dt&=\mathbb{E}I_n(\tau\wedge\tilde{\tau},\tau\wedge\tilde{\tau})\\&\le\mathbb{E}I_n(\tau,\tilde{\tau})=\int_0^\infty\int_0^\infty I_n(t,\tilde{t})e^{-t-\tilde{t}}\,dt\,d\tilde{t}\\&\le\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left[\int_0^\infty dt\,e^{-t}\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds_1\cdots ds_n\right]^{1+\beta^{-1}},\end{aligned}$$

where the last step follows from Jensen's inequality, the fact that $2^{-1}(1+\beta^{-1})<1$, and (3.4).

Let Σ_n be the set of permutations on $\{1,\dots,n\}$ and write

$$[0,t]^n_<=\big\{(s_1,\dots,s_n)\in[0,t]^n;\,s_1<\cdots<s_n\big\}.$$

We have

$$\begin{aligned}\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds_1\cdots ds_n&=\sum_{\sigma\in\Sigma_n}\int_{[0,t]^n_<}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_{\sigma(k)}\cdot B(s_k)}\right)ds_1\cdots ds_n\\&=\sum_{\sigma\in\Sigma_n}\int_{[0,t]^n_<}\mathbb{E}_0\exp\left\{i\sum_{k=1}^n\left(\sum_{j=k}^n\xi_{\sigma(j)}\right)\big(B(s_k)-B(s_{k-1})\big)\right\}ds_1\cdots ds_n\\&=\sum_{\sigma\in\Sigma_n}\int_{[0,t]^n_<}\prod_{k=1}^n\exp\left\{-\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2(s_k-s_{k-1})\right\}ds_1\cdots ds_n.\end{aligned}$$

Here we adopt the convention $s_0=0$. By Lemma 2.2.7, p. 39, [3], therefore,

$$\begin{aligned}\int_0^\infty dt\,e^{-t}\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds_1\cdots ds_n&=\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\int_0^\infty e^{-t}\exp\left\{-\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2t\right\}dt\\&=\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2\right\}^{-1}.\end{aligned}\tag{3.7}$$

Therefore, we have

$$\begin{aligned}2\int_0^\infty e^{-2t}I_n(t,t)\,dt&\le\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2\right\}^{-1}\right]^{1+\beta^{-1}}\\&\le(n!)^{\beta^{-1}}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2\right\}^{-(1+\beta^{-1})}\\&=(n!)^{1+\beta^{-1}}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_j\right|^2\right\}^{-(1+\beta^{-1})}\le(n!)^{1+\beta^{-1}}C^n,\end{aligned}\tag{3.8}$$

where the second inequality follows from Jensen's inequality and the last step follows from Lemma 3.2 below.

On the other hand, by (3.6)

$$2\int_0^\infty e^{-2t} I_n(t,t)\,dt = 2I_n(1,1)\int_0^\infty t^{n(1+\beta^{-1}-(d-H))}e^{2t}\,dt$$
$$= I_n(1,1)\left(\frac{1}{2}\right)^{n(1+\beta^{-1}-(d-H))}\Gamma(1+n\big(1+\beta^{-1}-(d-H)\big).$$

By Stirling formula, we have established the bound

$$I_n(1,1) \le C^n (n!)^{d-H}.$$

Finally, the desired (3.2) follows from (3.5).

In remaining of this section, we validate the last step in (3.8) by introducing two analytic lemmas. As the last step of our argument for Theorem 1.2, we achieve that by separating the rough and non-rough space components.

Lemma 3.1. *Let $f(\xi)$ and $g(\xi)$ be two non-negative definite functions on $\mathbb{R}^d$. Then for any $\xi \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} f(\eta)g(\eta-\xi)\,d\eta \le \int_{\mathbb{R}^d} f(\eta)g(\eta)\,d\eta,$$

whenever the right hand side is finite.

Proof. Let $\mu_f(dx)$ and $\mu_g(dx)$ be the spectral measures of f and g, respectively. By a standard procedure of approximation, we may assume that $\mu_f(dx)$ and $\mu_g(dx)$ are absolutely continuous. Assume $\mu_f(dx) = \hat{f}(x)\,dx$ and $\mu_g(dx) = \hat{g}(x)\,dx$ for some $\hat{f}, \hat{g} \ge 0$.

$$\int_{\mathbb{R}^d} f(\eta)g(\eta-\xi)\,d\eta = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\hat{f}(x)\hat{g}(x)\,dx \le \int_{\mathbb{R}^d}\hat{f}(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^d} f(\eta)g(\eta)\,d\eta. \qquad \square$$

Lemma 3.2. *For any*

$$\kappa > \frac{2(d-H)+(d_*-2H_*)}{2} \tag{3.9}$$

there is a constant $C>0$ such that

$$\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n \left(1+\frac{1}{2}\left|\sum_{k=j}^n \xi_k\right|^2\right)^{-\kappa} \mu(d\xi) \le C^n, \quad n=1,2,\ldots. \tag{3.10}$$

Proof. Notice that the right hand side of (3.9) is less than 2 and left hand side of (3.10) is non-increasing in κ. So we may assume that $\kappa \le 2$ in the following proof.

Recall that $J^* = \{1 \le j \le d;\, H_j \ge 1/2\}$ and $J_* = \{1 \le j \le d;\, H_j < 1/2\}$. In the notation $\xi_k = (\xi_{k,1},\ldots,\xi_{k,d})$, $\xi_k^+ = (\xi_{k,j})_{j\in J^*}$ and $\xi_k^- = (\xi_{k,j})_{j\in J_*}$

$$\mu(d\xi) = C^n \prod_{k=1}^n \left(\prod_{j=1}^d |\xi_{k,j}|^{1-2H_j}\right) d\xi_k = C^n \prod_{k=1}^n \left(\prod_{j\in J^*} |\xi_{k,j}|^{1-2H_j}\right)\left(\prod_{j\in J_*} |\xi_{k,j}|^{1-2H_j}\right) d\xi_k$$
$$= C^n \prod_{k=1}^n q^*\big(\xi_k^+\big) q_*\big(\xi_k^-\big)\,d\xi_k^+\,d\xi_k^- \quad \text{(say)}. \tag{3.11}$$

By translation,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n \left(1+\frac{1}{2}\left|\sum_{k=j}^n \xi_k\right|^2\right)^{-\kappa} \mu(d\xi) \\
&\quad = C^n \int_{(\mathbb{R}^{J_*}\times\mathbb{R}^{J^*})^n} \left\{\prod_{k=1}^n \left(1+\frac{1}{2}|\xi_k^-|^2+\frac{1}{2}|\xi_k^+|^2\right)^{-\kappa}\right\} \prod_{k=1}^n q^*(\xi_k^+-\xi_{k-1}^+) q_*(\xi_k^- - \xi_{k-1}^-)\, d\xi_k^+\, d\xi_k^- \\
&\quad \le C^n \int_{(\mathbb{R}^{J_*}\times\mathbb{R}^{J^*})^n} \left\{\prod_{k=1}^n \left(\left(1+|\xi_k^-|^2\right)^{\kappa/2}+|\xi_k^+|^\kappa\right)^{-2}\right\} \prod_{k=1}^n q^*(\xi_k^+-\xi_{k-1}^+) q_*(\xi_k^- - \xi_{k-1}^-)\, d\xi_k^+\, d\xi_k^-, \qquad (3.12)
\end{aligned}$$

where the last step follows from the bound

$$\left(1+\frac{a^2}{2}+\frac{b^2}{2}\right)^{\kappa/2} \ge C^{-1}\left\{(1+a)^{\kappa/2}+b^\kappa\right\}$$

uniformly over $a, b > 0$.

In addition, notice that the function $q^*(\eta)$ ($\eta \in \mathbb{R}^{J^*}$) is non-negative definite with spectral density that appears to be the constant multiple of

$$\prod_{j\in J^*} |x_j|^{-(2-2H_j)}, \quad x = (x_j)_{j\in J^*} \in \mathbb{R}^{J^*}.$$

We now claim for any $a > 0$, $f(\eta) = (a+|\eta|^\kappa)^{-1}$ ($\eta \in \mathbb{R}^{J^*}$) is non-negative definite. Indeed, let X_t ($t \ge 0$) be a d^*-dimensional κ-stable Lévy process (recall that $d^* = \#(J^*)$ and $0 < \kappa \le 2$) with the characteristic function

$$\mathbb{E}\exp\{i\eta\cdot X_t\} = e^{-|\eta|^\kappa}, \quad \eta \in \mathbb{R}^{J^*}$$

and let $g_t(x)$ be the density of X_t. One can see that the non-negative function

$$\hat{f}(x) = \int_0^\infty e^{-at} g_t(x)\, dt, \quad x \in \mathbb{R}^{J^*}$$

is the spectral density of $f(\eta)$. Consequently, $f^2(\eta) = (a+|\eta|^\kappa)^{-2}$ ($\eta \in \mathbb{R}^{J^*}$) is non-negative definite with the non-negative spectral density $\hat{f} * \hat{f}$.

By Lemma 3.1, for any $\zeta \in \mathbb{R}^{J^*}$

$$\begin{aligned}
\int_{\mathbb{R}^{J^*}} \left(a+|\eta|^\kappa\right)^{-2} q^*(\eta-\zeta)\, d\eta &\le \int_{\mathbb{R}^{J^*}} \left(a+|\eta|^\kappa\right)^{-2} q^*(\eta)\, d\eta \\
&= a^{-2+2\kappa^{-1}(d^*-H^*)} \int_{\mathbb{R}^{J^*}} \left(1+|\eta|^\kappa\right)^{-2} q^*(\eta)\, d\eta.
\end{aligned}$$

Notice that under (3.9), $\kappa + H^* > d^*$ which leads to

$$C \equiv \int_{\mathbb{R}^{J^*}} \left(1+|\eta|^\kappa\right)^{-2} q^*(\eta)\, d\eta = \int_{\mathbb{R}^{J^*}} \left(1+|\eta|^\kappa\right)^{-2} \prod_{j\in J^*} |\eta_j|^{1-2H_j}\, d\eta < \infty.$$

Consequently, for any $a_1, \dots, a_n > 0$,

$$\int_{(\mathbb{R}^{J^*})^n} \left(\prod_{k=1}^n \left(a_k+|\xi_k^+|^\kappa\right)^{-2}\right) \prod_{k=1}^n q^*(\xi_k^+-\xi_{k-1}^+)\, d\xi_k^+ \le C^n \prod_{k=1}^n a_k^{-2+2\kappa^{-1}(d^*-H^*)}. \qquad (3.13)$$

By Fubini's theorem and (3.12),

$$\int_{(\mathbb{R}^d)^n} \prod_{k=1}^n \left(1 + \frac{1}{2}\left|\sum_{k=j}^n \xi_k\right|^2\right)^{-\kappa} \mu(d\xi)$$
$$\le C^n \int_{(\mathbb{R}^{J_*})^n} \left(\prod_{k=1}^n \left(1 + |\xi_k|^2\right)^{-\kappa+(d^*-H^*)}\right) \prod_{k=1}^n q_*(\xi_k - \xi_{k-1})\, d\xi_k. \tag{3.14}$$

Here we use ξ_k instead of ξ_k^- on the right hand side for notation simplification.

Notice that

$$\prod_{k=1}^n q_*(\xi_k - \xi_{k-1}) = \prod_{k=1}^n \prod_{j\in J_*} |\xi_{k,j} - \xi_{k-1,j}|^{1-2H_j} \le \prod_{k=1}^n \prod_{j\in J_*} \left(|\xi_{k,j}|^{1-2H_j} + |\xi_{k-1,j}|^{1-2H_j}\right)$$
$$\le \sum_l \prod_{k=1}^n \prod_{j\in J_*} |\xi_{k,j}|^{l(k,j)(1-2H_j)},$$

where the summation is taken over all maps l: $\{1, \dots, n\} \times J_* \longmapsto \{0, 1, 2\}$. Notice that the number of the terms in the summation is at most $3^{nd_*} \le 3^{nd}$.

In view of (3.14), therefore, all we need to prove is that

$$\int_{\mathbb{R}^{J_*}} \left(1 + |\xi|^2\right)^{-\kappa+(d^*-H^*)} \prod_{j\in J_*} |\xi_j|^{l(1-2H_j)}\, d\xi < \infty, \quad l = 0, 1, 2. \tag{3.15}$$

Notice that $1 - 2H_j > 0$ for each $j \in J_*$. Obviously, only the case $l = 2$ needs to be checked. Indeed, by spherical substitution

$$\int_{\mathbb{R}^{J_*}} \left(1 + |\xi|^2\right)^{-\kappa+(d^*-H^*)} \prod_{j\in J_*} |\xi_j|^{2(1-2H_j)}\, d\xi = C \int_0^\infty \left(1 + \rho^2\right)^{-\kappa+(d^*-H^*)} \rho^{2(d_*-2H_*)} \rho^{d_*-1}\, d\rho < \infty,$$

where the last step follows from the assumption (3.9). □

4. Proof of Theorem 1.3

By the discussion in Section 2, all we need is the exponential integrability given in (2.19). By Taylor expansion, (2.15) and the Dalang's condition $d - H < 1$, it suffices to establish the moment bound

$$\int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^n} \mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B(s_k)-\widetilde{B}(s_k))}\, ds \le (n!)^{d-H} C^n t^{n(H+1-d)}, \quad n = 1, 2, \dots. \tag{4.1}$$

By the independence between B and $\widetilde{B}$,

$$\int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B(s_k)-\widetilde{B}(s_k))}\right) ds = \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)^2 ds$$
$$\le \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right) ds = t^{n(H+1-d)} \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right) ds,$$

where the last step follows from the Brownian scaling and homogeneity of the space covariance. Thus, it suffices to establish the bound

$$\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\le (n!)^{d-H}C^n,\quad n=1,2,\dots. \tag{4.2}$$

Using the Brownian scaling again

$$\begin{aligned}
&\int_0^\infty dt\, e^{-t}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\,ds\\
&\quad=\left\{\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right\}\int_0^\infty t^{n(H+1-d)}e^{-t}\,dt\\
&\quad=\Gamma\big(1+n(H+1-d)\big)\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds.
\end{aligned}\tag{4.3}$$

On the other hand, by (3.7)

$$\begin{aligned}
\int_0^\infty dt\, e^{-t}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds&=\sum_{\sigma\in\Sigma_n}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2\right\}^{-1}\\
&=n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_j\right|^2\right\}^{-1}\le n!C^n,
\end{aligned}$$

where the second step follows from permutation invariance and the last step from the bound given in Lemma 3.2 with $\kappa=1$.

Finally, (4.2) follows from (4.3) and Stirling formula.

5. Proof of Proposition 1.4

By the Cauchy–Schwartz's inequality $\mathbb{E}|\cdot|^{2k+1}\le(\mathbb{E}|\cdot|^{2k})^{1/2}(\mathbb{E}|\cdot|^{2k+2})^{1/2}$, we only need to establish the bound (1.20) for even and positive integers n. In view of the relation (2.14), (1.20) is equivalent to the proof of the bound

$$\begin{aligned}
&\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\\
&\quad\le C^n(n!)^{(d-H)+(2-2H_0)}t^{(H+2H_0-d)n},\quad t>0, n=1,2,\dots.
\end{aligned}\tag{5.1}$$

By the Brownian scaling and the homogeneity of the spectral measures $\mu_0(d\lambda)$ and $\mu(d\xi)$, the left hand side is equal to

$$t^{(H+2H_0-d)n}\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2.$$

Thus, all we need is to show

$$\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\le C^n(n!)^{(d-H)+(2-2H_0)} \tag{5.2}$$

for all even numbers $n\ge 2$.

Let n be even in the remaining of the proof. The implementation of time-exponentiation becomes more delicate, due to the fact that the quantity

$$\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds$$

is not monotonic in t. We carry it out in the following lemma.

Lemma 5.1. *For all even numbers $n\ge 2$,*

$$\begin{aligned}&2\int_0^\infty e^{-2t}\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2dt \\ &\quad\le(1+\sqrt{2})^{2n}\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_0^\infty dt\, e^{-t}\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2.\end{aligned}\tag{5.3}$$

Proof. To justify some of the computations, a strict procedure is to establish (5.3) first with $\mu_0(d\lambda)$ and $\mu(d\xi)$ being replaced by their smoothed versions

$$\mu_0^\delta(d\lambda)=e^{-\delta|\lambda|^2}\mu_0(d\lambda)\quad\text{and}\quad\mu^\epsilon(d\xi)=e^{-\epsilon|\xi|^2}\mu(d\xi)$$

(and accordingly, $\gamma_0(\cdot)$ and $\gamma(\cdot)$ being replaced by $\gamma_0^\delta(\cdot)$ and $\gamma^\epsilon(\cdot)$), respectively, and then to take the limit $\delta,\epsilon\to 0^+$ on the both sides of the inequality. To simplify the notation, we omit this procedure. For keeping rigorousness, the reader can treat the notations $\mu_0(d\lambda)$, $\mu(d\xi)$, $\gamma_0(\cdot)$ and $\gamma(\cdot)$ used in the proof as $\mu_0^\delta(d\lambda)$, $\mu^\epsilon(d\xi)$, $\gamma_0^\delta(\cdot)$ and $\gamma^\epsilon(\cdot)$, respectively.

For any $t,\tilde t>0$, write

$$\mathcal{Q}(t,\tilde t)=\int_0^t\int_0^{\tilde t}\gamma_0(s-r)\gamma\big(B(s)-\tilde B(r)\big)\,dr\,ds.$$

By Fourier transform,

$$\begin{aligned}\mathbb{E}_0\mathcal{Q}^n(t,\tilde t)=&\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg[\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg]\\ &\times\bigg[\int_{[0,\tilde t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{-i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg].\end{aligned}\tag{5.4}$$

Let $\tau,\tilde\tau$ be two independent exponential times with parameter 1 and independent of the Brownian motions B and $\widetilde B$. We extend the expectation "$\mathbb{E}_0$" and the probability "$\mathbb{P}_0$" to the probability space that includes τ and $\tilde\tau$. In view of the fact that $t\wedge\tilde t$ is exponential with parameter 2, the lemma can be restated as

$$\mathbb{E}_0\mathcal{Q}^n(\tau\wedge\tilde\tau,\tau\wedge\tilde\tau)\le(1+\sqrt{2})^{2n}\mathbb{E}_0\mathcal{Q}^n(\tau,\tilde\tau).\tag{5.5}$$

Consider the decomposition

$$\begin{aligned}\mathcal{Q}(\tau,\tilde\tau)=&\,\mathcal{Q}(\tau\wedge\tilde\tau,\tau\wedge\tilde\tau)+1_{\{\tau>\tilde\tau\}}\int_{\tilde\tau}^\tau\int_0^{\tilde\tau}\gamma_0(r-s)\gamma(B_r-\tilde B_s)\,dr\,ds\\ &+1_{\{\tau<\tilde\tau\}}\int_0^\tau\int_\tau^{\tilde\tau}\gamma_0(r-s)\gamma(B_r-\tilde B_s)\,dr\,ds.\end{aligned}$$

By the fact that last two terms on the right hand side are identically distributed and by the triangle inequality

$$\begin{aligned}
\big(\mathbb{E}_0\mathcal{Q}^n(\tau\wedge\tilde\tau,\tau\wedge\tilde\tau)\big)^{1/n} &\le \big(\mathbb{E}_0\mathcal{Q}^n_{\epsilon,\delta}(\tau,\tilde\tau)\big)^{1/n} \\
&\quad + 2\bigg(\mathbb{E}_0\bigg\{\bigg[\int_0^\tau\int_\tau^{\tilde\tau}\gamma_0(s-r)\gamma\big(B(s)-\tilde B(r)\big)\,dr\,ds\bigg]^n 1_{\{\tau<\tilde\tau\}}\bigg\}\bigg)^{1/n}. \qquad (5.6)
\end{aligned}$$

Besides,

$$\begin{aligned}
&\mathbb{E}_0\bigg\{\bigg[\int_0^\tau\int_\tau^{\tilde\tau}\gamma_0(s-r)\gamma\big(B(s)-\tilde B(r)\big)\,dr\,ds\bigg]^n 1_{\{\tau<\tilde\tau\}}\bigg\} \\
&\quad = \int_0^\infty e^{-t}\,dt\int_t^\infty e^{-\tilde t}d\tilde t\,\mathbb{E}_0\bigg[\int_0^t\int_t^{\tilde t}\gamma_0(s-r)\gamma\big(B(s)-\tilde B(r)\big)\,dr\,ds\bigg]^n \\
&\quad = \int_0^\infty dt\int_t^\infty d\tilde t\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg[\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg] \\
&\qquad \times\bigg[\mathbb{E}_0\int_{[t,\tilde t]^n}\bigg(\prod_{k=1}^n e^{-i(\lambda_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg].
\end{aligned}$$

Write

$$\begin{aligned}
&\int_{[t,\tilde t]^n}\bigg(\prod_{k=1}^n e^{-i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds \\
&\quad = \exp\bigg\{-i\sum_{k=1}^n\big(\lambda_k t+\xi_k\cdot B(t)\big)\bigg\}\int_{[0,\tilde t-t]^n}\bigg(\prod_{k=1}^n e^{-i(\lambda_k s_k+\xi_k\cdot(B(t+s_k)-B(t)))}\bigg)ds.
\end{aligned}$$

By the increment independence of the Brownian motion

$$\begin{aligned}
&\mathbb{E}_0\int_{[t,\tilde t]^n}\bigg(\prod_{k=1}^n e^{-i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds \\
&\quad = \mathbb{E}_0\exp\bigg\{-i\sum_{k=1}^n\big(\lambda_k t+\xi_k\cdot B(t)\big)\bigg\}\int_{[0,\tilde t-t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigg|\int_t^\infty e^{-\tilde t}\bigg[\mathbb{E}_0\int_{[t,\tilde t]^n}\bigg(\prod_{k=1}^n e^{-i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg]d\tilde t\bigg| \\
&\quad \le e^{-t}\bigg|\int_0^\infty e^{-\tilde t}d\tilde t\int_{[0,\tilde t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|.
\end{aligned}$$

Summarizing our computation,

$$\begin{aligned}
&\mathbb{E}_0\bigg\{\bigg[\int_0^\tau\int_\tau^{\tilde\tau}\gamma_0(s-r)\gamma\big(B(s)-\tilde B(r)\big)\,dr\,ds\bigg]^n 1_{\{\tau<\tilde\tau\}}\bigg\}\\
&\quad\le\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg[\int_0^\infty e^{-2t}\,dt\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|\bigg]\\
&\qquad\times\bigg|\int_0^\infty e^{-t}\,dt\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|\\
&\quad\le\bigg\{\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg[\int_0^\infty e^{-2t}\,dt\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|\bigg]^2\bigg\}^{1/2}\\
&\qquad\times\bigg\{\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_0^\infty e^{-t}\,dt\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2\bigg\}^{1/2}.
\end{aligned}$$

By (5.4),

$$\begin{aligned}
&\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_0^\infty e^{-t}\,dt\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2\\
&\quad=\int_0^\infty\int_0^\infty e^{-t-t'}\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg[\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg]\\
&\qquad\times\bigg[\int_{[0,\tilde t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{-i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg]dt\,d\tilde t\\
&\quad=\mathbb{E}_0\mathcal{Q}^n(\tau,\tilde\tau).
\end{aligned}$$

By Jensen's inequality

$$\begin{aligned}
&\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\bigg(\int_0^\infty e^{-2t}\,dt\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|\bigg)^2\\
&\quad\le\frac12\int_{(\mathbb{R}^{d+1})^n}\mu_0(d\lambda)\mu(d\xi)\int_0^\infty e^{-2t}\,dt\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2.
\end{aligned}$$

Recall the elementary fact that $\tau\wedge\tilde\tau$ is an exponential time with parameter 2. Consequently, the right hand side of the above inequality is equal to $4^{-1}\mathbb{E}_0\mathcal{Q}^n(\tau\wedge\tilde\tau,\tau\wedge\tilde\tau)$.

Summarizing our steps since (5.6),

$$\big(\mathbb{E}_0\mathcal{Q}^n(\tau\wedge\tilde\tau,\tau\wedge\tilde\tau)\big)^{1/n}\le\big(\mathbb{E}_0\mathcal{Q}^n(\tau,\tilde\tau)\big)^{1/n}+2\big(\mathbb{E}_0\mathcal{Q}^n(\tau,\tilde\tau)\big)^{\frac{1}{2n}}\big(\mathbb{E}_0\mathcal{Q}^n(\tau\wedge\tilde\tau,\tau\wedge\tilde\tau)\big)^{\frac{1}{2n}},$$

which leads to (5.5). □

We now come to the proof of (5.2). A computation similar to (3.7) leads to

$$\int_0^\infty dt\,e^{-t}\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds=\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\bigg\{1+\frac12\bigg|\sum_{j=k}^n\xi_{\sigma(j)}\bigg|^2-i\bigg(\sum_{j=k}^n\lambda_{\sigma(j)}\bigg)\bigg\}^{-1}.$$

Applying this to the right hand side of (5.3),

$$\begin{aligned}
&\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\int_0^\infty dt\, e^{-t}\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\\
&\quad=\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\left|\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\left\{1+\frac{1}{2}\left|\sum_{j=k}^n\xi_{\sigma(j)}\right|^2-i\left(\sum_{j=k}^n\lambda_{\sigma(j)}\right)\right\}^{-1}\right|^2\\
&\quad\le (n!)^2\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\prod_{k=1}^n\left|1+\frac{1}{2}\left|\sum_{j=k}^n\xi_j\right|^2-i\left(\sum_{j=k}^n\lambda_j\right)\}\right|^{-2}\\
&\quad=(n!)^2\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi)\prod_{k=1}^n\left\{\left(1+\frac{1}{2}\left|\sum_{j=k}^n\xi_j\right|^2\right)^2+\left(\sum_{j=k}^n\lambda_j\right)^2\right\}^{-1}, \qquad (5.7)
\end{aligned}$$

where the second step follows from Jensen's inequality and permutation invariance.

Let $(\xi_1,\dots,\xi_n)$ be fixed for a while and set

$$a_k=1+\frac{1}{2}\left|\sum_{j=k}^n\xi_j\right|^2,\quad k=1,\dots,n.$$

Recall that in our notation

$$\mu_0(d\lambda)=C^n\prod_{k=1}^n|\lambda_k|^{1-2H_0}\,d\lambda_k.$$

By the variable substitution, for any $a_k\ge 1$ $(k=1,\dots,n)$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\prod_{k=1}^n\left\{a_k^2+\left(\sum_{j=k}^n\lambda_j\right)^2\right\}^{-1}\mu_0(d\lambda)\\
&\quad=C^n\int_{\mathbb{R}^n}\left(\prod_{k=1}^n\{a_k^2+\lambda_k^2\}^{-1}\right)\prod_{k=1}^n|\lambda_k-\lambda_{k-1}|^{1-2H_0}\,d\lambda_k.
\end{aligned}$$

Here we adapt the convention $\lambda_0=0$.

Notice that $0<1-2H_0<1$. By the triangle inequality

$$\prod_{k=1}^n|\lambda_k-\lambda_{k-1}|^{1-2H_0}\le\prod_{k=1}^n\left\{|\lambda_k|^{1-2H_0}+|\lambda_{k-1}|^{1-2H_0}\right\}\le\sum_l\prod_{k=1}^n|\lambda_k|^{l_k(1-2H_0)},$$

where the summation is over all maps l: $\{1,\dots,n\}\longmapsto\{0,1,2\}$.

Noticing that the number of the terms in the summation of the right hand side is at most 3^n,

$$\begin{aligned}
\int_{\mathbb{R}^n}\prod_{k=1}^n\left\{a_k^2+\left(\sum_{j=k}^n\lambda_j\right)^2\right\}^{-1}\mu_0(d\lambda)&\le C^n\sum_l\prod_{k=1}^n\int_{-\infty}^\infty\frac{|\lambda|^{l_k(1-2H_0)}}{a_k^2+\lambda^2}\,d\lambda\\
&=C^n\sum_l\prod_{k=1}^n a_k^{l_k(1-2H_0)-1}\le C^n3^n\prod_{k=1}^n a_k^{-(4H_0-1)},
\end{aligned}$$

where the second step follows from the identity

$$\int_{-\infty}^{\infty} \frac{|\lambda|^{\beta}}{a^2+\lambda^2}\,d\lambda = \frac{1}{a^{1-\beta}} \int_{-\infty}^{\infty} \frac{|\lambda|^{\beta}}{1+\lambda^2}\,d\lambda \quad (\beta < 1)$$

and the last step follows from the fact that $a_k \geq 1$.

By (5.7) and Fubini's theorem, therefore,

$$\begin{aligned}
&\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi) \left| \int_0^{\infty} dt\, e^t \int_{[0,t]^n} \left(\mathbb{E}_0 \prod_{k=1}^{n} e^{i(\lambda_k s_k + \xi_k \cdot B(s_k))} \right) ds \right|^2 \\
&\quad \leq (n!)^2 C^n \int_{(\mathbb{R}^d)^n} \mu(d\xi) \prod_{k=1}^{n} \left(1 + \frac{1}{2} \left| \sum_{j=k}^{n} \xi_j \right|^2 \right)^{-(4H_0-1)} \leq (n!)^2 C^n,
\end{aligned} \tag{5.8}$$

where the last step follows from the assumption (1.19) and Lemma 3.2.

In connection to the left had side of the inequality (5.3), on the other hand, by the scaling property

$$\begin{aligned}
&2\int_0^{\infty} e^{-2t} \int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi) \left| \int_{[0,t]^n} \mathbb{E}_0 \prod_{k=1}^{n} e^{i(\lambda_k s_k + \xi_k \cdot B(s_k))}\, ds \right|^2 dt \\
&\quad = 2\int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi) \left| \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^{n} e^{i(\lambda_k s_k + \xi_k \cdot B(s_k))} \right) ds \right|^2 \int_0^{\infty} t^{n(H+2H_0-d)} e^{-t}\, dt \\
&\quad = \left(\frac{1}{2}\right)^{n(H+2H_0-d)} \Gamma\big(1 + n(H+2H_0-d)\big) \\
&\qquad \times \int_{(\mathbb{R}^{d+1})^n} \mu_0(d\lambda)\mu(d\xi) \left| \int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^{n} e^{i(\lambda_k s_k + \xi_k \cdot B(s_k))} \right) ds \right|^2.
\end{aligned}$$

Therefore, the desired bound (5.2) follows from (5.8) and Lemma 5.1 with a simple application of Stirling formula.

Remark 5.2. In the assumption of Theorem 1.2, the bound (5.1) can be obtained by a way essentially same as the one used here. Clearly, this bound is substantially worse than (3.1). This comparison indicates that the bound produced by the argument in this section might be not optimal for $H_0 < 1/2$ either.

6. Proof of Theorem 1.5

First notice that $u(t,x)$ is monotonic in the initial condition $u_0(x)$. By the assumption (1.12), therefore, $u(t,x) \geq 0$ for all (t,x). By linearity, further, $u(t,x)$ is between two possibly different constant multiples of the solution of (1.1) with the initial condition $u_0(x) \equiv 1$. Thus, we may assume $u_0(x) = 1$ in our proof. Notice that $u(t,x) \stackrel{d}{=} u(t,0)$ for any $x \in \mathbb{R}^d$ when $u_0(x) = 1$. So we take $x = 0$ in our proof.

The lower bounds for (1.26) and (1.27), i.e.,

$$\liminf_{t\to\infty} t^{-\frac{4-\alpha-2\alpha_0}{2-\alpha}} \log \mathbb{E} u^p(t,0) \geq p \left(\frac{p-1}{2}\right)^{\frac{2}{2-\alpha}} \mathcal{E}(H_0) \quad \forall p > 1 \tag{6.1}$$

and

$$\liminf_{m\to\infty} m^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E} u^m(t,0) \geq \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}} t^{\frac{4-\alpha-2\alpha_0}{2-\alpha}} \mathcal{E}(H_0) \quad \forall t > 0 \tag{6.2}$$

follow from the proof given in Section 3, [4] with the the integer $n \geq 2$ being replaced by the real number $p \geq 2$, with $V(t,x) = \theta \dot{W}^{\mathbf{H}}(t,x)$, and with the relation

$$\sup_{g\in\mathcal{A}_d}\left\{\theta\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|s-r|^{\alpha_0}}g^2(s,x)g^2(r,y)\,dx\,dy\,dr\,ds-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2x\,ds\right\}$$
$$=\theta^{\frac{2}{2-\alpha}}\mathcal{E}(H_0)\quad \forall\theta>0. \tag{6.3}$$

We remark that the covariance function $\gamma(\cdot)$ is non-negative and $p \geq 2$ is an integer in [4]. However, the argument of the lower bounds given in Section 3, [4] does not require that $\gamma(\cdot)$ be non-negative or that $p \geq 2$ be an integer.

On the other hand, the non-roughness assumption is essential to the proof of the upper bound given in [4]. Without it, the compactification by folding performed in [4] can not get through. The treatment given below is partially inspired by the recent work of Huang, Lê and Nualart (Proposition 5.1 in [17]), where the argument relies on a transition from Brownian system to Ornstein–Uhlenbeck system, an idea originated by Donsker and Varadhan [10].

Another difference in approach from [4]) is that we treat the spectral measure $\mu(d\xi)$ instead of the space covariance function $\gamma(\cdot)$ for $\gamma(\cdot)$ is not defined pointwise in the presence of spatial roughness. First, we develop the following sub-additive moment inequality.

6.1. *Sub-additivity*

Let $\nu(d\xi)$ be a measure on $\mathbb{R}^d$. Write

$$\mathcal{H}_t=\int_{\mathbb{R}^d}\nu(d\xi)\left[\int_0^t e^{i\xi\cdot B(s)}\,ds\right]\left[\int_0^t e^{-i\xi\cdot\tilde{B}(s)}\,ds\right].$$

Lemma 6.1. *For any $t_1, t_2 > 0$ and $\theta > 0$*

$$\mathbb{E}_0\exp\left\{\frac{\theta}{t_1+t_2}\mathcal{H}_{t_1+t_2}\right\}\leq\mathbb{E}_0\exp\left\{\frac{\theta}{t_1}\mathcal{H}_{t_1}\right\}\mathbb{E}_0\exp\left\{\frac{\theta}{t_2}\mathcal{H}_{t_2}\right\}, \tag{6.4}$$

whenever the right hand side is finite.

Proof. All we need is to show that for any integer $n \geq 1$,

$$\frac{1}{(t_1+t_2)^n}\mathbb{E}_0\mathcal{H}_{t_1+t_2}^n\leq\sum_{l=0}^n\binom{n}{l}\frac{1}{t_1^l}\frac{1}{t_2^{n-l}}\mathbb{E}_0|\mathcal{H}_{t_1}|^l\mathbb{E}_0\mathcal{H}_{t_2}^{n-l}. \tag{6.5}$$

We start with the computation

$$\mathbb{E}_0\mathcal{H}_{t_1+t_2}^n=\int_{(\mathbb{R}^d)^n}\nu(d\xi)\left|\mathbb{E}_0\prod_{k=1}^n\int_0^{t_1+t_2}e^{\xi_k\cdot B(s)}\,ds\right|^2$$
$$=\int_{(\mathbb{R}^d)^n}\nu(d\xi)\left|\sum_{j_1,\dots,j_n=1}^2\mathbb{E}_0\prod_{k=1}^n\Delta_{j_k}(\xi_k)\right|^2,$$

where

$$\Delta_1(\xi)=\int_0^{t_1}e^{i\xi\cdot B(s)}\,ds\quad\text{and}\quad\Delta_2(\xi)=\int_{t_1}^{t_1+t_2}e^{i\xi\cdot B(s)}\,ds.$$

Write

$$\left|\sum_{j_1,\dots,j_n=1}^2\mathbb{E}_0\prod_{k=1}^n\Delta_{j_k}(\xi_k)\right|^2=(t_1+t_2)^{2n}\left|\sum_{j_1,\dots,j_n=1}^2\left(\prod_{k=1}^n\frac{t_{j_k}}{t_1+t_2}\right)\mathbb{E}_0\prod_{k=1}^n\frac{1}{t_{j_k}}\Delta_{j_k}(\xi_k)\right|^2.$$

Notice that

$$\sum_{j_1,\dots,j_n=1}^{2}\left(\prod_{k=1}^{n}\frac{t_{j_k}}{t_1+t_2}\right)=1.$$

By Jensen's inequality,

$$\begin{aligned}\left|\sum_{j_1,\dots,j_n=1}^{2}\mathbb{E}_0\prod_{k=1}^{n}\Delta_{j_k}(\xi_k)\right|^2 &\le (t_1+t_2)^{2n}\sum_{j_1,\dots,j_n=1}^{2}\left(\prod_{k=1}^{n}\frac{t_{j_k}}{t_1+t_2}\right)\left|\mathbb{E}_0\prod_{k=1}^{n}\frac{1}{t_{j_k}}\Delta_{j_k}(\xi_k)\right|^2\\ &= (t_1+t_2)^{n}\sum_{j_1,\dots,j_n=1}^{2}\left(\prod_{k=1}^{n}\frac{1}{t_{j_k}}\right)\left|\mathbb{E}_0\prod_{k=1}^{n}\Delta_{j_k}(\xi_k)\right|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{(t_1+t_2)^n}\mathbb{E}_0\mathcal{H}_{1+t_2}^n &\le \sum_{j_1,\dots,j_n=1}^{2}\left(\prod_{k=1}^{n}\frac{1}{t_{j_k}}\right)\int_{(\mathbb{R}^d)^n}\nu(d\xi)\left|\mathbb{E}_0\prod_{k=1}^{n}\Delta_{j_k}(\xi_k)\right|^2\\ &=\sum_{l=0}^{n}\binom{n}{l}\frac{1}{t_1^l}\frac{1}{t_2^{n-l}}\int_{(\mathbb{R}^d)^n}\nu(d\xi)\left|\mathbb{E}_0\left\{\left(\prod_{k=1}^{l}\Delta_1(\xi_k)\right)\left(\prod_{k=l+1}^{n}\Delta_2(\xi_k)\right)\right\}\right|^2,\end{aligned}\tag{6.6}$$

where the last step follows from variable permutation.

Let $0\le l\le n$ be fixed. By the increment independence,

$$\begin{aligned}&\mathbb{E}_0\left\{\left(\prod_{k=1}^{l}\Delta_1(\xi_k)\right)\left(\prod_{k=l+1}^{n}\Delta_2(\xi_k)\right)\right\}\\ &\quad=\left\{\int_{[0,t_2]^n}\left(\mathbb{E}_0\prod_{k=l+1}^{n}e^{i\xi_k\cdot B(s_k)}\right)ds\right\}\mathbb{E}_0\left\{\left(\prod_{k=1}^{l}\Delta_1(\xi_k)\right)\exp\left(i\left(\sum_{k=l+1}^{n}\xi_k\right)\cdot B(t_1)\right)\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\mathbb{E}_0\left\{\left(\prod_{k=1}^{l}\Delta_1(\xi_k)\right)\left(\prod_{k=l+1}^{n}\Delta_2(\xi_k)\right)\right\}\right|^2\\ &\quad=\left|\int_{[0,t_2]^n}\left(\mathbb{E}_0\prod_{k=l+1}^{n}e^{i\xi_k\cdot B(s_k)}\right)ds\right|^2\\ &\qquad\times\mathbb{E}_0\left\{\left(\prod_{k=1}^{l}\Delta_1(\xi_k)\right)\left(\prod_{k=1}^{l}\widetilde{\Delta}_1(-\xi_k)\right)\exp\left(i\left(\sum_{k=l+1}^{n}\xi_k\right)\cdot\big(B(t_1)-\widetilde{B}(t_1)\big)\right)\right\}.\end{aligned}$$

Integrating against the first l variables $\xi_1,\dots,\xi_l$:

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^l}\nu(d\xi)\bigg|\mathbb{E}_0\bigg\{\bigg(\prod_{k=1}^{l}\Delta_1(\xi_k)\bigg)\bigg(\prod_{k=l+1}^{n}\Delta_2(\xi_k)\bigg)\bigg\}\bigg|^2\\
&=\bigg|\int_{[0,t_2]^n}\bigg(\mathbb{E}_0\prod_{k=l+1}^{n}e^{i\xi_k\cdot B(s_k)}\bigg)ds\bigg|^2\\
&\quad\times\mathbb{E}_0\bigg\{\exp\bigg(i\bigg(\sum_{k=l+1}^{n}\xi_k\bigg)\cdot\big(B(t_1)-\widetilde{B}(t_1)\big)\bigg)\int_{(\mathbb{R}^d)^l}\nu(d\xi)\bigg(\prod_{k=1}^{l}\Delta_1(\xi_k)\bigg)\bigg(\prod_{k=1}^{l}\widetilde{\Delta}_1(-\xi_k)\bigg)\bigg\}\\
&=\bigg|\int_{[0,t_2]^n}\bigg(\mathbb{E}_0\prod_{k=l+1}^{n}e^{i\xi_k\cdot B(s_k)}\bigg)ds\bigg|^2\mathbb{E}_0\bigg\{\exp\bigg(i\bigg(\sum_{k=l+1}^{n}\xi_k\bigg)\cdot\big(B(t_1)-\widetilde{B}(t_1)\big)\bigg)\mathcal{H}_{t_1}^l\bigg\}\\
&\le\bigg|\int_{[0,t_2]^n}\bigg(\mathbb{E}_0\prod_{k=l+1}^{n}e^{i\xi_k\cdot B(s_k)}\bigg)ds\bigg|^2\mathbb{E}_0|\mathcal{H}_{t_1}|^l.
\end{aligned}
$$

Integrating against $\xi_{l+1},\dots,\xi_n$,

$$
\begin{aligned}
&\int_{(R^d)^n}\nu(d\xi)\bigg|\mathbb{E}_0\bigg\{\bigg(\prod_{k=1}^{l}\Delta_1(\xi_k)\bigg)\bigg(\prod_{k=l+1}^{n}\Delta_2(\xi_k)\bigg)\bigg\}\bigg|^2\\
&\le\bigg\{\int_{(\mathbb{R}^d)^{n-l}}\nu(dxi)\bigg|\int_{[0,t_2]^n}\bigg(\mathbb{E}_0\prod_{k=l+1}^{n}e^{i\xi_k\cdot B(s_k)}\bigg)ds\bigg|^2\bigg\}\mathbb{E}_0|\mathcal{H}_{t_1}|^l\\
&=\mathbb{E}\mathcal{H}_{t_2}^{n-l}\cdot\mathbb{E}_0|\mathcal{H}_{t_1}|^l.
\end{aligned}
$$

By (6.6), this leads to (6.5). □

6.2. *Upper bound for* (1.26) *with* $p=2$

We start with the upper bound of (1.26) with $p=2$, i.e.,

$$
\limsup_{t\to\infty}t^{-\frac{4-\alpha-2\alpha_0}{2-\alpha}}\log\mathbb{E}u^2(t,0)\le 2\bigg(\frac{C_{H_0}\theta^2}{2}\bigg)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0). \tag{6.7}
$$

In view of the relation (1.14) with $m=2$, all we need is to show

$$
\limsup_{t\to\infty}t^{-\frac{4-\alpha-2\alpha_0}{2}}\log\mathbb{E}_0\exp\bigg\{C_{H_0}\theta^2\int_0^t\int_0^t\frac{\gamma(B(s)-\widetilde{B}(r))}{|s-r|^{\alpha_0}}\,ds\,dr\bigg\}\le 2\bigg(\frac{C_{H_0}\theta^2}{2}\bigg)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0).
$$

By Brownian scaling,

$$
\mathbb{E}_0\exp\bigg\{C_{H_0}\theta^2\int_0^t\int_0^t\frac{\gamma(B(s)-\widetilde{B}(r))}{|s-r|^{\alpha_0}}\,ds\,dr\bigg\}=\mathbb{E}_0\exp\bigg\{\frac{C_{H_0}\theta^2}{\bar t}\int_0^{\bar t}\int_0^{\bar t}\frac{\gamma(B(s)-\widetilde{B}(r))}{|\bar t^{-1}(s-r)|^{\alpha_0}}\,ds\,dr\bigg\}
$$

with $\bar t=t^{\frac{4-\alpha-2\alpha_0}{2-\alpha}}$. By time-changing, and by replacing $C_{H_0}\theta^2$ by θ, (6.7) is equivalent to the proof of

$$
\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\bigg\{\frac{\theta}{t}\int_0^t\int_0^t\frac{\gamma(B(s)-\widetilde{B}(r))}{|t^{-1}(s-r)|^{\alpha_0}}\,ds\,dr\bigg\}\le 2\bigg(\frac{\theta}{2}\bigg)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0)
$$

for every constant $\theta>0$.

By Fourier transform,

$$\begin{aligned}
&\int_0^t \int_0^t \frac{\gamma(B(s)-\widetilde{B}(r))}{|t^{-1}(s-r)|^{\alpha_0}}\,ds\,dr \\
&\quad = C_{H_0}^{-1}\int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\left[\int_0^t e^{i(t^{-1}\lambda s+\xi\cdot B(s))}\,ds\right]\left[\int_0^t e^{-i(t^{-1}\lambda s+\xi\cdot \tilde{B}(s))}\,ds\right] \\
&\quad = C_{H_0}^{-1}\int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi),
\end{aligned}$$

where

$$\eta_t(\lambda,\xi)=\int_0^t e^{i(t^{-1}\lambda s+\xi\cdot B(s))}\,ds \quad \text{and} \quad \tilde{\eta}_t(\lambda,\xi)=\int_0^t e^{-i(t^{-1}\lambda s+\xi\cdot \tilde{B}(s))}\,ds. \tag{6.8}$$

Therefore, (6.7) is equivalent to

$$\limsup_{t\to\infty}\log \mathbb{E}_0\exp\left\{\frac{\theta}{C_{H_0}t}\int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right\}\le 2\left(\frac{\theta}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0). \tag{6.9}$$

Given a small constant $\delta>0$, consider the decomposition

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi) \\
&\quad = \int_{\mathbb{R}^{d+1}} \mu_0^{\delta}(d\lambda)\mu^{\delta}(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)+\int_{\mathbb{R}^{d+1}} \bar{\mu}_0^{\delta}(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi),
\end{aligned} \tag{6.10}$$

where

$$\mu_0^{\delta}(d\lambda)=\int_{\mathbb{R}} e^{-\delta|\lambda|^2}\mu_0(d\lambda) \quad \text{and} \quad \bar{\mu}_0^{\delta}(d\lambda)=\int_{\mathbb{R}}\big(1-e^{-\delta|\lambda|^2}\big)\mu_0(d\lambda).$$

By Hölder inequality, for any two conjugate numbers $\beta,\bar{\beta}>1$

$$\begin{aligned}
&\mathbb{E}_0\exp\left\{\frac{\theta}{C_{H_0}t}\int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right\} \\
&\quad \le \left(\mathbb{E}_0\exp\left\{\frac{\beta\theta}{C_{H_0}t}\int_{\mathbb{R}^{d+1}} \mu_0^{\delta}(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right\}\right)^{1/\beta} \\
&\qquad \times \left(\mathbb{E}_0\exp\left\{\frac{\bar{\beta}\theta}{C_{H_0}t}\int_{\mathbb{R}^{d+1}} \bar{\mu}_0^{\delta}(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right\}\right)^{1/\bar{\beta}}.
\end{aligned} \tag{6.11}$$

Here we make β fixed but close to 1.

To show that the second exponential moment is negligible as δ is sufficiently small, we compute its moments. Indeed,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^{d+1}} \bar{\mu}_0^{\delta}(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right]^n \\
&\quad = \int_{(\mathbb{R}^{d+1})^n} \bar{\mu}_0^{\delta}(d\lambda)\mu(d\xi)\left|\mathbb{E}_0\prod_{k=1}^n \eta_t(\lambda_k,\xi_k)\right|^2 \\
&\quad \le \delta^{\frac{\bar{\alpha}_0-\alpha_0}{2}n}\int_{(\mathbb{R}^{d+1})^n} \hat{\mu}_0(d\lambda)\mu(d\xi)\left|\mathbb{E}_0\prod_{k=1}^n \eta_t(\lambda_k,\xi_k)\right|^2,
\end{aligned}$$

where $\alpha_0 < \bar{\alpha}_0 < 1$,

$$\hat{\mu}_0(d\lambda) = C^n \prod_{k=1}^n |\lambda_k|^{-(1-\bar{\alpha}_0)}\, d\lambda_k$$

for a constant $C > 0$ independent of n, δ and t, and the last step follows from

$$\begin{aligned}\bar{\mu}_0^\delta(d\lambda) &\le \int_{\mathbb{R}^n} \left(\prod_{k=1}^n \bigl(1 - e^{-\delta|\lambda_k|^2}\bigr)^{\frac{\bar{\alpha}_0-\alpha_0}{2}}\right) \mu_0(d\lambda) \\ &\le \delta^{\frac{\bar{\alpha}_0-\alpha_0}{2}n} \int_{\mathbb{R}^n} \left(\prod_{k=1}^n |\lambda_k|^{\bar{\alpha}_0-\alpha_0}\right) \mu_0(d\lambda) = \delta^{\frac{\bar{\alpha}_0-\alpha_0}{2}n} \hat{\mu}_0(d\lambda) \quad \text{(say).}\end{aligned}$$

Let $\bar{\alpha}_0$ be chosen by the constrain that $\bar{H}_0 \equiv 2^{-1}(2 - \bar{\alpha}_0)$ satisfies the second inequality in (1.13), i.e.,

$$4(1 - \bar{H}_0) + 2(d - H) + (d_* - 2H_*) < 4.$$

By the moment bound (3.1) with H_0 being replaced by $\bar{H}_0$,

$$\int_{(\mathbb{R}^{d+1})^n} \hat{\mu}_0(d\lambda)\mu(d\xi) \left| \int_{[0,t]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\lambda_k s_k + \xi_k \cdot B(s_k)} \right) ds \right|^2 \le C^n (n!)^{d-H} t^{(H+2\bar{H}_0-d)n}$$

or, by the Brownian scaling

$$\int_{(\mathbb{R}^{d+1})^n} \hat{\mu}_0(d\lambda)\mu(d\xi) \left| \mathbb{E}_0 \prod_{k=1}^n \eta_t(\lambda_k, \xi_k) \right|^2 \le C^n (n!)^{d-H} t^{(2-(d-H))n}.$$

Summarizing our computation,

$$\mathbb{E}\left[\int_{(\mathbb{R}^{d+1})^n} \bar{\mu}_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right]^n \le C^n \delta^{\frac{\bar{\alpha}_0-\alpha_0}{2}n} (n!)^{d-H} t^{(2-(d-H))n}.$$

For even n, it gives the bound for $\mathbb{E}_0|\cdot|^n$. Further, the same bound for $\mathbb{E}_0|\cdot|^n$ can be extended to odd number $n = 2k+1$ by the Cauchy–Schwartz inequality

$$\mathbb{E}_0|\cdot|^{2k+1} \le \bigl(\mathbb{E}_0[\cdot]^{2k}\bigr)^{1/2}\bigl(\mathbb{E}_0[\cdot]^{2k+2}\bigr)^{1/2}.$$

By Stirling formula, the bound can be reformulated as

$$\mathbb{E}_0 \left| \int_{(\mathbb{R}^{d+1})^n} \bar{\mu}_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi) \right|^{\frac{n}{d-H}} \le C^n n! \delta^{n\frac{\bar{\alpha}_0-\alpha_0}{2(d-H)}} t^{n\frac{2-(d-H)}{d-H}}$$

with $C > 0$ independent of δ, n and t. By Taylor expansion, for any $\delta > 0$

$$\sup_{t\ge1} \mathbb{E}_0 \exp\left\{ \frac{1}{2} C^{-1} \delta^{-\frac{\bar{\alpha}_0-\alpha_0}{2(d-H)}} t^{-\frac{2-(d-H)}{d-H}} \left| \int_{\mathbb{R}^{d+1}} \bar{\mu}_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi) \right|^{\frac{1}{d-H}} \right\} < \infty.$$

On the other hand, for any $A \ge 1$

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{ \frac{\bar{\beta}\theta}{C_{H_0} t} \int_{\mathbb{R}^{d+1}} \bar{\mu}_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi) \right\} \\ &\quad \le \exp\left\{ \frac{\theta\bar{\beta}}{C_{H_0}} A^{-1} t \right\} + \mathbb{E}_0 \exp\left\{ \frac{\bar{\beta}\theta}{C_{H_0}} A^{\frac{1-(d-H)}{d-H}} t^{-\frac{2-(d-H)}{d-H}} \left| \int_{\mathbb{R}^{d+1}} \bar{\mu}_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi) \right|^{\frac{1}{d-H}} \right\}.\end{aligned}$$

Taking δ sufficiently small making the second term on the right hand side bounded in t, therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{\frac{\bar\beta\theta}{C_{H_0}t}\int_{\mathbb{R}^{d+1}}\bar\mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi)\right\}\le\frac{\theta\bar\beta}{C_{H_0}}A^{-1}.$$

The right hand side can be arbitrarily small if A is sufficiently large (it requires that δ be sufficiently small).

Since $\beta>1$ can be made arbitrarily close to 1 in (6.11), we have reduced (6.9) to the proof of

$$\limsup_{t\to\infty}\log\mathbb{E}_0\exp\left\{\frac{\theta}{C_{H_0}t}\int_{\mathbb{R}^{d+1}}\mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi)\right\}\le 2\left(\frac{\theta}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0)\tag{6.12}$$

for any $\delta>0$.

Truncating the space spectral measure $\mu(d\xi)$ is a more delicate issue. Due to the reason that the bound (3.1) is H-sensitive, applying the above procedure to the space component would decrease the space Hurst parameter H and therefore produce a worse bound through (3.1). To prevent it from happening, we carry out a different approach. Let $\delta>0$ be fixed. Given $M\ge 1$, consider the decomposition

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}}\mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi)\\&\quad=\int_{\mathbb{R}\times[-M,M]^d}\mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi)+\int_{\mathbb{R}\times([-M,M]^d)^c}\mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi).\end{aligned}\tag{6.13}$$

Notice that for any integer $n\ge 1$

$$\begin{aligned}&\mathbb{E}_0\left[\int_{\mathbb{R}\times([-M,M]^d)^c}\mu_0(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi)\right]^n\\&\quad=\int_{(\mathbb{R}\times([-M,M]^d)^c)^n}\mu_0^\delta(d\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\prod_{k=1}^n e^{it^{-1}\lambda_k s_k}\right)\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right|^2\\&\quad\le\mu_0^\delta(\mathbb{R})^n\int_{(([-M,M]^d)^c)^n}\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}\right)ds\right|^2\\&\quad=\mu_0^\delta(\mathbb{R})^n\mathbb{E}_0\mathcal{H}_t\big(([-M,M]^d)^c\big)^n,\end{aligned}$$

where

$$\mathcal{H}_t(B)=\int_B\mu(d\xi)\left[\int_0^t e^{i\xi\cdot B(s)}\,ds\right]\left[\int_0^t e^{-i\xi\cdot\tilde B(s)}\,ds\right],\qquad B\subset\mathbb{R}^d$$

and the inequality follows from the fact that

$$\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B(s_k)}>0.$$

Hence, for any $\theta>0$,

$$\begin{aligned}&\mathbb{E}_0\exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times([-M,M]^d)^c}\mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde\eta_t(\lambda,\xi)\right\}\\&\quad\le\mathbb{E}_0\exp\left\{\mu_0^\delta(\mathbb{R})\frac{\theta}{t}\mathcal{H}_t\big(([-M,M]^d)^c\big)\right\}\le\big(\mathbb{E}_0\exp\big\{\mu_0^\delta(\mathbb{R})\theta\big|\mathcal{H}_1\big(([-M,M]^d)^c\big)\big|\big\}\big)^t,\end{aligned}\tag{6.14}$$

where the last step follows from Lemma 6.1.

To complete our procedure of truncating space spectral measure $\mu(d\xi)$, we claim that

$$\lim_{M\to\infty} \mathbb{E}_0 \exp\left\{\mu_0^\delta(\mathbb{R})\theta\left|\mathcal{H}_1\left(\left([-M,M]^d\right)^c\right)\right|\right\} = 1. \tag{6.15}$$

First notice that for any $n \ge 1$, the nth moment

$$\mathbb{E}_0\left[\mathcal{H}_1(B)\right]^n = \int_{B^n} \mu(d\xi) \left|\int_{[0,1]^n} \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B(s_k)}\right) ds\right|^2$$

is monotonic in $B \subset \mathbb{R}^d$.

For the even $n = 2k$,

$$\mathbb{E}_0\left|\mathcal{H}_1\left(\left([-M,M]^d\right)^c\right)\right|^n = \mathbb{E}_0\left[\mathcal{H}_1\left(\left([-M,M]^d\right)^c\right)\right]^n \le \mathbb{E}_0\left[\mathcal{H}_1\left(\mathbb{R}^d\right)\right]^n.$$

As for odd number $n = 2k+1$, by Cauchy–Schwartz inequality,

$$\begin{aligned}
\mathbb{E}_0\left|\mathcal{H}_1\left(\left([-M,M]^d\right)^c\right)\right|^n &\le \left(\mathbb{E}_0\left[\mathcal{H}_1\left(\left([-M,M]^d\right)^c\right)\right]^{2k}\right)^{1/2}\left(\mathbb{E}_0\left[\mathcal{H}_1\left(\left([-M,M]^d\right)^c\right)\right]^{2k+2}\right)^{1/2} \\
&\le \left(\mathbb{E}_0\left[\mathcal{H}_1\left(\mathbb{R}^d\right)\right]^{2k}\right)^{1/2}\left(\mathbb{E}_0\left[\mathcal{H}_1\left(\mathbb{R}^d\right)\right]^{2k+2}\right)^{1/2}.
\end{aligned}$$

Notice the bound (3.1) can be extended easily to the case when $H_0 = 1$ (i.e., the setting of time-independence). So we have the bound

$$\mathbb{E}_0\left[\mathcal{H}_1\left(\mathbb{R}^d\right)\right]^n \le C^n (n!)^{d-H}, \quad n = 1, 2, \ldots.$$

Thus, (6.15) follows from the dominated convergence. By (6.15) and (6.14)

$$\limsup_{M\to\infty}\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}_0 \exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times([-M,M]^d)^c} \mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right\} = 0$$

for every $\theta > 0$. In view of the decomposition (6.13), therefore, an exponential approximation by Hölder inequality reduces (6.12) to the proof of

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}_0 \exp\left\{\frac{\theta}{C_{H_0} t}\int_{\mathbb{R}\times[M,M]^d} \mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi)\right\} \le 2\left(\frac{\theta}{2}\right)^{\frac{2}{2-\alpha}} \mathcal{E}(H_0)$$

for any $\delta > 0$ and $M \ge 1$.

By the relation

$$\begin{aligned}
&\int_{\mathbb{R}\times[M,M]^d} \mu_0^\delta(d\lambda)\mu(d\xi)\eta_t(\lambda,\xi)\tilde{\eta}_t(\lambda,\xi) \\
&\quad \le \frac{1}{2}\left\{\int_{\mathbb{R}\times[M,M]^d} \mu_0^\delta(d\lambda)\mu(d\xi)\left|\eta_t(\lambda,\xi)\right|^2 + \int_{\mathbb{R}\times[M,M]^d} \mu_0^\delta(d\lambda)\mu(d\xi)\left|\tilde{\eta}_t(\lambda,\xi)\right|^2\right\}
\end{aligned}$$

and independence, the problem is further reduced to the proof of

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}_0 \exp\left\{\frac{\theta}{2C_{H_0} t}\int_{\mathbb{R}\times[M,M]^d} \left|\eta_t(\lambda,\xi)\right|^2 \mu_0^\delta(d\lambda)\mu(d\xi)\right\} \le \left(\frac{\theta}{2}\right)^{\frac{2}{2-\alpha}} \mathcal{E}(H_0) \tag{6.16}$$

for any $\delta > 0$ and $M \ge 1$.

On the other hand, for each $n \geq 1$,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}\times[M,M]^d}\big|\eta_t(\lambda,\xi)\big|^2\mu_0^\delta(d\lambda)\mu(d\xi)\bigg]^n \\
&= \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^{2n}}\left(\prod_{k=1}^n\gamma_0^\delta\left(\frac{s_k-r_k}{t}\right)\right)\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot(B(s_k)-B(r_k))}\right)ds\,dr \\
&= \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^{2n}}\left(\prod_{k=1}^n\gamma_0^\delta\left(\frac{s_k-r_k}{t}\right)\right)\exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^n\xi_k\cdot\big(B(s_k)-B(r_k)\big)\right)\right\}ds\,dr. \qquad (6.17)
\end{aligned}$$

Given a small constant $\kappa > 0$, let $\mathbb{P}^\kappa$ and $\mathbb{E}^\kappa$ be the law and expectation, respectively, of an d-dimensional Ornstein–Uhlenbeck process starting from 0 with the infinitesimal generator $2^{-1}\Delta - \kappa x\cdot\nabla$. Whenever associated with $\mathbb{P}^\kappa$ or $\mathbb{E}^\kappa$, $B(s)$ represents an Ornstein–Uhlenbeck process. By Girsanov's theorem,

$$\begin{aligned}
\frac{d\mathbb{P}^\kappa}{d\mathbb{P}_0}\bigg|_{[0,t]} &= \exp\left\{-\kappa\int_0^t B(s)\cdot dB(s) - \frac{\kappa^2}{2}\int_0^t\big|B(s)\big|^2 ds\right\} \\
&= \exp\left\{-\frac{\kappa}{2}\big|B(t)\big|^2 + \frac{\kappa d}{2}t - \frac{\kappa^2}{2}\int_0^t\big|B(s)\big|^2 ds\right\}. \qquad (6.18)
\end{aligned}$$

In particular,

$$\frac{d\mathbb{P}^\kappa}{d\mathbb{P}_0}\bigg|_{[0,t]} \leq \exp\left\{\frac{\kappa d}{2}t\right\}. \qquad (6.19)$$

Applying Lemma 3.9, [10] to the Gaussian laws $\mathbb{P}^\kappa|_{[0,t]}$ and $\mathbb{P}_0|_{[0,t]}$, in connection to (6.17) we have

$$\operatorname{Var}\left(\sum_{k=1}^n\xi_k\cdot\big(B(s_k)-B(r_k)\big)\right) \geq \operatorname{Var}_\kappa\left(\sum_{k=1}^n\xi_k\cdot\big(B(s_k)-B(r_k)\big)\right),$$

where $\operatorname{Var}_\kappa(\cdot)$ represents the variance under the law $\mathbb{P}^\kappa$. Notice that (6.17) remains true when the Brownian motion is replaced by the Ornstein–Uhlenbeck process (or, $\mathbb{E}_0$ and Var are replaced by $\mathbb{E}^\kappa$ and $\operatorname{Var}^\kappa$, resp.). By the fact that $\gamma_0^\delta(\cdot) \geq 0$,

$$\mathbb{E}_0\bigg[\int_{\mathbb{R}\times[M,M]^d}\big|\eta_t(\lambda,\xi)\big|^2\mu_0^\delta(d\lambda)\mu(d\xi)\bigg]^n \leq \mathbb{E}^\kappa\bigg[\int_{\mathbb{R}\times[M,M]^d}\big|\eta_t(\lambda,\xi)\big|^2\mu_0^\delta(d\lambda)\mu(d\xi)\bigg]^n \qquad (6.20)$$

for $n = 1, 2, \dots$.

By Taylor expansion again and parameter substitution, (6.16) is further reduced to the proof of

$$\limsup_{\kappa\to 0^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times[-M,M]^d}\big|\eta_t(\lambda,\xi)\big|^2\mu_0^\delta(d\lambda)\mu(d\xi)\right\} \leq (C_{H_0}\theta)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0). \qquad (6.21)$$

To compute the limit "$t\to\infty$", let $\kappa > 0$ be fixed for a while. As a process with strong ergodicity, Ornstein–Uhlenbeck process has much better property than Brownian motion in terms of the tightness. Indeed, by (6.18),

$$\begin{aligned}
\mathbb{E}^\kappa\exp\left\{\int_0^t\big|B(s)\big|ds\right\} &\leq \mathbb{E}_0\exp\left\{\int_0^t\left(\big|B(s)\big| - \frac{\kappa^2}{2}\big|B(s)\big|^2\right)ds + \frac{\kappa d}{2}t\right\} \\
&\leq \exp\left\{\left(\frac{1}{2\kappa^2} + \frac{\kappa d}{2}\right)t\right\},
\end{aligned}$$

where the last step follows from the elementary relation $\lambda - 2^{-1}\kappa^2\lambda^2 \le (2\kappa^2)^{-1}$ ($\lambda \in \mathbb{R}$). A standard application of Chebyshev inequality shows that for any number $l > 0$ one can make $R > 0$ sufficiently large such that

$$\mathbb{P}^\kappa\left\{\frac{1}{t}\int_0^t \left|B(s)\right| ds \ge R\right\} \le e^{-lt} \quad \forall t > 0.$$

On the other hand, consider the decomposition

$$\begin{aligned}&\mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0^\delta(d\lambda)\mu(d\xi)\right\}\\ &\quad= \mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0^\delta(d\lambda)\mu(d\xi)\right\}1_{\Omega_{t,R}}\\ &\qquad+ \mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0^\delta(d\lambda)\mu(d\xi)\right\}1_{\Omega_{t,R}^c},\end{aligned}$$

where

$$\Omega_{t,R} = \left\{\frac{1}{t}\int_0^t \left|B(s)\right| ds \le R\right\}.$$

For the first term on the right hand side, it is bounded by

$$\exp\left\{\theta t\mu_0^\delta\left([-N,N]^c\right)\mu\left([-M,M]^d\right)\right\}\mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\int_{[-N,N]\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0(d\lambda)\mu(d\xi)\right\}1_{\Omega_{t,R}}$$

for any $N > 0$.

As for the second term, it is bounded by

$$\exp\left\{\theta t\mu_0^\delta(\mathbb{R})\mu\left([-M,M]^d\right)\right\}\mathbb{P}^\kappa\left\{\frac{1}{t}\int_0^t \left|B(s)\right| ds \ge R\right\},$$

which is negligible in comparison to the first term as R is sufficiently large. Therefore

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0^\delta(d\lambda)\mu(d\xi)\right\}\\ &\quad\le \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\int_{[-N,N]\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0(d\lambda)\mu(d\xi)\right\}1_{\Omega_{t,R}}\\ &\qquad+ \theta\mu_0^\delta\left([-N,N]^c\right)\mu\left([-M,M]^d\right)\end{aligned} \tag{6.22}$$

for sufficiently large R and N.

Let R and N be fixed. Consider the Hilbert space $\mathcal{H}$ of all possibly complex valued functions $f(u,x)$ on $[-N,N]\times[-M,M]^d$ such that $f(-\lambda,-\xi) = \overline{h(\lambda,\xi)}$ a.e. and that

$$\|h\|_{\mathcal{H}}^2 \equiv \int_{[-N,N]\times[-M,M]^d}\left|f(\lambda,\xi)\right|^2\mu_0(d\lambda)\mu(d\xi) < \infty.$$

Despite that the functions in $\mathcal{H}$ are allowed to take complex valued, $\mathcal{H}$ is a real space in the sense that for any real numbers c_1, c_2, $c_1h_1 + c_2h_2 \in \mathcal{H}$ as soon as $h_1, h_2 \in \mathcal{H}$.

Notice that $\|h\|_{\mathcal{H}}^2 \ge -\|f\|_{\mathcal{H}}^2 + 2\langle f,h\rangle_{\mathcal{H}}$ for any $h, f \in \mathcal{H}$. Further, if the subspace $\mathcal{H}_0 \subset \mathcal{H}$ is dense in $\mathcal{H}$, then

$$\|h\|_{\mathcal{H}}^2 = \sup_{f\in\mathcal{H}_0}\left\{-\|f\|_{\mathcal{H}}^2 + 2\langle f,h\rangle_{\mathcal{H}}\right\}. \tag{6.23}$$

With $\eta_t(\cdot,\cdot)$ being viewed as a $\mathcal{H}$-valued stochastic process, a crucial fact is that there is a compact set $\mathcal{K}\subset\mathcal{H}$ such that $t^{-1}\eta_t\in\mathcal{K}$ on $\Omega_{t,R}$ for every $t\ge 1$. Indeed, by Arzelá–Ascoli theorem, the class

$$\mathcal{C}=\Big\{h(\cdot,\cdot)\in\mathcal{H};\ \sup_{\lambda,\xi}\big|h(\lambda,\xi)\big|\le 1 \text{ and } \big|h(\lambda_1,\xi_1)-h(\lambda_2,\xi_2)\big|\le(1+R)\big|(\lambda_1,\xi_1)-(\lambda_2,\xi_2)\big|$$
$$\text{for any } (\lambda_1,\xi_1),(\lambda_2,\xi_2)\in[-N,N]\times[-M,M]^d\Big\}$$

is relatively compact in $C([-N,N]\times[-M,M]^d)$ under the uniform topology. Consequently, $\mathcal{C}$ is also relatively compact in $\mathcal{H}$ as the uniform convergence leads to the $\mathcal{L}^2$-convergence.

It is easy to see that for any $t\ge 1$, $t^{-1}\eta_t\in\mathcal{C}$ on $\Omega_{t,R}$. Therefore, one can take $\mathcal{K}$ as the closure of $\mathcal{C}$ in $\mathcal{H}$.

Take $\mathcal{H}_0$ in (6.23) as all bounded functions in $\mathcal{H}$. Given $\epsilon>0$, the sets

$$\mathcal{O}_f=\big\{h\in\mathcal{H};\ \|h\|_{\mathcal{H}}^2<-\|f\|_{\mathcal{H}}^2+2\langle f,h\rangle_{\mathcal{H}}+\epsilon\big\}\quad f\in\mathcal{H}_0$$

form an open cover of $\mathcal{H}$, and therefore of $\mathcal{K}$. Let $\mathcal{O}_{f_1},\dots,\mathcal{O}_{f_m}$ be be a finite sub-cover of $\mathcal{K}$. By the fact that $t^{-1}\eta_t\in\mathcal{K}$ on $\Omega_{t,R}$,

$$\big\|t^{-1}\eta_t\big\|_{\mathcal{H}}^2<\epsilon+\max_{1\le j\le m}\big\{-\|f_j\|_{\mathcal{H}}^2+2\langle f_j,\mathcal{Z}_t\rangle_{\mathcal{H}}\big\}$$

on $\Omega_{t,R}$. Hence,

$$\mathbb{E}^\kappa\exp\bigg\{\frac{\theta}{t}\|\eta_t\|_{\mathcal{H}}^2\bigg\}1_{\Omega_{t,R}}\le e^{\theta\epsilon t}\sum_{j=1}^m e^{-\theta t\|f_j\|_{\mathcal{H}}^2}\mathbb{E}^\kappa\exp\big\{2\theta\langle f_j,\eta_t\rangle_{\mathcal{H}}\big\}.$$

Consequently,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\bigg\{\frac{\theta}{t}\|\eta_t\|_{\mathcal{H}}^2\bigg\}1_{\Omega_{t,R}}\\&\quad\le\theta\epsilon+\max_{1\le j\le m}\bigg\{-\theta\|f_j\|_{\mathcal{H}}^2+\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\kappa\exp\big\{2\theta\langle f_j,\eta_t\rangle_{\mathcal{H}}\big\}\bigg\}.\end{aligned}\tag{6.24}$$

Notice that

$$\langle f_j,\eta_t\rangle_{\mathcal{H}}=\int_{[-N,N]\times[-M,M]^d}f_j(\lambda,\xi)\overline{\eta_t(\lambda,\xi)}\mu_0(d\lambda)\mu(d\xi)=\int_0^t\bar{f}_j\bigg(\frac{s}{t},B(s)\bigg)ds,$$

where

$$\bar{f}_j(s,x)=\int_{[-N,N]\times[-M,M]^d}f(\lambda,\xi)\exp\big\{-i(\lambda s+\xi\cdot x)\big\}\mu_0(d\lambda)\mu(d\xi)$$

is a real-valued function, thank to the symmetry $f_j(-\lambda,-\xi)=\overline{f_j(\lambda,\xi)}$.

In view of (6.19), therefore, we transform the problem from the Ornstein–Uhlenbeck system back to the Brownian system:

$$\mathbb{E}^\kappa\exp\big\{2\theta\langle f_j,Z_t\rangle_{\mathcal{H}}\big\}\le\exp\bigg\{\frac{\kappa d}{2}t\bigg\}\mathbb{E}_0\exp\bigg\{2\theta\int_0^t\bar{f}_j\bigg(\frac{s}{t},B(s)\bigg)ds\bigg\}.\tag{6.25}$$

By the boundedness of $f_j(\lambda,\xi)$, the function $\bar{f}_j(s,x)$ satisfies the regularities assumed in Proposition 3.1, [6]. Hence,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\bigg\{2\theta\int_0^t\bar{f}_j\bigg(\frac{s}{t},B(s)\bigg)ds\bigg\}\\&\quad=\sup_{g\in\mathcal{A}_d}\bigg\{2\theta\int_0^1\int_{\mathbb{R}^d}\bar{f}_j(s,x)g^2(s,x)\,dx\,ds-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\big|\nabla_x g(s,x)\big|^2\,dx\,ds\bigg\}.\end{aligned}\tag{6.26}$$

We point out here that in the variation,

$$\int_0^1 \int_{\mathbb{R}^d} \bar{f}_j(s,x) g^2(s,x)\,dx\,ds = \langle f_j, \mathcal{F}(g^2)\rangle_{\mathcal{H}},$$

where

$$\mathcal{F}(g^2)(\lambda,\xi) = \int_0^1 \int_{\mathbb{R}^d} g^2(s,x) e^{i(\lambda s+\xi\cdot x)}\,dx\,ds.$$

Summarizing the computation since (6.24),

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\|\eta_t\|_{\mathcal{H}}^2\right\} 1_{\Omega_{t,R}} \\
&\quad \le \theta\epsilon + \frac{\kappa d}{2} + \max_{1\le j\le m}\left\{-\theta\|f_j\|_{\mathcal{H}}^2 + \sup_{g\in\mathcal{A}_d}\left\{2\theta\langle f_j, \mathcal{F}(g^2)\rangle_{\mathcal{H}} - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d} \left|\nabla_x g(s,x)\right|^2 dx\,ds\right\}\right\}.
\end{aligned}$$

Noticing that

$$\begin{aligned}
&-\theta\|f_j\|_{\mathcal{H}}^2 + \sup_{g\in\mathcal{A}_d}\left\{2\theta\langle f_j, \mathcal{F}(g^2)\rangle_{\mathcal{H}} - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d} \left|\nabla_x g(s,x)\right|^2 dx\,ds\right\} \\
&\quad = \sup_{g\in\mathcal{A}_d}\left\{\theta\left(-\|f_j\|_{\mathcal{H}}^2 + 2\langle f_j, \mathcal{F}(g^2)\rangle_{\mathcal{H}}\right) - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d} \left|\nabla_x g(s,x)\right|^2 dx\,ds\right\} \\
&\quad \le \sup_{g\in\mathcal{A}_d}\left\{\theta\left\|\mathcal{F}(g^2)\right\|_{\mathcal{H}}^2 - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d} \left|\nabla_x g(s,x)\right|^2 dx\,ds\right\}
\end{aligned}$$

and that (1.10)

$$\begin{aligned}
\left\|\mathcal{F}(g^2)\right\|_{\mathcal{H}}^2 &= \int_{[-N,N]\times[-M,M]^d} \left|\mathcal{F}(g^2)\right|^2 \mu_0(d\lambda)\mu(d\xi) \le \int_{\mathbb{R}^{d+1}} \left|\mathcal{F}(g^2)\right|^2 \mu_0(d\lambda)\mu(d\xi) \\
&= C_{H_0}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\gamma(x-y)}{|s-r|^{\alpha_0}} g^2(s,x) g^2(r,y)\,dx\,dy\,ds\,dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}^\kappa \exp\left\{\frac{\theta}{t}\|\eta_t\|_{\mathcal{H}}^2\right\} 1_{\Omega_{t,R}} \\
&\quad \le \theta\epsilon + \frac{\kappa d}{2} + \sup_{g\in\mathcal{A}_d}\left\{C_{H_0}\theta\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\gamma(x-y)}{|s-r|^{\alpha_0}} g^2(s,x)g^2(r,y)\,dx\,dy\,ds\,dr\right. \\
&\qquad \left. - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2 dx\,ds\right\} = \theta\epsilon + \frac{\kappa d}{2} + (C_{H_0}\theta)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0),
\end{aligned}$$

where the last step follows from the time-space homogeneity and the variational substitution

$$g(s,x) \mapsto (C_{H_0}\theta)^{\frac{d}{2(2-\alpha)}} g\left(s, (C_{H_0}\theta)^{\frac{1}{2-\alpha}} x\right).$$

By (6.22),

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\kappa}\exp\left\{\frac{\theta}{t}\int_{\mathbb{R}\times[-M,M]^d}\left|\eta_t(\lambda,\xi)\right|^2\mu_0^{\delta}(d\lambda)\mu(d\xi)\right\}$$
$$\le\theta\epsilon+\frac{\kappa d}{2}+\theta\mu_0^{\delta}\big([-N,N]^c\big)\mu\big([-M,M]^d\big)+(C_{H_0}\theta)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0).$$

Letting $\epsilon\to 0^+$ and $N\to\infty$, the bound on the right hand side tends to

$$\frac{\kappa d}{2}+(C_{H_0}\theta)^{\frac{2}{2-\alpha}}\mathcal{E}(H_0).$$

Thus, the desired upper bound (6.21) is obtained.

6.3. *Upper bounds for* (1.26) *and* (1.27)

Thank to a recent hypercontractivity inequality by Lê [18], the proof becomes a simple corollary of (6.7). To state Lê's result, we introduce the notation $u_\lambda(t,x)$ for the parabolic Anderson equation

$$\begin{cases}\frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\Delta u(t,x)+\sqrt{\lambda}\dot{W}(t,x)\diamond u(t,x), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ u(0,x)=1, & x\in\mathbb{R}^d\end{cases}\tag{6.27}$$

for any $\lambda>0$. As a special case of his result (Theorem 1, [18]) Lê proves that for any $p\ge 2$,

$$\left\|u_\lambda(t,x)\right\|_{\mathcal{L}^p(\Omega)}\le\left\|u_{(p-1)\lambda}(t,x)\right\|_{\mathcal{L}^2(\Omega)}\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\tag{6.28}$$

whenever the right hand side is finite.

In the setting of this paper (in particular, $\lambda=\theta^2$), therefore,

$$\mathbb{E}u^p(t,x)\le\left(\mathbb{E}_0\exp\left\{(p-1)\theta^2C_{H_0}\int_0^t\int_0^t\frac{\gamma(B(s)-\widetilde{B}(r))}{|s-r|^{\alpha_0}}\,ds\,dr\right\}\right)^{p/2}$$
$$=\left(\mathbb{E}_0\exp\left\{\theta^2C_{H_0}\int_0^{t(p-1)^{\frac{2}{4-\alpha-2\alpha_0}}}\int_0^{t(p-1)^{\frac{2}{4-\alpha-2\alpha_0}}}\frac{\gamma(B(s)-\widetilde{B}(r))}{|s-r|^{\alpha_0}}\,ds\,dr\right\}\right)^{p/2}$$
$$=\left(\mathbb{E}u^2\big(t(p-1)^{\frac{2}{4-\alpha-2\alpha_0}},x\big)\right)^{p/2}.$$

Thus, the upper bounds of (1.26) and (1.27) follow from (6.7).

6.4. *Proof of* (1.28)

Existing evidence shows that the hypercontractivity inequality (6.28) is no longer asymptotically sharp when comes to the time-white setting. Besides, the large t intermittency is not fully understood even for the case $p=2$. Differently, (1.28) is given as a consequence of the free-energy problem studied in [7].

By a usual practice of Hölder inequality, one need only to evaluate the limit along the integers. By the Feynman–Kac moment representation (1.17), therefore, the problem is to prove

$$\lim_{m\to\infty}m^{-\frac{4-\alpha}{2-\alpha}}\log\mathbb{E}_0\exp\left\{\theta^2\sum_{1\le j<k\le m}\int_0^t\gamma\big(B_j(s)-B_k(s)\big)\,ds\right\}=\left(\frac{\theta^2}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}.\tag{6.29}$$

Notice that (4.1) leads to exponential integrability described by (2.19). By Theorem 1.1, [7],

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{mt_m}\log\mathbb{E}_0\exp\left\{\frac{\theta^2}{m}\sum_{1\le j<k\le m}\int_0^{t_m}\gamma\big(B_j(s)-B_k(s)\big)\,ds\right\}\\
&\quad=\sup_{g\in\mathcal{F}_d}\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,dx\,dy-\frac{1}{2}\int_{\mathbb{R}^d}\big|\nabla g(x)\big|^2\,dx\right\}\\
&\quad=\left(\frac{\theta^2}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}
\end{aligned}
\tag{6.30}
$$

for any positive sequence with $t_m\to\infty$, where the last step follows from homogeneity and suitable variable substitution.

Hence, (6.29) follows from the identity in law

$$
\sum_{1\le j<k\le m}\int_0^t\gamma(B_j(s)-B_k(s)\overset{d}{=}\frac{1}{m}\sum_{1\le j<k\le m}\int_0^{tm^{\frac{2}{2-\alpha}}}\gamma\big(B_j(s)-B_k(s)\big)\,ds
$$

and (6.30) with $t_m=m^{\frac{\alpha}{2-\alpha}}$.

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