DOI: 10.37863/umzh.v72i9.6278

UDC 519.21

Xia Chen (Univ. Tennessee, USA)

CONDITION FOR INTERSECTION OCCUPATION MEASURE TO BE ABSOLUTELY CONTINUOUS*

УМОВА АБСОЛЮТНОЇ НЕПЕРЕВНОСТІ МІРИ ВІДВІДУВАНЬ ПЕРЕТИНІВ

Given the i.i.d. $\mathbb{R}^d$-valued stochastic processes $X_1(t), \ldots, X_p(t),\ p \geq 2$, with the stationary increments, a minimal condition is provided for the occupation measure

$$\mu_t(B) = \int_{[0,t]^p} 1_B\big(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p)\big) ds_1 \ldots ds_p, \quad B \subset \mathbb{R}^{d(p-1)},$$

to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{d(p-1)}$. An isometry identity related to the resulting density (known as intersection local time) is also established.

Для незалежних однаково розподільних $\mathbb{R}^d$-значних випадкових процесів $X_1(t), \ldots, X_p(t),\ p \geq 2$ зі стаціонарними приростами наведено мінімальну умову, коли міра відвідувань перетинів

$$\mu_t(B) = \int_{[0,t]^p} 1_B\big(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p)\big) ds_1 \ldots ds_p, \quad B \subset \mathbb{R}^{d(p-1)},$$

абсолютно неперервна відносно міри Лебега на $\mathbb{R}^{d(p-1)}$. Також доведено ізометричну тотожність, пов'язану із відповідною щільністю (відомою як локальний час перетинів).

1. Main theorem. Let $X(t)$ be a stochastic process taking values in $\mathbb{R}^d$ with $X(0) = 0$ and let $p_t(x)$ $(x \in \mathbb{R}^d)$ be the density function of $X(t)$. Assume that, for any $0 \leq s < t$,

$$X(t) - X(s) \overset{d}{=} X(t-s). \tag{1.1}$$

Let $X_1(t), \ldots, X_p(t)$ be independent copies of $X(t)$. Given $t_1, \ldots, t_p \geq 0$ and $\mathbf{x} \in \mathbb{R}^{d(p-1)}$, the intersection local time $\alpha(t_1, \ldots, t_p, \mathbf{x})$ of $X_1(t), \ldots, X_p(t)$ formally written as

$$\alpha(t_1, \ldots, t_p, \mathbf{x}) = \int_0^{t_1} \ldots \int_0^{t_p} \delta_{\mathbf{x}}\big(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p)\big) ds_1 \ldots ds_p \tag{1.2}$$

is defined as Radon – Nikodym derivative of the occupation measure

$$\mu_{t_1,\ldots,t_p}(B) = \int_0^{t_1} \ldots \int_0^{t_p} 1_B\big(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p)\big) ds_1 \ldots ds_p \tag{1.3}$$

with respect to the Lebesgue measure on $\mathbb{R}^{d(p-1)}$. The most-investigated setting is when $X(t)$ is a Brownian motion. The criteria on the mutual intersection of independent Brownian motions was completed by Dvoretzsky, Erdös and Kakutani [3, 4] in 1950s. Their work was followed by the extensive investigations either on the trajectory properties of the Brownian intersection local times (see, e.g., [1, 7, 8]), or on the extension to some other stochastic processes (see, e.g., [2, 5, 6]).

* This paper was partially supported by the Simons (Foundation No. 585506).

A critical step to construct intersection local times is to establish the absolute continuity of $\mu_{t_1,\ldots,t_p}(\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^{d(p-1)}$. In literature, this is mostly done either for Gaussian processes [2, 6] or for Markov processes [5] with Gaussian/Markovian property being used as the main tool. In this short notes, we carry this step out without the help from Gaussian/Markovian property. The main result of this paper is the following theorem.

Theorem 1.1. *Under* (1.1), *assume that there exists* $\theta > 0$ *such that*

$$\int\limits_{\mathbb{R}^d} \left[\int\limits_0^\infty e^{-\theta t} p_t(x) ds \right]^p dx < \infty. \tag{1.4}$$

Then

$$\mathbb{P}\left\{\mu_{t_1,\ldots,t_p} \text{ is absolutely continuous for all } t_1,\ldots,t_p \ge 0\right\} = 1. \tag{1.5}$$

Further, the density $\alpha(t_1,\ldots,t_p,\mathbf{x})$ *given in* (1.2) *lives in* $\mathcal{L}^2$*-space, i.e.,*

$$\mathbb{P}\left\{ \int\limits_{\mathbb{R}^{d(p-1)}} \big[\alpha(t_1,\ldots,t_p,\mathbf{x}\big]^2 d\mathbf{x} < \infty \text{ for all } t_1,\ldots,t_p \ge 0 \right\} = 1 \tag{1.6}$$

and satisfies the isometry identity

$$\mathbb{E} \int\limits_{\mathbb{R}^{d(p-1)}} \big[\alpha(t_1,\ldots,t_p,\mathbf{x})\big]^2 d\mathbf{x} = \int\limits_{\mathbb{R}^d} \left[\prod_{j=1}^p \int\limits_0^{t_j} (t_j - s)\big\{p_s(x) + p_s(-x)\big\} ds \right] dx \tag{1.7}$$

for any $t_1,\ldots,t_p \ge 0$.

Remark **1.1.** In the special case, when $X(t)$ is symmetric, i.e., $X(-t) \stackrel{d}{=} X(t)$ (or $p_s(x) = = p_s(-x)$) for every $t \ge 0$, the isometry identity (1.7) becomes

$$\mathbb{E} \int\limits_{\mathbb{R}^{d(p-1)}} \big[\alpha(t_1,\ldots,t_p,\mathbf{x})\big]^2 d\mathbf{x} = 2^p \int\limits_{\mathbb{R}^d} \left[\prod_{j=1}^p \int\limits_0^{t_j} (t_j - s) p_s(x) ds \right] dx. \tag{1.8}$$

Proof of Theorem **1.1.** For any measure μ on $\mathbb{R}^{d(p-1)}$, its Fourier transform is defined as

$$\hat{\mu}(\lambda_1,\ldots,\lambda_{p-1}) = \int\limits_{\mathbb{R}^{d(p-1)}} \exp\left\{ i \sum_{j=1}^{p-1} \lambda_j x_j \right\} \mu(dx_1 \ldots dx_{p-1}).$$

For any $\theta > 0$, define the random measure

$$\mu_\theta(B) = \int\limits_{(\mathbb{R}^+)^p} \exp\left\{ -\theta \sum_{j=1}^p t_j \right\} \mu_{t_1,\ldots,t_p}(B) dt_1 \ldots dt_p.$$

Notice the fact that

$$\mu_{t_1,\ldots,t_p}(\cdot) \le \exp\left\{\theta\sum_{j=1}^{p} t_j\right\}\mu_\theta(\cdot), \quad t_1,\ldots,t_p \ge 0.$$

To show (1.5) and (1.6), all we need is to establish the almost sure absolute continuity for μ_θ (for some $\theta > 0$) and square integrability for the consequential density of the measure $\mu_\theta(\cdot)$. According to Plancherel – Parseval theorem (Theorem B.3, in [1, p. 302]), this is validated by the integrability

$$\int_{(\mathbb{R}^d)^{p-1}} \mathbb{E}|\hat\mu_\theta(\lambda_1,\ldots,\lambda_{p-1})|^2 d\lambda_1\ldots d\lambda_{p-1} < \infty. \tag{1.9}$$

To establish (1.9) and the isometry (1.7), we first prove that

$$\int_{(\mathbb{R}^d)^{p-1}} \mathbb{E}|\hat\mu_{t_1,\ldots,t_p}(\lambda_1,\ldots,\lambda_{p-1})|^2 d\lambda_1\ldots d\lambda_{p-1} =$$

$$= (2\pi)^{d(p-1)}\int_{\mathbb{R}^d}\left[\prod_{j=1}^{p}\int_0^{t_j}(t_j - s)\big\{p_s(x) + p_s(-x)\big\}ds\right]dx, \quad t_1,\ldots,t_p \ge 0. \tag{1.10}$$

Notice that

$$\hat\mu_{t_1,\ldots,t_p}(\lambda_1,\ldots,\lambda_{p-1}) = \int_{[0,t_1]\times\ldots[0,t_p]} \exp\left\{i\sum_{j=1}^{p-1}\lambda_j\big(X_j(s_j) - X_{j+1}(s_{j+1})\big)\right\}ds_1\ldots ds_p =$$

$$= \prod_{j=1}^{p}\int_0^{t_j}\exp\big\{i(\lambda_j - \lambda_{j-1})X_j(s)\big\}ds.$$

Here and elsewhere we follow the convention that $\lambda_0 = \lambda_p = 0$.

Therefore,

$$|\hat\mu_{t_1,\ldots,t_p}(\lambda_1,\ldots,\lambda_{p-1})|^2 = \hat\mu_{t_1,\ldots,t_p}(\lambda_1,\ldots,\lambda_{p-1})\overline{\hat\mu_{t_1,\ldots,t_p}(\lambda_1,\ldots,\lambda_{p-1})} =$$

$$= \prod_{j=1}^{p}\int_0^{t_j}\int_0^{t_j}\exp\big\{i(\lambda_j - \lambda_{j-1})\big(X_j(s) - X_j(r)\big)\big\}dsdr.$$

Take expectation on the both sides. By the independence of $X_1,\ldots,X_p$ and by the increment stationarity given in (1.1), we get

$$\mathbb{E}|\hat\mu(\lambda_1,\ldots,\lambda_{p-1})|^2 =$$

$$= \prod_{j=1}^{p}\int_0^{t_j}\int_0^{t_j}\mathbb{E}\exp\big\{i(\lambda_j - \lambda_{j-1})\big(X_j(s) - X_j(r)\big)\big\}dsdr =$$

$$= \prod_{j=1}^{p}\left\{\int\int_{\{0\le r<s\le t_j\}}\mathbb{E}\exp\big\{i(\lambda_j - \lambda_{j-1})X(s - r)\big\} + \right.$$

$$+\int\int_{\{0\le s<r\le t_j\}} \mathbb{E}\exp\big\{-i(\lambda_j-\lambda_{j-1})X(r-s)\big\}dsdr\Bigg\}=$$

$$=2^p\prod_{j=1}^{p}\int_0^{t_j}(t_j-s)\varphi_s(\lambda_j-\lambda_{j-1})ds=2^p\prod_{j=1}^{p}Q_{t_j}(\lambda_j-\lambda_{j-1}), \tag{1.11}$$

where

$$\varphi_t(\lambda)=\frac{1}{2}\left\{\mathbb{E}e^{i\lambda X(t)}+\mathbb{E}e^{-i\lambda X(t)}\right\}=\mathbb{E}e^{iR\lambda X(t)},\quad Q_t(\lambda)=\int_0^t(t-s)\varphi_s(\lambda)dt,$$

and R is a Bernoulli random variable independent of $X(t)$ with the distribution

$$\mathbb{P}\{R=1\}=\mathbb{P}\{R=-1\}=1/2.$$

Integrating on the both sides, we get

$$\int_{(\mathbb{R}^d)^{p-1}}\mathbb{E}|\hat{\mu}_{t_1,\ldots,t_p}(\lambda_1,\ldots,\lambda_{p-1})|^2d\lambda_1\ldots d\lambda_{p-1}=$$

$$=2^p\int_{(\mathbb{R}^d)^{p-1}}\prod_{j=1}^{p}Q_{t_j}(\lambda_j-\lambda_{j-1})d\lambda_1\ldots d\lambda_{p-1}=$$

$$=2^p\int_{(\mathbb{R}^d)^{p-1}}Q_{t_p}(-\gamma_1-\ldots-\gamma_{p-1})\prod_{j=1}^{p-1}Q_{t_j}(\gamma_j)d\gamma_1\ldots d\gamma_{p-1}, \tag{1.12}$$

where the last step follows from the substitution $\gamma_1=\lambda_1,\ \gamma_2=\lambda_2-\lambda_1,\ldots,\gamma_{p-1}=\lambda_{p-1}-\lambda_{p-2}$ (recall the convention $\lambda_0=\lambda_p=0$),

Set

$$G_t(x)=\int_0^t(t-s)\frac{p_s(x)+p_s(-x)}{2}ds.$$

The following steps are set for the justification of using Fubini's theorem. Notice that

$$\int_{\mathbb{R}^d}G_t(x)dx=\int_0^t(t-s)\left[\int_{\mathbb{R}^d}\frac{p_s(x)+p_s(-x)}{2}dx\right]ds=\int_0^t(t-s)ds=\frac{t^2}{2}<\infty.$$

Since

$$Q_t(\lambda)=\int_{\mathbb{R}^d}e^{i\lambda\cdot x}G_t(x)dx,$$

we have $|Q_t(\lambda)|\le t^2/2$. In particular, for any $\varepsilon>0$,

$$\int\limits_{(\mathbb{R}^d)^{p-1}\times\mathbb{R}^d} G_{t_p}(x)\prod_{j=1}^{p-1}|Q_{t_j}(\lambda_j)|\exp\left\{-\frac{\varepsilon}{2}|\lambda_k|^2\right\}d\lambda_1\ldots d\lambda_{p-1}dx\le$$

$$\le\left(\prod_{j=1}^{p}\frac{t_j^2}{2}\right)\left(\int\limits_{\mathbb{R}^d}\exp\left\{-\frac{\varepsilon}{2}|\lambda|^2\right\}d\lambda\right)^{p-1}<\infty.$$

This justifies the use of Fubini's theorem in the following way:

$$\int\limits_{(\mathbb{R}^d)^{p-1}} Q_{t_p}(-\gamma_1-\ldots-\gamma_{p-1})\prod_{j=1}^{p-1}Q_{t_j}(\gamma_j)\exp\left\{-\frac{\varepsilon}{2}|\gamma_j|_2^2\right\}d\gamma_1\ldots d\gamma_{p-1}=$$

$$=\int\limits_{(\mathbb{R}^d)^{p-1}}\left[\int\limits_{\mathbb{R}^d}e^{-i(\gamma+\ldots+\gamma_{p-1})}G_{t_p}(x)dx\right]\prod_{j=1}^{p-1}Q_{t_j}(\gamma_j)\exp\left\{-\frac{\varepsilon}{2}|\gamma_j|_2^2\right\}d\gamma_1\ldots d\gamma_{p-1}=$$

$$=\int\limits_{\mathbb{R}^d}G_{t_p}(x)\left[\prod_{j=1}^{p-1}\int\limits_{\mathbb{R}^d}e^{-i\lambda\cdot x}Q_{t_j}(\lambda)\exp\left\{-\frac{\varepsilon}{2}|\lambda|_2^2\right\}d\lambda\right]dx=$$

$$=(2\pi)^{d(p-1)}\int\limits_{\mathbb{R}^d}G_{t_p}(x)\prod_{j=1}^{p-1}(G_{t_j}*\phi_\varepsilon)(x)dx,$$

where $\phi_\varepsilon(x)$ is the density of the d-dimensional normal distribtion $N(\mathbf{0},\varepsilon\mathbf{I}_{d\times d})$ ($I_{d\times d}$ is the $(d\times d)$ identity metrix) and the last step follows from Fourier inverse transform.

We now let $\varepsilon\to 0^+$ on the both sides. First, from (1.11) one can see that

$$Q_{t_p}(-\gamma_1-\ldots-\gamma_{p-1})\prod_{j=1}^{p-1}Q_{t_j}(\gamma_j)\ge 0\quad\forall(\gamma_1,\ldots,\gamma_{p-1})\in\mathbb{R}^{d(p-1)}$$

with a proper variable substitution. By monotonic convergence, therefore, the left-hand side increases to

$$\int\limits_{(\mathbb{R}^d)^{p-1}}Q_{t_p}(-\gamma_1-\ldots-\gamma_{p-1})\prod_{j=1}^{p-1}Q_{t_j}(\gamma_j)d\gamma_1\ldots d\gamma_{p-1}$$

regardless finity or infinity of the limit.

In view of (1.12), to prove (1.10), it suffices to show that

$$\lim_{\varepsilon\to0^+}\int\limits_{\mathbb{R}^d}G_{t_p}(x)\prod_{j=1}^{p-1}(G_{t_j}*\phi_\varepsilon)(x)dx=\int\limits_{\mathbb{R}^d}\left(\prod_{j=1}^{p}G_{t_j}(x)\right)dx. \tag{1.13}$$

Indeed, for any $\theta>0$

$$\int\limits_0^\infty e^{-\theta t}G_t(x)dt=\left(\int\limits_0^\infty te^{-\theta t}dt\right)\left(\int\limits_0^\infty e^{-\theta t}\frac{p_t(x)+p_s(-x)}{2}dt\right)=$$

$$= \theta^{-2} \int\limits_0^\infty e^{-\theta t} \frac{p_t(x) + p_s(-x)}{2} dt. \tag{1.14}$$

Therefore, by the condition (1.4), we obtain

$$\int\limits_{\mathbb{R}^d} G_t^p(x) dx < \infty \quad \forall t \geq 0.$$

By Lemma 2.2.2 in [1, p. 28],

$$\lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^d} \left| (G_{t_j} * \phi_\varepsilon)(x) - G_{t_j}(x) \right|^p dx = 0, \quad j = 1, \ldots, p-1.$$

This clearly leads to (1.13).

It remains to prove (1.9) and (1.7). For (1.9), simply notice that

$$\int\limits_{(\mathbb{R}^d)^{p-1}} \mathbb{E} |\hat{\mu}_\theta(\lambda_1, \ldots, \lambda_{p-1})|^2 d\lambda_1 \ldots d\lambda_{p-1} =$$

$$= \int\limits_{(\mathbb{R}^d)^{p-1}} \mathbb{E} \left| \int\limits_{(\mathbb{R}^+)^p} dt_1 \ldots dt_p \exp\left\{ -\theta \sum_{j=1}^p t_j \right\} \hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \right|^2 d\lambda_1 \ldots d\lambda_{p-1} \leq$$

$$\leq \theta^{-p} \int\limits_{(\mathbb{R}^+)^p} dt_1 \ldots dt_p \exp\left\{ -\theta \sum_{j=1}^p t_j \right\} \int\limits_{(\mathbb{R}^d)^{p-1}} \mathbb{E} \left| \hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \right|^2 d\lambda_1 \ldots d\lambda_{p-1} =$$

$$= (2\pi)^{d(p-1)} \theta^{-p} \int\limits_{(\mathbb{R}^+)^p} dt_1 \ldots dt_p \exp\left\{ -\theta \sum_{j=1}^p t_j \right\} \int\limits_{\mathbb{R}^d} \left[\prod_{j=1}^p \int\limits_0^{t_j} (t_j - s) \{ p_s(x) + p_s(-x) \} ds \right] dx =$$

$$= (2\pi)^{d(p-1)} \theta^{-3p} \int\limits_{\mathbb{R}^d} \left[\prod_{j=1}^p \int\limits_0^\infty e^{-\theta t} \{ p_t(x) + p_t(-x) \} dt \right] dx \leq$$

$$\leq 2^p (2\pi)^{d(p-1)} \theta^{-3p} \int\limits_{\mathbb{R}^d} \left[\int\limits_0^\infty e^{-\theta t} p_t(x) dt \right]^p dx,$$

where the second, the third and the fouth steps follow from Jensen inequality, (1.10) and (1.14), respectively. Therefore, (1.9) follows from (1.4).

We have established (1.5) and (1.6). Further, by Parseval's identity, we have

$$\int\limits_{\mathbb{R}^{d(p-1)}} \left[\alpha(t_1, \ldots, t_p, \mathbf{x}) \right]^2 d\mathbf{x} = (2\pi)^{-d(p-1)} \int\limits_{(\mathbb{R}^d)^{p-1}} \left| \hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \right|^2 d\lambda_1 \ldots d\lambda_{p-1}.$$

This, together with (1.10), proves the identity (1.7).

Theorem 1.1 is proved.

We end this section with the following comment: The density $\alpha(t_1,\ldots,t_p,\mathbf{x})$ addressed in Theorem 1.1 exists only in the form of equivalent class — a fact that brings some inconvinience when it comes to application. For instance, it becomes ambguous to talk about $\alpha(t_1,\ldots,t_p,0)$ for given $t_1,\ldots,t_p$ as $\alpha(t_1,\ldots,t_p,0)$ represents a class of random variables such that any two members of this class are equal to each other with probability 1. The treatment is to find a continuous modification of $\alpha(t_1,\ldots,t_p,x)$. A standard procedure by Kolmogorov extension theory requires the local Hölder type of moment continuity

$$\mathbb{E}\Big|\alpha(t_1,\ldots,t_p,\mathbf{x})-\alpha(s_1,\ldots,s_p,\mathbf{y})\Big|^m\le C\left\{\|\mathbf{t}-\mathbf{s}\|+\|\mathbf{x}-\mathbf{y}\|\right\}^\beta \tag{1.15}$$

for some $m>0$ and $\beta>d+1$. This can not be achieved without extra assumption. In the setting when $X(t)$ is Gaussian, for example, (1.15) can be installed under some nonlocal determinism conditions (Theorem 2.8 in [6]).

2. Applications to Gaussian processes. Let $X(t)$ be a $\mathbb{R}^d$-valued stochastic process satisfying our point-wise increment-stationarity given in (1.1). In addition, assume that $X(t)$ is point-wisely Gaussian:

$$X(t)\sim\mathcal{N}(\mathbf{0},\mathbf{\Sigma}(t)),\quad t\ge 0, \tag{2.1}$$

where $\mathbf{\Sigma}(t))$ is a nonnegative definite $(d\times d)$-matrix.

Theorem 2.1. *Assume that* (1.1) *and* (2.1) *are true. The condition* (1.4) *is satisfied if*

$$\int_0^\infty e^{-\theta t}\det\big(\mathbf{\Sigma}(t)\big)^{-\frac{p-1}{2p}}dt<\infty \quad \textit{for some } \theta>0. \tag{2.2}$$

Consequently, all statements in Theorem 1.1 *hold under* (2.2).

Proof. Notice that

$$\int_{\mathbb{R}^d}\left[\int_0^\infty e^{-\theta t}p_t(x)dt\right]^p dx=\int_0^\infty\ldots\int_0^\infty dt_1\ldots dt_p e^{-(t_1+\ldots+t_p)}\int_{\mathbb{R}^d}\prod_{j=1}^p p_{t_j}(x)dx\le$$

$$\le\int_0^\infty\ldots\int_0^\infty dt_1\ldots dt_p\prod_{j=1}^p\left\{\int_{\mathbb{R}^d}\Big(p_{t_j}(x)\Big)^p dx\right\}^{1/p}=$$

$$=\left\{\int_0^\infty e^{-\theta t}\left[\int_{\mathbb{R}^d}\Big(p_t(x)\Big)^p dx\right]^{1/p}dt\right\}^p.$$

From (2.2), $\Sigma(t)$ is positive-definite allmost everywhere in t. Therefore,

$$\int_{\mathbb{R}^d}\Big(p_t(x)\Big)^p dx=(2\pi)^{-dp/2}\det\big(\mathbf{\Sigma}(t)\big)^{-p/2}\int_{\mathbb{R}^d}\exp\left\{-\frac{p}{2}\langle x,\mathbf{\Sigma}(t)^{-1}x\rangle\right\}dx=$$

$$= p^{-d/2}(2\pi)^{-\frac{d(p-1)}{2}} \det\big(\mathbf{\Sigma}(t)\big)^{-\frac{p-1}{2}} \quad \text{a.e.}$$

Hence, by the condition (1.14) we get

$$\int_{\mathbb{R}^d} \left[\int_0^\infty e^{-\theta t} p_t(x)dt \right]^p dx \le p^{-d/2}(2\pi)^{-\frac{d(p-1)}{2}} \left\{ \int_0^\infty e^{-\theta t} \det\big(\mathbf{\Sigma}(t)\big)^{-\frac{p-1}{2p}} dt \right\}^p < \infty.$$

In the rest of this section, we consider two examples.

***Example* 2.1.** Let $X(t)$ be a d-dimensional fractional Brownian motion with the Hurst parameter $(H_1, \ldots, H_d)$, $(H_1, \ldots, H_d) \in (0,1)$. That is, the components $B_1^{H_1}(t), \ldots, B_d^{H_d}(t)$ of $X(t)$ are independent mean-zero Gaussian process with the covariance functions given as

$$\operatorname{Cov}\big(B_j^{H_j}(t), B_j^{H_j}(s)\big) = \frac{1}{2}\big(|t|^{2H_j} + |s|^{2H_j} - |t-s|^{2H_j}\big), \quad j = 1, \ldots, p.$$

In particular, $\mathbf{\Sigma}(t)$ is diagonal with diagonal elements $|t|^{2H_1}, \ldots |t|^{2H_d}$. Hence

$$\det\big(\mathbf{\Sigma}(t)\big) = \prod_{j=1}^p |t|^{2H_j} = t^{2(H_1 + \ldots + H_d)}, \quad t \ge 0.$$

Thus, the condition (2.2) is equivalent to

$$H_1 + \ldots + H_d < \frac{p}{p-1}.$$

To compare it to the known result, we consider the special case when $H_1 = \ldots = H_d = H$. In this case the above inequality becomes

$$dH < \frac{p}{p-1}.$$

This is the condition given in [2] ((5.7)) for existence of intersection local times of fractional Brownian motions with identically distributed components.

***Example* 2.2.** An 1-dimensional Ornstein – Uhlenbeck process $U_1(t)$ is a mean-zero stationary Gaussian process with covariance function

$$\operatorname{Cov}\big(U_1(0), U_1(t)\big) = e^{-t/2}, \quad t \ge 0. \tag{2.3}$$

A d-dimensional Ornstein – Uhlenbeck process $U(t) = \big(U_1(t), \ldots, U_d(t)\big)$ takes i.i.d. 1-dimensional Ornstein – Uhlenbeck processes $U_1(t), \ldots, U_d(t)$ as components. In the following discussion, $U(t)$ is a d-dimensional Ornstein – Uhlenbeck process. Set

$$X(t) = \int_0^t U(s)ds, \quad t \ge 0.$$

Then $X(t)$ satisfies (1.1) and (2.1). To compute $\det\big(\mathbf{\Sigma}(t)\big)$, notice that

$$\mathbf{\Sigma}(t) = \int_0^t \int_0^t \operatorname{Cov}\big(U(s), U(r)\big) ds dr = \int_0^t \int_0^t \exp\left\{-\frac{1}{2}|s-r|\right\} \mathbf{I}_{d\times d} ds dr =$$

$$= 4\Big[t - 2\big(1 - e^{-t/2}\big)\Big]\mathbf{I}_{d\times d},$$

where $\mathbf{I}_{d\times d}$ is the $(d \times d)$ identity matrix and the second inequality follows from (2.3). Hence,

$$\det\big(\mathbf{\Sigma}(t)\big) = 4^d\Big[t - 2\big(1 - e^{-t/2}\big)\Big]^d.$$

In particular,

$$\det\big(\mathbf{\Sigma}(t)\big) \sim t^{2d} \quad (t \to 0^+).$$

So, the condition (2.2) is equivalent to

$$d < \frac{p}{p-1}.$$

In other words, (2.2) holds if and only if $d = 1$.

References

1. X. Chen, *Random Walk intersections: large deviations and related topics.* Math. Surveys and Monogr., **157**, Amer. Math. Soc., Providence (2009).
2. X. Chen, W. V. Li, J. Rosiński, Q.-M. Shao, *Large deviations for local times and intersection local times of fractional Brownian motions and Riemann – Liouville processes*, Ann. Probab., **39**, 727 – 778 (2011).
3. A. Dvoretzky, P. Erdös, S. Kakutani, *Double points of paths of Brownian motions in n-space*, Acta Sci. Math. (Szeged), **12**, 75 – 81.
4. A. Dvoretzky, P. Erdös, S. Kakutani, *Multiple points of paths of Brownian motions in the plane*, Bull. Res. Counceil Israel, **3**, 364 – 371 (1954).
5. E. B. Dynkin, *Additive functionals of several time-reversable Markov processes*, J. Funct. Anal., **42**, 64 – 101 (1981).
6. D. Geman, J. Horowitz, J. Rosen, *A local time analysis of intersection of Brownian paths in the plane*, Ann. Probab., **12**, 86 – 107 (1984).
7. J.-F. Le Gall, *Propriétés d'intersection des marches aléatoires. I. Convergence vers le temps local d'intersection*, Commun. Math. Phys., **104**, 471 – 507 (1986).
8. J.-F. Le Gall, *Propriétés d'intersection des marches aléatoires. II. Étude des cas critiques*, Commun. Math. Phys., **104**, 509 – 528 (1986).

Received 20.08.20