Annealed asymptotics for Brownian motion of renormalized potential in mobile random medium

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Abstract

Motivated by the study of the directed polymer model with mobile Poissonian traps or catalysts and the stochastic parabolic Anderson model with time dependent potential, we investigate the asymptotic behavior of

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\pm\theta \int_0^t \bar{V}(s, B_s) ds\right\} \qquad (t \to \infty)$$

where $\theta > 0$ is a constant, $\overline{V}$ is the renormalized Poisson potential of the form

$$\overline{V}(s, x) = \int_{\mathbb{R}^d} \frac{1}{|y - x|^p} \left(\omega_s(dy) - dy\right),$$

and ω_s is the measure-valued process consisting of independent Brownian particles whose initial positions form a Poisson random measure on $\mathbb{R}^d$ with Lebesgue measure as its intensity. Different scaling limits are obtained according to the parameter p and dimension d. For the logarithm of the negative exponential moment, the range of $\frac{d}{2} < p < d$ is divided into 5 regions with various scaling rates of the orders $t^{d/p}$, $t^{3/2}$, $t^{(4-d-2p)/2}$, $t \log t$ and t, respectively. For the positive exponential moment, the limiting behavior is studied according to the parameters p and d in three regions. In the sub-critical region ($p < 2$), the double logarithm of the exponential moment has a rate of t. In the critical region ($p = 2$), it has different behavior over two parts decided according to the comparison of θ with the best constant in the Hardy inequality. In the super-critical region ($p > 2$), the exponential moments become infinite for all $t > 0$.

Key-words: renormalization, Poisson field, Brownian motion, parabolic Anderson model.

AMS subject classification (2010): 60J45, 60J65, 60K37, 60K37, 60G55.

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1 Introduction

The model of Brownian motion in a static random medium has been thoroughly investigated. For the directed polymer with immobile Poissonian traps or catalysts, we refer to Sznitman's book [39] for general collection (up to the year 1998), [30] for a survey and [2], [3], [28], [36] for specific topics. For the results motivated by the parabolic Anderson models with time-independent random potentials, we cite [7], [8], [14], [16], [20], [26], [27], [38] as a partial list of the publications on this subject. In addition, we also point out the very recent papers [10], [11], [12] [13] and [21] for the investigation related to the topic of this paper.

The case of mobile environment is much less understood. In the models considered in this paper, the environment consists of moving particles initially distributed in the space $\mathbb{R}^d$ according to a Poisson field $\omega_0(dx)$ with the Lebesgue measure dx as its intensity. The particles move in $\mathbb{R}^d$ independently so the spatial distribution of the obstacles at the time t is of the form

$$\omega_t(dx) = \int_{\mathbb{R}^d} \delta_{X_y(t)}(dx)\omega_0(dy) \tag{1.1}$$

where the stochastic flow $\{X_y(t)\}$ represents the paths of the random obstacles with $X_y(0) = y$. Unless stated otherwise, throughout this paper, $X_y(t)$ are independent Brownian motions with the variance $\sigma^2 = Var(X_y(1)) > 0$. More precisely, the processes $X_y(t) - y$ are i.i.d., independent of $\omega_0(dx)$ and distributed the same as the σ-multiple of a standard d-dimensional Brownian motion. The notations "$\mathbb{P}$" and "$\mathbb{E}$" are used for the probability law and the expectation, respectively, generated by the measure-valued process $\{\omega_t(dx);\ \ t \geq 0\}$.

Let $K(x) \geq 0$ be a properly chosen function (known as shape function) on $\mathbb{R}^d$. The random field

$$V(t,x) = \int_{\mathbb{R}^d} K(y-x)\omega_t(dy) \tag{1.2}$$

represents the total potential at $x \in \mathbb{R}^d$ generated by the Poisson obstacles.

In the model of directed polymer of the potential $V(t,x)$, all possible trajectories of the random path $\{B_s\}_{0\leq s\leq t}$ are re-weighted by some properly defined Gibbs measure (see (1.3) and (1.4) below) that favors trajectories in the time-space region less populated with random traps $\{X_y(t)\}$. In this paper, $\{B_t\}_{t\geq 0}$ is a d-dimensional Brownian motion independent of the environment $\{\omega_t(dx);\ t \geq 0\}$. Throughout this article, the notations "$\mathbb{P}_x$" and "$\mathbb{E}_x$" are for the probability law and the expectation, respectively, of the Brownian motion B_s with $B_0 = x$.

In the quenched setting, where the set-up is conditioned on the random environment created by the stochastic flow $\{X_y(t)\}$, the survival path is selected by the Gibbs measure

$$\frac{d\mu_{t,\omega}}{d\mathbb{P}_0} = \frac{1}{Z_{t,\omega}} \exp\left\{ -\theta \int_0^t V(s,B_s)ds\right\} \tag{1.3}$$

defined on the space $C\left([0,t];\ \mathbb{R}^d\right)$ of the continuous functions $f\colon\ [0,t] \longrightarrow \mathbb{R}^d$.

In the annealed setting, where the model averages on both the Brownian motion and the environment, the Gibbs measure is given as

$$\frac{d\mu_t}{d(\mathbb{P}_0\otimes\mathbb{P})}=\frac{1}{Z_t}\exp\Big\{-\theta\int_0^t V(s,B_s)ds\Big\}. \tag{1.4}$$

In (1.3) and (1.4), the integral

$$\int_0^t V(s,B_s)ds$$

represents the accumulated potential of the Brownian path $\{B_s\}_{0\le s\le t}$ with respect to the moving traps described by $\omega_t(dx)$. Under the law $\mu_{t,\omega}$ or μ_t, therefore, the Brownian trajectories heavily impacted by the Poisson obstacles are penalized and become less likely.

The Brownian motion in mobile random medium corresponds to the parabolic Anderson model

$$\begin{cases} \partial_t u(t,x)=\kappa\Delta u(t,x)+\xi(t,x)u(t,x) \\ \\ u(0,x)=1, \qquad x\in\mathbb{R}^d, \end{cases} \tag{1.5}$$

with the time-dependent potential $\xi(t,x)$. Take $\xi(t,x)=-\theta V(t,x)$ and consider two types of particles A and B in the space $\mathbb{R}^d$. The B-particles evolve according to the measure-valued process $\omega_t(dx)$. The A-particles diffuse from region of high concentration to the region of low concentration according to Fick's law, and is destroyed by nearby B-particles at the annihilation rate $\theta V(t,x)$ which measures the total potential generated by B-particles at (t,x). The solution $u(t,x)$ of (1.5) represents the time-space density of A-particles which are uniformly distributed at the beginning. The story is called the model of trapping reactions and the annihilation mechanism described here is labeled as "$A+B\longrightarrow B$" in the physical literature (see e.g., [32]),

The case when $\xi(t,x)=\theta V(t,x)$ corresponds to the branching Brownian motion in the catalytic medium. The A-particles (reactant) diffuse according to Fick's law while under the stimulation of the nearby B-particles (catalyst), each of them split into two at the rate $\theta V(t,x)$. In this model, $u(t,x)$ represents the time-space density of A-particles. The interested reader is referred to the survey [23] for detail physical background of this model.

The existing literature (see, e.g., [18], [23], [32]) on the models (1.3), (1.4) and (1.5) mainly consider the setting of the lattice space (instead of $\mathbb{R}^d$) combined with pure jump random walks (instead of Brownian motions). In these publications, the shape function is $K(x)=\delta_0(x)$ (Dirac function). To the model of directed polymer, this means that the only traps or catalysts that located on the path $\{B_s\}_{0\le s\le t}$ contribute to the re-shape of the path $\{B_s\}_{0\le s\le t}$. To the parabolic Anderson model with $\xi(t,x)=\pm\theta V(t,x)$, it means that the A-particles and B-particles react only in the case of collision.

Motivated by Newton's law of the universal attraction, a renormalized Poisson potential, formally written as

$$\overline{V}(x)=\int_{\mathbb{R}^d}\frac{1}{|y-x|^p}\big[\omega_0(dy)-dy\big], \qquad x\in\mathbb{R}^d, \tag{1.6}$$

is introduced in [11] in the case of the static medium (or, the case when $\sigma = 0$). Among other things, it has been shown (Theorem 1.1, [11]) that $\overline{V}(x)$ is well defined as a random field if and only if $d/2 < p < d$ and in this case the annealed moment

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ -\theta \int_0^t \overline{V}(B_s) ds \right\} < \infty \tag{1.7}$$

for every $\theta > 0$ and $t > 0$.

We intend to apply this idea to the case of mobile random medium by constructing the renormalized potential

$$\overline{V}(t,x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p} \big[\omega_t(dy) - dy\big] \qquad t \ge 0, \quad x \in \mathbb{R}^d \tag{1.8}$$

as the replacement of $V(t,x)$ in (1.3), (1.4) and in the models of the trapping reactions and the branching Brownian motion in catalytic media. Indeed, by a calculation of characteristic function (see Proposition 1.3 in van den Berg et al [41]) we have that for any $t > 0$

$$\omega_t(dx) \stackrel{d}{=} \omega_0(dx), \tag{1.9}$$

and hence, the field $\overline{V}(t,\cdot)$ is well defined and has the same distribution as $\overline{V}(0,\cdot)$. Further, a slight modification of the argument for Proposition 2.8, [11] shows that $\overline{V}(t,x)$ is continuous in probability as the function of (t,x). Consequently (Chapter 6, [4]), the family of random variables (defined as equivalent classes) $\Big\{\overline{V}(t,x);\ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\Big\}$ yields a measurable (joint in (t,x)) modification. In the remaining of the paper, the same notation $\overline{V}(t,x)$ is used for the measurable modification. Taking $K(x) = |x|^{-p}$ in Lemma 3.1 below, the integral

$$\int_0^t \overline{V}(s, B_s) ds$$

converges almost surely under $d/2 < p < d$.

The necessity of renormalization comes from the fact that

$$\int_{\mathbb{R}^d} \frac{1}{|y-x|^p} \omega_t(dy) = \infty \qquad a.s.\ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{1.10}$$

as $p \le d$. Without the renormalization, the Gibbs measure will be indeterminant which is $\frac{0}{0}$ for trapping model and $\frac{\infty}{\infty}$ for branching model.

The rationale of renormalization is based on the symbolic decomposition

$$\overline{V}(t,x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p} \omega_t(dy) - \int_{\mathbb{R}^d} \frac{1}{|y-x|^p} dy = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p} \omega_t(dy) - \int_{\mathbb{R}^d} \frac{1}{|y|^p} dy.$$

The Lebesgue integral part (if treated as an ordinary constant) does not change the mechanism of the Gibbs measures in (1.3) and (1.4) as it is "renormalized" into the partition functions $Z_{t,\omega}$ and Z_t. Thus, the directed polymer of the renormalized potential $\overline{V}(t,x)$ can be modeled in terms of the quenched Gibbs measure

$$\frac{d\overline{\mu}_{t,\omega}}{d\mathbb{P}_0}=\frac{1}{\overline{Z}_{t,\omega}}\exp\left\{-\theta\int_0^t\overline{V}(s,B_s)ds\right\} \tag{1.11}$$

and the annealed Gibbs measure

$$\frac{d\overline{\mu}_t}{d(\mathbb{P}_0\otimes\mathbb{P})}=\frac{1}{\overline{Z}_t}\exp\left\{-\theta\int_0^t\overline{V}(s,B_s)ds\right\} \tag{1.12}$$

in the case when the partition functions $\overline{Z}_{t,\omega}$ and $\overline{Z}_t$ are finite, the condition taken neither for granted (given the non-locality and singularity of the shape function $K(x)=|x|^{-p}$), nor as the corollary to (1.7).

Taking $\xi(t,x)=-\theta\overline{V}(t,x)$ in the parabolic Anderson model (1.5), we obtained a renormalized version of the model of trapping reactions where the B-particles are annihilated at the rate proportional to a Newton-type-potential and are generated at a constant rate. With $\xi(t,x)=\theta\overline{V}(t,x)$, the branching Brownian motions (reactants) split at the time-space rate the same as the one in the model of trapping reactions.

For a sufficiently smooth and well-bounded function $\xi(t,x)$, by Feynman-Kac formula (cf. Theorem 3.2 in Durrett [19] p138) the equation (1.5) is solved by the function

$$u(t,x)=\mathbb{E}_x\exp\left\{\int_0^t\xi(t-s,B_{2\kappa s})ds\right\}. \tag{1.13}$$

For the case when $\xi(t,x)=\pm\theta\overline{V}(t,x)$, it follows from the same argument as in [12] that $u(t,x)$ is a mild solution to (1.5) (see (2.13) below). Therefore, the investigation of the random field $u(t,x)$ defined in (1.13) is closely associated to our understanding of the parabolic Anderson model (1.5).

This paper is to study the partition function $\overline{Z}_t$ in the annealed directed polymer defined in (1.12) and the expectation of the Feynman-Kac exponential moment given in (1.13) with $\xi(t,x)=\pm\theta\overline{V}(t,x)$. By the identities in law:

$$\overline{V}(t,x)\overset{d}{=}\overline{V}(t,0) \tag{1.14}$$

and

$$\Big\{\omega_{t-s}(dx);\quad 0\le s\le t\Big\}\overset{d}{=}\Big\{\omega_s(dx);\quad 0\le s\le t\Big\}, \tag{1.15}$$

we derive that for any $x\in\mathbb{R}^d$, $\theta>0$ and $t>0$,

$$\begin{aligned}\mathbb{E}_x\exp\left\{\pm\theta\int_0^t\overline{V}(t-s,B_{2\kappa s})ds\right\}&\overset{d}{=}\mathbb{E}_0\exp\left\{\pm\theta\int_0^t\overline{V}(t-s,B_{2\kappa s})ds\right\}\\ \overset{d}{=}\mathbb{E}_0\exp\left\{\pm\theta\int_0^t\overline{V}(s,B_{2\kappa s})ds\right\}&=\mathbb{E}_0\exp\left\{\pm\frac{\theta}{2\kappa}\int_0^{2\kappa t}\overline{V}((2\kappa)^{-1}s,B_s)ds\right\}.\end{aligned} \tag{1.16}$$

To simplify our notation we first consider the case when $\kappa = 1/2$. Therefore, our objective is summarized into the investigation of the annealed exponential moments

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ \pm \theta \int_0^t \overline{V}(s, B_s) ds \right\} \tag{1.17}$$

with the concerns on integrability and long term asymptotics.

The existing literature in the setting of mobile medium deals with the model on the lattice with the Dirac function as the shape function (i.e., $K(x) = \delta_0(x)$). In the setting of trapping reactions, the annealed moment

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ -\theta \int_0^t V(s, B_s) ds \right\}$$

is called the surviving probability in Bramson and Lebowitz ([5] and [6]), Drewitz *et al* [18], where it is proved that

$$\log \mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ -\theta \int_0^t V(s, B_s) ds \right\} \sim \begin{cases} -c_1(\theta)\sqrt{t} & d = 1 \\ -c_2(\theta)\frac{t}{\log t} & d = 2 \\ -c_3(\theta)t & d \geq 3 \end{cases} \tag{1.18}$$

where $0 < c_i(\theta) < \infty$ are constants and the $c_2(\theta)$ and $c_3(\theta)$ are explicitly identified. Similar results for a related model is also obtained by Gärtner *et al* [25]. Here the notation $a_t \sim b_t$ means that $a_t/b_t \to 1$ as $t \to \infty$.

As for the branching random walks in catalytic medium, Gärtner and den Hollander ([22]) observe a double exponential growth given as

$$\lim_{t\to\infty} \frac{1}{t} \log \log \mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ \theta \int_0^t V(s, B_s) ds \right\} = C(\theta) \tag{1.19}$$

with an extended constant $0 \leq C(\theta) \leq \infty$ (see, Theorem 1.4, [22] for the discussion on the limit $C(\theta)$). We also refer the paper [24] for the branching random walks with the voter model as the catalyst.

Another relevance in literature is the recent study ([10], [11], [12] and [13]) on the Brownian motion of the renormalized potential $\overline{V}(x)$ (defined in (1.6)) in a static Poisson medium. It is shown ([12]) that under $d/2 < p < d$,

$$\lim_{t\to\infty} t^{-d/p} \log \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ -\theta \int_0^t \overline{V}(B_s) ds \right\} = \theta^{d/p} \omega_d \frac{p}{d-p} \Gamma\Big(\frac{2p-d}{p}\Big) \tag{1.20}$$

for all $\theta > 0$, and ([11]) that

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \theta \int_0^t \overline{V}(B_s) ds \right\} = \infty \tag{1.21}$$

for all $\theta > 0$ and $t > 0$, where ω_d is the volume of the d-dimensional unit ball. See [10] and [13] for the investigation on the quenched setting. In view of the stationarity (1.9) of the measure-valued process $\omega_t(dx)$, we have that $\overline{V}(t,\cdot) \overset{d}{=} \overline{V}(\cdot)$ for each $t \geq 0$. A natural question is to ask what will remain and what will change when the static Poisson $\omega_0(dx)$ is replaced by the mobile medium $\{\omega_t(dx);\ \ t \geq 0\}$.

Finally, we point out that $V(t,x)$ is well-defined for the case of $p > d$ (we refer the reader to Proposition 2.1 of [11] for a proof of this fact). In this case, renormalization is not needed. Because of the length of the paper, we will treat the case of $p > d$ elsewhere.

2 Main theorems

We adopt all notations introduced in the previous section: the measure-valued process $\omega_t(dx)$ is defined in (1.1) where $\{X_y(t)\}$ are the independent Brownian motions with the covariance matrix σI_d and initial location $X_y(0) = y$, where $\sigma^2 > 0$ and I_d is the $d \times d$ identical matrix. The renormalized potential $\overline{V}(t,x)$ is well defined in (1.8) under the condition

$$d/2 < p < d \tag{2.1}$$

Two functions $\psi(a)$ and $\Psi(a)$ on $\mathbb{R}^+$ frequently appearing in the remaining of this paper are

$$\psi(a) = e^{-a} - 1 + a \ \text{ and } \ \Psi(a) = e^{a} - 1 - a, \qquad a \geq 0. \tag{2.2}$$

It is easy to see that $\psi(a)$ and $\Psi(a)$ are non-negative, increasing and convex on $\mathbb{R}^+$ with $\psi(a) \leq \Psi(a)$.

For the negative exponential moment asymptotic behavior, the condition (2.1) is divided into five disjoint regimes attached with the constants $\rho_i(\theta,\sigma^2)$ $(1 \leq i \leq 5)$, respectively.

Regime I. $p < 2$. Write

$$\rho_1(\theta,\sigma^2) = \theta^{d/p}\omega_d \frac{p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big).$$

Regime II. $p = 2$. By (2.1) $d = 3$ in this case. Write

$$\rho_2(\theta,\sigma^2) = \int_{\mathbb{R}^3} \mathbb{E}\psi\bigg(\theta\int_0^1 \frac{ds}{|x + X_0(s)|^2}\bigg)dx.$$

The exact value of $\rho_2(\theta,\sigma^2)$ remains unknown but can be explicitly bounded (see Lemma 4.1 below).

Regime III. $2 < p < \frac{d+2}{2}$. Write

$$\rho_3(\theta,\sigma^2) = \frac{2^{\frac{2+d-2p}{2}}\theta^2 d\omega_d}{(2+d-2p)(4+d-2p)\sigma^{2p-d}} \frac{\Gamma^2\Big(\frac{d-p}{2}\Big)\Gamma\Big(\frac{2p-d}{2}\Big)}{\Gamma^2\Big(\frac{p}{2}\Big)}.$$

Regime IV. $p = \frac{d+2}{2} > 2$. Write

$$\rho_4(\theta, \sigma^2) = 2^{\frac{d+4}{2}} d\omega_d \Big(\frac{\theta}{(d-2)\sigma}\Big)^2.$$

Regime V. $p > \max\Big\{2, \frac{d+2}{2}\Big\}$. The constant $0 < \rho_5(\theta, \sigma^2) < \infty$ remains unknown but is explicitly bounded (see (4.12) below).

The next theorem gives the negative exponential moment asymptotic behavior.

Theorem 2.1 *Under (2.1), we have*

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\Big\{ -\theta \int_0^t \overline{V}(s, B_s) ds \Big\} < \infty \qquad \theta > 0, \quad t > 0. \tag{2.3}$$

Further,

$$\log \mathbb{E}_0 \otimes \mathbb{E} \exp\Big\{ -\theta \int_0^t \overline{V}(s, B_s) ds \Big\} \sim \begin{cases} \rho_1(\theta, \sigma^2) t^{d/p} & \textit{in Regime I} \\ \rho_2(\theta, \sigma^2) t^{3/2} & \textit{in Regime II} \\ \rho_3(\theta, \sigma^2) t^{\frac{4+d-2p}{2}} & \textit{in Regime III} \\ \rho_4(\theta, \sigma^2) t \log t & \textit{in Regime IV} \\ \rho_5(\theta, \sigma^2) t & \textit{in Regime V} \end{cases} \tag{2.4}$$

as $t \to \infty$.

Recall that the random potential $\overline{V}(x)$ is defined in (1.6). The comparison between (1.20) and (2.4) shows that moving from static environment to mobile environment, the asymptotic behavior remains the same only in the regime I (which contains the cases of $d = 1$ or 2). Further, notice that counting from regimes I to V, the deviation scales

$$t^{d/p}, \ t^{3/2}, \ t^{\frac{4+d-2p}{2}}, \ t \log t \text{ and } t$$

decrease in the following sense: For p_1, p_3 in region I, III, respectively, and for t large,

$$t^{d/p_1} > t^{3/2} > t^{\frac{4+d-2p_3}{2}} > t \log t > t.$$

Since the deviation scale is positively related to the survival probability, it is generally harder for the Brownian particle B_s to avoid the Poisson traps in the mobile medium than in the static medium. The fact that the deviation scale decreases in p suggests that the total impact of the obstacles over B_s increases when the tail of the shape function gets heavier. This observation provides the evidence showing that the major contribution comes from the vast number of the traps located a distance away from the Brownian particle, rather than a few in a close neighborhood of the Brownian particle.

We now move to the case of catalytic medium. Let $W^{1,2}(\mathbb{R}^d)$ be the Sobolev space defined as

$$W^{1,2}(\mathbb{R}^d) = \Big\{ f \in \mathcal{L}^2(\mathbb{R}^d);\ \ \nabla f \in \mathcal{L}^2(\mathbb{R}^d) \Big\}.$$

By Lemma 7.2, [10], for any $d/2 < p < \min\{2, d\}$ there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^2} dx \le C \|f\|_2^{2-p} \|\nabla f\|_2^p \quad f \in W^{1,2}(\mathbb{R}^d). \tag{2.5}$$

Let $\gamma(d,p) > 0$ be the best constant in the above inequality.

Theorem 2.2 *Assume (2.1).*

(1) When $p < 2$, for any $\theta > 0$ and $t > 0$, we have

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \theta \int_0^t \overline{V}(s, B_s) ds \right\} < \infty. \tag{2.6}$$

Further,

$$\lim_{t\to\infty} \frac{1}{t} \log\log \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \theta \int_0^t \overline{V}(s, B_s) ds \right\} = \frac{2-p}{2} \Big(\frac{p}{\sigma^2}\Big)^{\frac{p}{2-p}} \Big(\theta\gamma(d,p)\Big)^{\frac{2}{2-p}}. \tag{2.7}$$

(2). When $p = 2$ (and therefore $d = 3$ by (2.1)), for any $t > 0$,

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \theta \int_0^t \overline{V}(s, B_s) ds \right\} \begin{cases} < \infty & \text{when } \theta < \frac{\sigma^2}{8} \\ = \infty & \text{when } \theta > \frac{\sigma^2}{8}. \end{cases} \tag{2.8}$$

Further, for any $\theta < \sigma^2/8$, we have

$$\lim_{t\to\infty} t^{-3/2} \log \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \theta \int_0^t \overline{V}(s, B_s) ds \right\} = \int_{\mathbb{R}^3} \mathbb{E}\Psi\Big(\theta \int_0^1 \frac{ds}{|x + X_0(s)|^2} \Big) dx, \tag{2.9}$$

and the right hand side is finite.

(3). When $p > 2$, for any $\theta > 0$ and $t > 0$, we have

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \theta \int_0^t \overline{V}(s, B_s) ds \right\} = \infty. \tag{2.10}$$

The comparison between (1.21) and Theorem 2.2 shows that in the catalytic medium, the mobile Poisson particles are less attractable to the Brownian particles than static Poisson particles. Another interesting observation is on the role played by the parameter p. It is evident that the singularity of the shape function $K(x) = |x|^{-p}$ at $x = 0$ is the only reason for the positive exponential moment to blow-up. Consequently, the model is more likely to blow-up for greater p. As for the large-t behavior, the singularity of the shape function is no longer the sole driving force, as indicated by the comparison of the double exponential growth in the sub-critical case $p < 2$ versus the single exponential growth in the critical case $p = 2$.

In connection to the parabolic Anderson model posted in (1.5), write

$$u_{\pm}(t,x) = \mathbb{E}_x \exp\left\{ \pm\theta \int_0^t \overline{V}(t-s, B_{2\kappa s}) ds \right\}. \tag{2.11}$$

By Theorems 2.1-2.2 (with σ^2 be replaced by $(2\kappa)^{-1}\sigma^2$) and the relation (1.16), $\mathbb{E}u_-(t,x) < \infty$ and $\mathbb{E}u_+(t,x) < \infty$ for all $\theta > 0$ and $t > 0$ under $d/2 < p < d$ and $d/2 < p < \min\{2, d\}$, respectively. In the case $p = 2$ and $d = 3$, $\mathbb{E}u_+(t,x) < \infty$ if $\theta < \frac{\sigma^2}{16\kappa}$; and $= \infty$ if $\theta > \frac{\sigma^2}{16\kappa}$. Finally, $\mathbb{E}u_+(t,x) = \infty$ for all $\theta > 0$ and $t > 0$ when $p > 2$.

Unfortunately, given the fact (derived from Proposition 2.9 in [11] and the relation (1.9)) that for each $t > 0$, the random field $\overline{V}(t,\cdot)$ is unbounded and therefore discontinuous in any neighborhood in $\mathbb{R}^d$ with positive probability, it is unlikely that with $\xi(t,x) = \pm\theta\overline{V}(t,x)$, the equation (1.5) has path-wise solution. On the other hand, by an argument the same as in the proofs of Proposition 1.2 and Proposition 1.6, [11], we can show that whenever $u_{\pm}(t,x) < \infty$ a.s., particularly when $\mathbb{E}u_{\pm}(t,x) < \infty$, $u_{\pm}(t,x)$ is the mild solution to (1.5) (with $\xi(t,x) = \pm\theta\overline{V}(t,x)$), in the sense that

$$\int_0^t \int_{\mathbb{R}^d} p_{2\kappa(t-s)}(x-y) |\overline{V}(s,y) u_{\pm}(s,y)| \, dy ds < +\infty, \quad x \in \mathbb{R}^d, \ t > 0 \tag{2.12}$$

and

$$u_{\pm}(t,x) = 1 \pm \theta \int_0^t \int_{\mathbb{R}^d} p_{2\kappa(t-s)}(x-y) \overline{V}(s,y) u_{\pm}(s,y) \, dy ds, \quad x \in \mathbb{R}^d, \ t > 0, \tag{2.13}$$

where $p_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/2)$ is the transition probability density of B_t.

Further, by Theorem 2.1 (with σ^2 be replaced by $(2\kappa)^{-1}\sigma^2$) and the relation (1.16),

$$\log \mathbb{E}u_-(t,x) = \log \mathbb{E}u_-(t,0) \sim \begin{cases} \rho_1\Big(\frac{\theta}{2\kappa}, \frac{\sigma^2}{2\kappa}\Big) t^{d/p} & \text{in Regime I} \\ \rho_2\Big(\frac{\theta}{2\kappa}, \frac{\sigma^2}{2\kappa}\Big) t^{3/2} & \text{in Regime II} \\ \rho_3\Big(\frac{\theta}{2\kappa}, \frac{\sigma^2}{2\kappa}\Big) t^{\frac{4+d-2p}{2}} & \text{in Regime III} \\ \rho_4\Big(\frac{\theta}{2\kappa}, \frac{\sigma^2}{2\kappa}\Big) t \log t & \text{in Regime IV} \\ \rho_5\Big(\frac{\theta}{2\kappa}, \frac{\sigma^2}{2\kappa}\Big) t & \text{in Regime V} \end{cases} \quad (t \to \infty). \tag{2.14}$$

By Theorem 2.2 and the relation (1.16), under $d/2 < p < \min\{2, d\}$

$$\lim_{t\to\infty} \frac{1}{t} \log\log \mathbb{E}u_+(t,x) = \frac{2-p}{4\kappa} \Big(\frac{p}{\sigma^2}\Big)^{\frac{p}{2-p}} \Big(\theta\gamma(d,p)\Big)^{\frac{2}{2-p}}. \tag{2.15}$$

Under $p = 2$ and $d = 3$, we have

$$\lim_{t\to\infty} t^{-3/2}\log \mathbb{E}u_+(t,x) = (2\kappa)^{3/2}\int_{\mathbb{R}^3}\mathbb{E}\Psi\left(\frac{\theta}{2\kappa}\int_0^1 \frac{ds}{|x+X_0(s)|^2}\right)dx \tag{2.16}$$

for every $\theta < \sigma^2/(16\kappa)$.

Our approach is guided by the idea known as Pascal principle, which first introduced by Moreau et al ([31], ([32]) and reformulated by Drewitz et al ([18]) in their context. According to Pascal principle, the best way for the Brownian particle(s) to avoid Poisson traps in the direct trapping reaction model or, to split at the requested level in catalytic model, is to stay in a neighborhood of 0 (see Lemma 3.2 for some mathematical implementation of Pascal principle). The mathematical formulation is the relation

$$\mathbb{E}\otimes\mathbb{E}_0\exp\left\{\pm\theta\int_0^t \overline{V}(s,B_s)ds\right\} \le \mathbb{E}\exp\left\{\pm\theta\int_0^t \overline{V}(s,0)ds\right\}.$$

Different from [31], ([32] and [18], our realization (Lemma 3.2) of Pascal principle relies on the Gaussian property (rather than Markovian property) of the system.

Based on Pascal principle and in view of the equalities in Lemma 3.1, much of our attention is on the asymptotic behaviors of the integrals

$$\int_{\mathbb{R}^d}\psi\left(\theta\int_0^t \frac{ds}{|x+X_0|^p}\right)dx \quad\text{and}\quad \int_{\mathbb{R}^d}\Psi\left(\theta\int_0^t \frac{ds}{|x+X_0|^p}\right)dx.$$

It should be pointed out that our treatment is very different in the trapping and catalytic settings and in different regimes marked by the combinations of d and p.

The rest of this paper is organized as follows: In Section 3, we establish moment identities and Pascal principle for random potential function $\overline{V}(t,B_t)$. Both of these two results will play key roles in the proofs of Theorems 2.1 and 2.2 presented in Sections 4 and 5, respectively. An appendix which includes the proof of a technique result is given at the end of the paper.

3 Moment of occupation time with random potential

As preparation for the proof of the main results, in this section, we prove a key identity for the exponential moment of the occupation time with random potential, and the Pascal principle. The identity essentially says that the expectation (with respect to the environment) of the exponential of the occupation time is the same as the exponential of the expectation of a related random variable. The Pascal principle essentially says that this expectation just mentioned in last sentence is maximized when the particle does not move in the random medium. This principle is useful in deriving upper bound for the asymptotic behavior of the occupation time.

According to Propositions 2.1 and 2.8 of [11] and (1.9), for any Borel measurable function $K(x)\ge 0$, the compensated Poisson integral

$$\overline{V}_K(t,x) = \int_{R^d} K(y-x)\big[\omega_t(dy) - dy\big] \tag{3.1}$$

is defined under the condition

$$\int_{\mathbb{R}^d} \psi\Big(K(x)\Big)dx < \infty. \tag{3.2}$$

Lemma 3.1 *Let $K(x) \geq 0$ satisfy (3.2). The time integral*

$$\int_0^t \overline{V}_K(s, B_s)ds \tag{3.3}$$

converges almost surely for any $t > 0$ and

$$\begin{aligned}&\mathbb{E}_0 \otimes \mathbb{E} \exp\Big\{ -\int_0^t \overline{V}_K(s, B_s)ds\Big\} \\ &= \mathbb{E}_0 \exp\Big\{ \int_{\mathbb{R}^d} \mathbb{E}\psi\Big(\int_0^t K(x + X_0(s) - B_s)ds\Big)dx\Big\}.\end{aligned} \tag{3.4}$$

In addition, the equality

$$\begin{aligned}&\mathbb{E}_0 \otimes \mathbb{E} \exp\Big\{ \int_0^t \overline{V}_K(s, B_s)ds\Big\} \\ &= \mathbb{E}_0 \exp\Big\{ \int_{\mathbb{R}^d} \mathbb{E}\Psi\Big(\int_0^t K(x + X_0(s) - B_s)ds\Big)dx\Big\}\end{aligned} \tag{3.5}$$

holds.

Proof: By (1.9) and Fubini's theorem,

$$\mathbb{E}_0 \otimes \mathbb{E}\Big(\int_0^t |\overline{V}_K(s, B_s)|ds\Big) = \mathbb{E}_0 \int_0^t \mathbb{E}|\overline{V}_K(0, B_s)|ds = t\mathbb{E}|\overline{V}_K(0, 0)| < \infty,$$

where the first step follows from (1.9) and the last step follows from Proposition 2.6 of [11]. So the integral in (3.3) a.s. converges.

First, we claim that the right hand side of (3.4) is finite. Indeed, by Jensen inequality

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi\Big(\int_0^t K(x + X_0(s) - B_s)ds\Big)dx \\ \leq\ & \frac{1}{t}\int_0^t \int_{\mathbb{R}^d} \psi\big(tK(x + X_0(s) - B_s)\big)dxds \\ =\ & \int_{\mathbb{R}^d} \psi\big(tK(x)\big)dx < \infty,\end{aligned} \tag{3.6}$$

where the last step follows from (3.2), monotonicity of $\psi(\cdot)$ and the fact that $\psi(2\theta) \leq C\psi(\theta)$ $(\forall\theta > 0)$ for some constant $C > 0$.

To prove the equality in (3.4), we first consider the case when $K(x)$ is bounded and locally supported. By (1.1) and Fubini theorem,

$$\int_0^t \left[\int_{\mathbb{R}^d} K(x-B_s)\omega_s(dx)\right]ds = \int_{\mathbb{R}^d}\left[\int_0^t K(X_y(s)-B_s)ds\right]\omega_0(dy)$$

Hence, by the independence between $X_y(s)$ and $\omega_0(dy)$, and by the independence among the Brownian motions $X_y(s)$, we have

$$\begin{aligned}
&\mathbb{E}\exp\left\{-\int_0^t\left[\int_{\mathbb{R}^d} K(x-B_s)\omega_s(dx)\right]ds\right\}\\
=\ &\mathbb{E}\exp\left\{\int_{\mathbb{R}^d} h(y)\omega_0(dy)\right\}\\
=\ &\exp\left\{\int_{\mathbb{R}^d}\left(e^{h(y)}-1\right)dy\right\}\\
=\ &\exp\left\{\int_{\mathbb{R}^d}\left[\mathbb{E}\exp\left(-\int_0^t K(y+X_0(s)-B_s)ds\right)-1\right]dy\right\},
\end{aligned}$$

where

$$h(y)=\log\mathbb{E}\exp\left(-\int_0^t K(y+X_0(s)-B_s)ds\right).$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\exp\left\{-\int_0^t\left[\int_{\mathbb{R}^d} K(x-B_s)\big[\omega_s(dx)-dx\big]\right]ds\right\}\\
&=\exp\left\{\int_{\mathbb{R}^d}\left[\mathbb{E}\exp\left(-\int_0^t K(y+X_0(s)-B_s)ds\right)-1+tK(y)\right]dy\right\}.
\end{aligned}$$

Thus, (3.4) (with bounded and locally supported $K(x)$) follows from the equality

$$\int_{\mathbb{R}^d}\left[\int_0^t K(y+X_0(s)-B_s)ds\right]dx = t\int_{\mathbb{R}^d} K(y)dy. \tag{3.7}$$

We now prove (3.4) with full generality. Let $K_n(x)\geq 0$ be a non-decreasing sequence of bounded, locally supported function such that $K_n(x)\uparrow K(x)$ point-wise. Notice that

$$\mathbb{E}_0\otimes\mathbb{E}\left|\int_0^t\left(\overline{V}_{K_n}(s,B_s)-\overline{V}_K(s,B_s)\right)ds\right|\leq t\mathbb{E}_0\otimes\mathbb{E}\left|\overline{V}_{K_n}(0,0)-\overline{V}_K(0,0)\right|.$$

Using the decomposition in the proof of Proposition 2.6 of [11], one can show that the right hand side tends to zero as $n\to\infty$.

We now take $n\to\infty$ on the both sides of (3.4) with K_n replacing K. The right hand side passes from K_n to K by the monotonic convergence. On the other hand, we note that

$$\begin{aligned}
&\mathbb{E}_0\otimes\mathbb{E}\exp\Big\{-2\int_0^t\overline{V}_{K_n}(s,B_s)ds\Big\}\\
=\;&\mathbb{E}_0\exp\Big\{\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\int_0^t 2K_n(x+X_0(s)-B_s)ds\Big)dx\Big\}\\
\le\;&\mathbb{E}_0\exp\Big\{\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\int_0^t 2K(x+X_0(s)-B_s)ds\Big)dx\Big\}\\
<\;&\infty,
\end{aligned}$$

where the last step follows from (3.6). Therefore, the left hand side of (3.4) passes from K_n to K by the uniform integrability of $\Big\{\exp\Big\{-\int_0^t\overline{V}_{K_n}(s,B_s)ds\Big\}:\ n\ge 1\Big\}$. Hence, (3.4) holds for K. The proof of (3.5) follows from a similar argument. □

Lemma 3.2 (Pascal principle) *Assume that $d/2<p<d$. For any deterministic continuous function b_s in $C\big([0,t],\mathbb{R}^d\big)$,*

$$\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\int_0^t\frac{ds}{|x+X_0(s)-b_s|^p}\Big)dx\le\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\int_0^t\frac{ds}{|x+X_0(s)|^p}\Big)dx \tag{3.8}$$

and

$$\mathbb{E}\int_{\mathbb{R}^d}\bigg[\int_0^t\frac{ds}{|x+X_0(s)-b_s|^p}\bigg]^m dx\le\mathbb{E}\int_{\mathbb{R}^d}\bigg[\int_0^t\frac{ds}{|x+X_0(s)|^p}\bigg]^m dx,\qquad m=2,3,\cdots. \tag{3.9}$$

Consequently (from (3.9), for any $\theta>0$

$$\mathbb{E}\int_{\mathbb{R}^d}\Psi\Big(\theta\int_0^t\frac{ds}{|x+X_0(s)-b_s|^p}\Big)dx\le\mathbb{E}\int_{\mathbb{R}^d}\Psi\Big(\theta\int_0^t\frac{ds}{|x+X_0(s)|^p}\Big)dx. \tag{3.10}$$

Proof: Let $K(x)\ge 0$ be bounded, locally supported and radially symmetric. As a corollary of Theorem 1.5, [17],

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big[\mathbb{E}\exp\Big(-\int_0^t K(x+X_0(s)-b_s)ds\Big)-1\Big]dx\\
&\le\int_{\mathbb{R}^d}\Big[\mathbb{E}\exp\Big(-\int_0^t K(x+X_0(s))ds\Big)-1\Big]dx.
\end{aligned}$$

Subtracting

$$t\int_{\mathbb{R}^d}K(x)dx$$

on the both sides and by (3.7) (with B_s being replaced by b_S or 0),

$$\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\int_0^t K(x+X_0(s)-b_s)ds\Big)dx\le\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\int_0^t K(x+X_0(s))ds\Big)dx.$$

Taking a sequence of such function K's approximating the function $|\cdot|^{-p}$, we obtain the desired inequality (3.8).

Our approach for (3.9) relies on Fourier transformation and the fact that

$$\int_{\mathbb{R}^d} \frac{1}{|x|^p} e^{i\lambda\cdot x} dx = C\frac{1}{|\lambda|^{d-p}}, \tag{3.11}$$

where $C > 0$ is a constant.

Write

$$h(y_1, \cdots, y_{m-1}) = \int_{\mathbb{R}^d} \frac{1}{|x|^p} \prod_{k=1}^{m-1} \frac{1}{|x+y_k|^p} dx.$$

Then,

$$\begin{aligned}
&\widehat{h}(\lambda_1, \cdots, \lambda_{m-1}) = \int_{(\mathbb{R}^d)^{m-1}} h(y_1, \cdots, y_{m-1}) \exp\Big\{ i\sum_{k=1}^{m-1} \lambda_k\cdot y_k \Big\} dy_1\cdots dy_{m-1} \\
&= \int_{\mathbb{R}^d} \frac{1}{|x|^p} \exp\Big\{ -i(\lambda_1+\cdots+\lambda_{m-1})\cdot x\Big\} dx \prod_{k=1}^{m-1} \int_{\mathbb{R}^d} \frac{1}{|y|^p} e^{i\lambda_k\cdot y} dy \\
&= C^m \frac{1}{|\lambda_1+\cdots+\lambda_{m-1}|^{d-p}} \prod_{k=1}^{m-1} \frac{1}{|\lambda_k|^{d-p}} > 0.
\end{aligned}$$

Write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left[\int_0^t \frac{ds}{|x+X_0(s)-b_s|^p}\right]^m dx \\
&= \int_{[0,t]^m} \left[\int_{\mathbb{R}^d} \prod_{k=1}^m \frac{1}{|x+X_0(s_k)-b_{s_k}|^p} dx\right] ds_1\cdots ds_m \\
&= \int_{[0,t]^m} \left[\int_{\mathbb{R}^d} \frac{1}{|x|^p} \prod_{k=1}^m \frac{1}{|x+\big(X_0(s_k)-X_0(s_m)\big)-(b_{s_k}-b_{s_m})|^p} dx\right] ds_1\cdots ds_m \\
&= \int_{[0,t]^m} h\Big(\big(X_0(s_1)-X_0(s_m)\big)-(b_{s_1}-b_{s_m}), \cdots, \\
&\qquad\qquad \big(X_0(s_{m-1})-X_0(s_m)\big)-(b_{s_{m-1}}-b_{s_m})\Big) ds_1\cdots ds_m.
\end{aligned}$$

By Fourier inversion,

$$\begin{aligned}
&h\Big(\big(X_0(s_1)-X_0(s_m)\big)-(b_{s_1}-b_{s_m}), \cdots, \big(X_0(s_{m-1})-X_0(s_m)\big)-(b_{s_{m-1}}-b_{s_m})\Big) \\
&= \frac{1}{(2\pi)^{(m-1)d}} \int_{(\mathbb{R}^d)^{m-1}} d\lambda_1\cdots d\lambda_{m-1} \widehat{h}(\lambda_1, \cdots, \lambda_{m-1}) \\
&\times \exp\Big\{ -i\sum_{k=1}^{m-1} \lambda_k\cdot\Big(\big(X_0(s_k)-X_0(s_m)\big)-(b_{s_k}-b_{s_m})\Big)\Big\}.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^d}\left[\int_0^t \frac{ds}{|x+X_0(s)-b_s|^p}\right]^m dx$$
$$=\frac{1}{(2\pi)^{(m-1)d}}\int_{(\mathbb{R}^d)^{m-1}} d\lambda_1\cdots d\lambda_{m-1}\widehat{h}(\lambda_1,\cdots,\lambda_{m-1})\int_{[0,t]^m} ds_1\cdots ds_m$$
$$\times\exp\left\{-i\sum_{k=1}^{m-1}\lambda_k\cdot\Big(\big(X_0(s_k)-X_0(s_m)\big)-(b_{s_k}-b_{s_m})\Big)\right\}.$$

Hence,

$$\int_{\mathbb{R}^d}\mathbb{E}\left[\int_0^t \frac{ds}{|x+X_0(s)-b_s|^p}\right]^m dx$$
$$=\frac{1}{(2\pi)^{(m-1)d}}\int_{(\mathbb{R}^d)^{m-1}} d\lambda_1\cdots d\lambda_{m-1}\widehat{h}(\lambda_1,\cdots,\lambda_{m-1})\int_{[0,t]^m} ds_1\cdots ds_m$$
$$\times\exp\left\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^{m-1}\lambda_k\cdot\Big(\big(X_0(s_k)-X_0(s_m)\big)\Big)\right\}\exp\left\{i\sum_{k=1}^{m-1}\lambda_k\cdot(b_{s_k}-b_{s_m})\right\}$$
$$\le\frac{1}{(2\pi)^{(m-1)d}}\int_{(\mathbb{R}^d)^{m-1}} d\lambda_1\cdots d\lambda_{m-1}\widehat{h}(\lambda_1,\cdots,\lambda_{m-1})\int_{[0,t]^m} ds_1\cdots ds_m$$
$$\times\exp\left\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^{m-1}\lambda_k\cdot\Big(\big(X_0(s_k)-X_0(s_m)\big)\Big)\right\}$$
$$=\int_{\mathbb{R}^d}\mathbb{E}\left[\int_0^t \frac{ds}{|x+X_0(s)|^p}\right]^m dx.$$

The inequality (3.10) follows from the Taylor expansion of Ψ at $a=0$. □

Remark 3.3 *The asymptotic version of Pascal principle (3.8) for a different shape function has been obtained by [34] and the full Pascal principle for that setting was obtained by [35].*

Finally, we state a variety of the inequality (3.9) which will be used later. For any $a>0$ write $K_a(x)=|x|^{-p}1_{\{|x|\ge a\}}$. It is easy to check that

$$\widehat{K_a}(\lambda)\equiv\int_{\mathbb{R}^d}K_a(x)e^{i\lambda\cdot x}dx\ge 0\qquad \lambda\in\mathbb{R}^d.$$

By the same argument as the one for (3.9),

$$\int_{\mathbb{R}^d}\mathbb{E}\left[\int_0^t K_a\big(x+X_0(s)-b_s\big)ds\right]^m dx\le\int_{\mathbb{R}^d}\mathbb{E}\left[\int_0^t K_a\big(x+X_0(s)\big)ds\right]^m dx \tag{3.12}$$

for $m=2,3,\cdots$.

4 Model of trapping reactions

We prove Theorem 2.1 in this section. Throughout, we assume that $d/2 < p < d$. The integrability assertion (2.3) follows from (3.4) and (3.6) with $K(x) = \theta|x|^{-p}$. In the following subsections, we establish the asymptotics given in (2.4) in different regimes.

The key is to estimate the exponent

$$\bar{\psi}_t(B) \equiv \int_{\mathbb{R}^d} \mathbb{E}\psi\left(\theta\int_0^t \frac{ds}{|x+X_0(s)-B_s|^p}\right)dx,$$

which depends on the initial obstacles (IO), moving obstacles (MO) and the moving particle (MP).

It follows from Lemma 3.1 that

$$\mathbb{E}\otimes\mathbb{E}_0\exp\left\{-\theta\int_0^t V(s,B_s)ds\right\} = \mathbb{E}_0\exp\left(\bar{\psi}_t(B)\right). \tag{4.1}$$

By Pascal principle, we have

$$\log\mathbb{E}\otimes\mathbb{E}_0\exp\left\{-\theta\int_0^t V(s,B_s)ds\right\} \le \bar{\psi}_t(0). \tag{4.2}$$

So the proof of the upper bound is reduced to the estimate of the deterministic quantity $\bar{\psi}_t(0)$.

4.1 Regime I: $p < 2$

The main idea of the proof for this region is to consider the asymptotic scales of the three components affecting $\bar{\psi}_t$. From the shape function, we see that the IO is of the order $t^{1/p}$ (see (4.3) below). On the other hand, the MO and MP are Brownian motions, and hence, of order $t^{1/2}$. When $p < 2$, the MO and MP are negligible.

By (3.6) (with $K(x) = \theta|x|^{-p}$),

$$\bar{\psi}_t(0) \le \int_{\mathbb{R}^d}\psi\Big(\frac{t\theta}{|x|^p}\Big)dx = (t\theta)^{d/p}\int_{\mathbb{R}^d}\psi\Big(\frac{1}{|x|^p}\Big)dx. \tag{4.3}$$

By Lemma 7.1 of [10], we have

$$\int_{\mathbb{R}^d}\psi\Big(\frac{1}{|x|^p}\Big)dx = \omega_d\frac{p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big), \qquad \forall\ d/2 < p < d. \tag{4.4}$$

By (4.2)-(4.4), we obtain the upper bound

$$\limsup_{t\to\infty} t^{-d/p}\log\mathbb{E}\otimes\mathbb{E}_0\exp\left\{-\theta\int_0^t \overline{V}(s,B_s)ds\right\} \le \theta^{d/p}\omega_d\frac{p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big).$$

On the other hand, for $|x| > \delta t^{\frac{1}{p}}$ and $|y| \le 2M\sqrt{t}$, we have as $t \to \infty$,

$$\frac{|x+y|}{|x|} \le 1 + \frac{|y|}{|x|} \le 1 + \frac{2M}{\delta} t^{\frac{1}{2}-\frac{1}{p}} \to 1,$$

and hence, for t large enough,

$$\frac{1}{|x+y|^p} \ge \frac{1-\delta}{|x|^p}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_0 \exp\Big\{\bar{\psi}_t(B)\Big\} \\
\ge\ & \mathbb{E}_0 \int_{|x|\ge \delta t^{1/p}} \mathbb{E}\psi\left(\theta \int_0^t \frac{(1-\delta)}{|x|^p} ds\right) dx 1_{\{\sup_{s\le t}(|B_s|, |X_0(s)|)\le M\sqrt{t}\}} \\
\ge\ & \mathbb{P}_0\Big\{\max_{s\le t}|B_s| \le M\sqrt{t}\Big\} \exp\left\{\mathbb{P}\Big\{\max_{s\le t}|X_0(s)| \le M\sqrt{t}\Big\}\int_{|x|\ge\delta t^{1/p}} \psi\left(\frac{\theta t(1-\delta)}{|x|^p}\right) dx\right\} \\
=\ & \mathbb{P}_0\Big\{\max_{s\le t}|B_s| \le M\sqrt{t}\Big\} \exp\left\{\big(\theta t(1-\delta)\big)^{d/p}\mathbb{P}\Big\{\max_{s\le t}|X_0(s)| \le M\sqrt{t}\Big\}\right. \\
& \left.\times \int_{\{|x|\ge \delta(1-\delta)^{-1/p}\theta^{-1/p}\}} \psi\left(\frac{1}{|x|^p}\right) dx\right\},
\end{aligned}$$

for any $0 < \delta < 1$ and $M > 0$, as t is large.

The probabilities

$$\mathbb{P}_0\Big\{\max_{s\le t}|B_s| \le M\sqrt{t}\Big\} \quad\text{and}\quad \mathbb{P}\Big\{\max_{s\le t}|X_0(s)| \le M\sqrt{t}\Big\}$$

can be made arbitrarily close to 1 by taking $M > 0$ arbitrarily large. Thus,

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-d/p} \log \mathbb{E}_0 \exp\Big\{\bar{\psi}_t(B)\Big\} \\
&\ge \big(\theta(1-\delta)\big)^{d/p} \int_{\{|x|\ge\delta(1-\delta)^{-1/p}\theta^{-1/p}\}} \psi\left(\frac{1}{|x|^p}\right) dx
\end{aligned}.$$

Letting $\delta \to 0^+$ on the right hand side, the desired lower bound follows from (3.4) and (4.4).
□

4.2 Regime II: $p = 2$ and $d = 3$

In this regime, the scales of the three components are the same so the limit is obtained by a scaling argument.

By scaling, we have

$$\mathbb{E}_0 \exp\Big\{\bar{\psi}_t(B)\Big\} = \mathbb{E}_0 \exp\Big\{t^{3/2}\bar{\psi}_1(B)\Big\}.$$

By (4.1) and (4.2), all we need to show is the lower bound.

Note that the estimate

$$\begin{aligned}&\mathbb{E}_0 \exp\Big\{t^{3/2}\bar\psi_1(B)\Big\}\\ \geq\ & \mathbb{P}_0\Big\{\max_{0\le s\le 1}|B_s|\le\epsilon\Big\}\exp\left(t^{3/2}\int_{\mathbb{R}^3}\mathbb{E}\psi\Big(\theta\int_0^1\frac{ds}{\big(|x+X_0(s)|+\epsilon\big)^2}\Big)dx\right),\end{aligned}$$

implies that

$$\liminf_{t\to\infty} t^{-3/2}\log\mathbb{E}_0\exp\Big\{\bar\psi_t(B)\Big\}\geq\int_{\mathbb{R}^3}\mathbb{E}\psi\Big(\theta\int_0^1\frac{ds}{\big(|x+X_0(s)|+\epsilon\big)^2}\Big)dx.$$

Letting $\epsilon\to 0^+$ and applying Fatou's lemma we obtain the desired lower bound. □

Next, we give an explicit upper bound for ρ_2.

Lemma 4.1

$$\rho_2(\theta,\sigma^2)\le\frac{8}{3}\pi^{3/2}\theta^{3/2}. \tag{4.5}$$

Proof: The conclusion follows from the following estimate

$$\begin{aligned}&\int_{\mathbb{R}^3}\psi\Big(\theta\int_0^1\frac{ds}{|x+X_0(s)|^2}\Big)dx\le\int_0^1\Big[\int_{\mathbb{R}^3}\psi\Big(\frac{\theta}{|x+X_0(s)|^2}\Big)dx\Big]ds\\ &=\int_{\mathbb{R}^3}\psi\Big(\frac{\theta}{|x|^2}\Big)dx=\theta^{3/2}\int_{\mathbb{R}^3}\psi\Big(\frac{1}{|x|^2}\Big)dx=\frac{8}{3}\pi^{3/2}\theta^{3/2},\end{aligned}$$

where the last step follows from (4.4). □

4.3 Regime III: $2 < p < \frac{d+2}{2}$

In this regime, we consider the Taylor expansion of the function ψ. It turns out that the quadratic term dominates the others. The upper bound estimate is again obtained by (4.2). The lower bound is obtained by roughly making the particle motionless, namely, to restrict to the event $\{\sup_{s\le t}|B_s|\le t^{1/p}\}$. Although the probability of this event tends to 0, its exponential rate is less than that of the scale determined in the upper bound.

By the fact that $\psi(a)\le 2^{-1}a^2$, $(a\ge 0)$, we have

$$\begin{aligned}\bar\psi_t(0)&\le\int_{\mathbb{R}^d}\frac{\theta^2}{2}\mathbb{E}\Big[\int_0^t\frac{ds}{|x+X_0(s)|^p}\Big]^2dx\\ &=\frac{\theta^2}{2}C(d,p)\mathbb{E}\int_0^t\int_0^t\frac{drds}{|X_0(r)-X_0(s)|^{2p-d}},\end{aligned}$$

where the second step follows from (6.7).

Let U be a d-dimensional standard normal random vector, i.e., $U \sim N(0, I_d)$. By Brownian scaling

$$\mathbb{E}\int_0^t\!\!\int_0^t \frac{drds}{|X_0(r)-X_0(s)|^{2p-d}} = \sigma^{-(2p-d)}\mathbb{E}|U|^{-(2p-d)}\int_0^t\!\!\int_0^t |r-s|^{-\frac{2p-d}{2}}drds$$
$$= t^{\frac{4-2p+d}{2}}\frac{2^{\frac{4+d-2p}{2}}d\omega_d}{(2+d-2p)(4+d-2p)\sigma^{2p-d}}\pi^{-d/2}\Gamma(d-p).$$

Summarizing our computation, we have obtained the desired upper bound

$$\limsup_{t\to\infty} t^{-\frac{4+d-2p}{2}}\log\mathbb{E}\otimes\mathbb{E}_0\exp\Big\{-\theta\int_0^t \overline{V}(s,B_s)ds\Big\} \le \rho_3(\theta,\sigma^2). \tag{4.6}$$

On the other hand, for any $M>1$

$$\mathbb{E}_0\exp\Big\{\bar{\psi}_t(B)\Big\} \tag{4.7}$$
$$\ge \mathbb{E}_0\exp\Big\{\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t \frac{1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(s)-B_s|^p}ds\Big)dx\Big\}$$
$$\ge \mathbb{P}_0\Big\{\max_{s\le t}|B_s|\le t^{1/p}\Big\}\exp\Big\{\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\frac{M^p\theta}{(M+1)^p}\int_0^t \frac{1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(s)|^p}ds\Big)dx\Big\}.$$

Given $\epsilon>0$ there is a $\delta>0$ such that $\psi(a)\ge 2^{-1}(1-\epsilon)a^2$ whenever $0\le a\le\delta$. Take $M>0$ sufficiently large so

$$\frac{M^p\theta}{(M+1)^p}\int_0^t \frac{1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(s)|^p}ds \le \frac{M^p\theta}{(M+1)^p}M^{-p}\le\delta.$$

Consequently,

$$\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\frac{M^p\theta}{(M+1)^p}\int_0^t \frac{1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(s)|^p}ds\Big)dx \tag{4.8}$$
$$\ge \frac{1}{2}(1-\epsilon)\Big(\frac{M^p\theta}{(M+1)^p}\Big)^2\int_{\mathbb{R}^d}\mathbb{E}\Big[\int_0^t \frac{1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(s)|^p}ds\Big]^2dx.$$

Notice that

$$\int_{\mathbb{R}^d}\mathbb{E}\Big[\int_0^t \frac{1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(s)|^p}ds\Big]^2dx \tag{4.9}$$
$$= \int_0^t\!\!\int_0^t \mathbb{E}\Big[\int_{\mathbb{R}^d}\frac{1_{\{|x+X_0(r)|\ge Mt^{1/p}\}}1_{\{|x+X_0(s)|\ge Mt^{1/p}\}}}{|x+X_0(r)|^p|x+X_0(s)|^p}dx\Big]drds$$
$$= \int_0^t\!\!\int_0^t\Big[\int_{\{|x|\ge Mt^{1/p}\}}\frac{1}{|x|^p}\mathbb{E}\frac{1_{\{|x+X_0(s)-X_0(r)|\ge Mt^{1/p}\}}}{|x+X_0(s)-X_0(r)|^p}dx\Big]drds.$$

Recall that $U \sim N(0, I_d)$. By Brownian scaling, the right hand side of (4.9) becomes

$$\int_0^t\int_0^t \left[\int_{\{|x|\geq Mt^{1/p}\}} \frac{1}{|x|^p}\mathbb{E}\frac{1_{\{|x+|r-s|^{1/2}\sigma U|\geq Mt^{1/p}\}}}{|x+|r-s|^{1/2}\sigma U|^p}dx\right]drds$$
$$= \sigma^{-(2p-d)}\int_0^t\int_0^t |r-s|^{-\frac{2p-d}{2}}Q\Big(M\sigma^{-1}t^{1/p}|r-s|^{-1/2}\Big)drds$$
$$\geq \sigma^{-(2p-d)}Q\Big(M\sigma^{-1}u^{-1/2}t^{\frac{1}{p}-\frac{1}{2}}\Big)\iint_{\{[0,t]^2\cap\{|r-s|\geq ut\}\}} |r-s|^{-\frac{2p-d}{2}}drds,$$

where $Q(b)$ is defined by (6.4) in the Appendix. Note that

$$\iint_{\{[0,t]^2\cap\{|r-s|\geq ut\}\}} |r-s|^{-\frac{2p-d}{2}}drds = t^{\frac{4-2p+d}{2}}\iint_{\{[0,1]^2\cap\{|r-s|\geq u\}\}} |r-s|^{-\frac{2p-d}{2}}drds.$$

Therefore,

$$\begin{aligned}&\liminf_{t\to\infty} t^{-\frac{4-2p+d}{2}}\int_{\mathbb{R}^d}\mathbb{E}\left[\int_0^t \frac{1_{\{|x+X_0(s)|\geq Mt^{1/p}\}}}{|x+X_0(s)|^p}ds\right]^2dx \\ \geq\ & \sigma^{-(2p-d)}\lim_{b\to 0}Q(b)\iint_{\{[0,1]^2\cap\{|r-s|\geq u\}\}} |r-s|^{-\frac{2p-d}{2}}drds. \end{aligned} \tag{4.10}$$

Note that

$$\lim_{u\to 0+}\iint_{\{[0,1]^2\cap\{|r-s|\geq u\}\}} |r-s|^{-\frac{2p-d}{2}}drds = \frac{8}{(2+d-2p)(4+d-2p)}. \tag{4.11}$$

By (4.8), (4.10), (4.11) and Lemma 6.3, we get

$$\liminf_{t\to\infty} t^{-\frac{4-2p+d}{2}}\int_{\mathbb{R}^d}\mathbb{E}\psi\bigg(\frac{M^p\theta}{(M+1)^p}\int_0^t \frac{1_{\{|x+X_0(s)|\geq Mt^{1/p}\}}}{|x+X_0(s)|^p}ds\bigg)dx \geq (1-\epsilon)\Big(\frac{M}{M+1}\Big)^p\rho_3(\theta,\sigma^2).$$

Bringing this back to (4.7) and by the classic estimate

$$\log\mathbb{P}_0\Big\{\max_{s\leq t}|B_s|\leq t^{1/p}\Big\} \geq -Ct^{\frac{p-2}{p}} = -o\Big(t^{\frac{4+d-2p}{2}}\Big)$$

for large t and taking into account that ϵ can be arbitrarily small and M can be arbitrarily large, we obtain the desired lower bound. □

4.4 Regime V: $p > \max\left\{2, \frac{d+2}{2}\right\}$

The existence of the limit is obtained by super-additivity in t, which implies the existence of the limit. To prove the non-triviality of the limit, $\bar{\psi}$ is written into two part by comparing the distance between obstacles and the particle with a constant. When the distance is large, we have $\psi(u) \sim \frac{1}{2}u^2$. When the distance is small, we have $\psi(u) \sim u$. The upper bound can be estimated directly. The lower bound is obtained using Jensen's inequality and the scaling property of the Brownian motion.

It is straightforward to check that the function $\psi(\cdot)$ is super-additive on $\mathbb{R}^+$: $\psi(\alpha+\beta) \geq \psi(\alpha)+\psi(\beta)$ for $\alpha, \beta \geq 0$. In particular, for any $t_1, t_2 > 0$,

$$\int_{\mathbb{R}^d} \psi\Big(\theta \int_0^{t_1+t_2} \frac{ds}{|x+X_0(s)-B_s|^p}\Big)dx$$
$$\geq \int_{\mathbb{R}^d} \psi\Big(\theta \int_0^{t_1} \frac{ds}{|x+X_0(s)-B_s|^p}\Big)dx + \int_{\mathbb{R}^d} \psi\Big(\theta \int_{t_1}^{t_1+t_2} \frac{ds}{|x+X_0(s)-B_s|^p}\Big)dx.$$

Notice that the two terms on the right hand side are independent and,

$$\int_{\mathbb{R}^d} \psi\Big(\theta \int_{t_1}^{t_1+t_2} \frac{ds}{|x+X_0(s)-B_s|^p}\Big)dx \overset{d}{=} \int_{\mathbb{R}^d} \psi\Big(\theta \int_0^{t_2} \frac{ds}{|x+X_0(s)-B_s|^p}\Big)dx.$$

So we have that

$$\mathbb{E}_0 \exp\Big\{\bar{\psi}_{t_1+t_2}(B)\Big\} \geq \mathbb{E}_0 \exp\Big\{\bar{\psi}_{t_1}(B)\Big\}\mathbb{E}_0 \exp\Big\{\bar{\psi}_{t_2}(B)\Big\}.$$

Therefore, the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{\bar{\psi}_t(B)\Big\} = \rho_5(\theta, \sigma^2)$$

exists as an extended constant $0 < \rho_5(\theta, \sigma^2) \leq \infty$.

Lemma 4.2 *There are two finite constants $C_1(\sigma^2)$ and $C_2(\sigma^2)$ such that*

$$C_1(\sigma^2)\theta^{(d-2)/(p-2)} \leq \rho_5(\theta, \sigma^2) \leq C_2(\sigma^2)\theta^{(d-2)/(p-2)}. \tag{4.12}$$

Proof: Let $a > 0$ and $0 < \gamma < 1$ be fixed but arbitrary. By Pascal principle and convexity,

$$\bar{\psi}_t(B) \leq \bar{\psi}_t(0) \leq \gamma \int_{\mathbb{R}^d} \mathbb{E}\psi\Big(\theta\gamma^{-1} \int_0^t \frac{1\{|x+X_0(s)| \geq a\}}{|x+X_0(s)|^p} ds\Big)dx$$
$$+(1-\gamma) \int_{\mathbb{R}^d} \mathbb{E}\psi\Big(\frac{\theta}{1-\gamma} \int_0^t \frac{1\{|x+X_0(s)| < a\}}{|x+X_0(s)|^p} ds\Big)dx. \tag{4.13}$$

For the first term on the right hand side

$$\int_{\mathbb{R}^d} \mathbb{E}\psi\Big(\theta\gamma^{-1} \int_0^t \frac{1\{|x+X_0(s)| \geq a\}}{|x+X_0(s)|^p} ds\Big)dx \tag{4.14}$$
$$\leq \frac{\theta^2}{2\gamma^2} \int_{\mathbb{R}^d} \mathbb{E}\Big[\int_0^t \frac{1\{|x+X_0(s)| \geq a\}}{|x+X_0(s)|^p} ds\Big]^2 dx.$$

By the computation next to (4.9),

$$\int_{\mathbb{R}^d} \mathbb{E}\Big[\int_0^t \frac{1\{|x+X_0(s)| \geq a\}}{|x+X_0(s)|^p} ds\Big]^2 dx \tag{4.15}$$
$$= \int_0^t\int_0^t \Big[\int_{\{|x|\geq a\}} \frac{1}{|x|^p} \mathbb{E} \frac{1_{\{|x+X_0(s)-X_0(r)|\geq a\}}}{|x+X_0(s)-X_0(r)|^p} dx\Big] drds$$
$$= \sigma^{-(2p-d)} \int_0^t\int_0^t |r-s|^{-\frac{2p-d}{2}} Q\Big(a\sigma^{-1}|r-s|^{-1/2}\Big) drds,$$

where $Q(b)$ is defined in (6.4), and the equation follows from Brownian scaling and variable substitution.

Notice that

$$\int_0^t\int_0^t |r-s|^{-\frac{2p-d}{2}} Q\Big(a\sigma^{-1}|r-s|^{-1/2}\Big)drds = 2\int_0^t (t-s)s^{-\frac{2p-d}{2}}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds. \quad (4.16)$$

According to Lemma 6.4, $Q(b)=O(b^{-(2p-d)})$ as $b\to\infty$. By (6.5)

$$\int_0^\infty s^{-\frac{2p-d}{2}}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds<\infty$$

as $p>\frac{d+2}{2}$. Consequently, as $t\to\infty$,

$$\begin{aligned}&\int_0^t (t-s)s^{-\frac{2p-d}{2}}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds \sim t\int_0^\infty s^{-\frac{2p-d}{2}}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds\\ &=2\Big(\frac{\sigma}{a}\Big)^{2p-d-2}t\int_0^\infty s^{2p-d-3}Q(s)ds.\end{aligned}$$

Summarizing the computation, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathbb{E}\bigg[\int_0^t \frac{1\{|x+X_0(s)|\ge a\}}{|x+X_0(s)|^p}ds\bigg]^2dx \\ &\le \big(1+o(1)\big)a^{-(2p-d-2)}D(d,p,\sigma^2)t\end{aligned} \quad (4.17)$$

as $t\to\infty$, where

$$D(d,p,\sigma^2)=4\sigma^{-2}\int_0^\infty s^{2p-d-3}Q(s)ds.$$

As for the second term on the right hand side of (4.13), taking $K(x)=\theta(1-\gamma)^{-1}|x|^{-p}1_{\{|x|<a\}}$ in (3.6) gives

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathbb{E}\psi\bigg(\frac{\theta}{1-\gamma}\int_0^t\frac{1\{|x+X_0(s)|<a\}}{|x+X_0(s)|^p}ds\bigg)dx \quad (4.18)\\ &\le\int_{\{|x|<a\}}\psi\Big(\frac{\theta}{1-\gamma}\frac{t}{|x|^p}\Big)dx\\ &=\Big(\frac{\theta t}{1-\gamma}\Big)^{d/p}\int_{\{|x|\le a(1-\gamma)^{1/p}(\theta t)^{-1/p}\}}\psi\Big(\frac{1}{|x|^p}\Big)dx\\ &\sim\Big(\frac{\theta t}{1-\gamma}\Big)^{d/p}\int_{\{|x|\le a(1-\gamma)^{1/p}(\theta t)^{-1/p}\}}\frac{1}{|x|^p}dx\\ &=\Big(\frac{\theta t}{1-\gamma}\Big)^{d/p}\frac{d\omega_d}{d-p}a^{d-p}\Big((1-\gamma)^{1/p}(\theta t)^{-1/p}\Big)^{d-p}\\ &=\frac{\theta t}{1-\gamma}\frac{d\omega_d}{d-p}a^{d-p},\qquad (t\to\infty).\end{aligned}$$

Combining (4.14), (4.17) and (4.18), we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{\bar{\psi}_t(B)\right\} \le (2\gamma)^{-1}\theta^2 a^{-(2p-d-2)}D(d,p,\sigma^2)+\frac{\theta d\omega_d}{d-p}a^{d-p}. \tag{4.19}$$

Taking $\gamma\to 1^-$ and choosing a to minimize the right hand side of (4.19), we see that there is a constant $C_2(\sigma^2)$ (drop its dependency on $d,\ p$) such that

$$\rho_5(\theta,\sigma^2)\le C_2(\sigma^2)\theta^{(d-2)/(p-2)}.$$

This proves the upper bound in (4.12), and hence, the finiteness of $\rho_5(\theta,\sigma^2)$. In fact, by elementary calculus, it is easy to show that $C_2(\sigma^2)=K(d,p)\sigma^{-2(d-p)/(p-2)}$ with

$$K(d,p)=\left\{\left(\frac{d-p}{2p-d-2}\right)^{\frac{2p-d-2}{p-2}}+\left(\frac{2p-d-2}{p-2}\right)^{\frac{2p-d-2}{d-p}}\right\}D_1(d,p)^{\frac{2p-d-2}{d-p}}\left(\frac{d\omega_d}{d-p}\right)^{\frac{2p-d-2}{p-2}},$$

where $D_1(d,p)=\frac{1}{2}\sigma^2D(d,p,\sigma^2)$.

On the other hand, let α be a constant to be decided later. By Brownian scaling, we have

$$\begin{aligned}
&\frac{1}{t}\log\mathbb{E}_0\exp\left\{\bar{\psi}_t(B)\right\}\\
=\ &\frac{1}{t}\mathbb{E}_0\int_{\mathbb{R}^d}\mathbb{E}\psi\left(\theta\int_0^t\frac{ds}{|x+X_0(s)-B_s|^p}\right)dx\\
=\ &\frac{1}{t}\mathbb{E}_0\otimes\mathbb{E}\int_{\mathbb{R}^d}\psi\left(\theta^{1-\alpha}\int_0^{\theta^\alpha t}\frac{ds}{|x+\theta^{-\alpha/2}(X_0(s)-B_s)|^p}\right)dx\\
=\ &\frac{1}{t}\mathbb{E}_0\otimes\mathbb{E}\int_{\mathbb{R}^d}\psi\left(\theta^{1-\alpha}\theta^{\alpha p/2}\int_0^{\theta^\alpha t}\frac{ds}{|x\theta^{\alpha/2}+X_0(s)-B_s|^p}\right)dx\\
=\ &\theta^{-\alpha d/2}\theta^{\alpha}\frac{1}{\theta^\alpha t}\mathbb{E}_0\otimes\mathbb{E}\int_{\mathbb{R}^d}\psi\left(\theta^{1-\alpha}\theta^{\alpha p/2}\int_0^{\theta^\alpha t}\frac{ds}{|x+X_0(s)-B_s|^p}\right)dx.
\end{aligned}$$

Taking $\alpha=-\frac{2}{p-2}$ and letting $t\to\infty$, we see that

$$\begin{aligned}
\rho_5(\theta,\sigma^2)&\ge\theta^{(d-2)/(p-2)}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_0\otimes\mathbb{E}\int_{\mathbb{R}^d}\psi\left(\int_0^t\frac{ds}{|x+X_0(s)-B_s|^p}\right)dx\\
&\ge\theta^{(d-2)/(p-2)}C_1(\sigma^2)
\end{aligned}$$

where the existence of the limit follows from the superadditivity, and

$$C_1(\sigma^2)=\mathbb{E}_0\otimes\mathbb{E}\int_{\mathbb{R}^d}\psi\left(\int_0^1\frac{ds}{|x+X_0(s)-B_s|^p}\right)dx.$$

□

4.5 Regime IV: $p = \frac{d+2}{2} > 2$

One of the differences of Regimes IV and V is that the second term on the right hand side of (**??**) does make contribution in Regime IV. A much more delicate treatment is needed for this critical case. We first prove the upper bound.

Proposition 4.3

$$\limsup_{t\to\infty} \frac{1}{t\log t} \log \mathbb{E}_0 \exp\Big\{\bar{\psi}_t(B)\Big\} \le \rho_4(\theta,\sigma^2). \tag{4.20}$$

Proof: We continuing the calculation of (4.16) of the last subsection Setting $p = \frac{d+2}{2}$, we consider the decomposition

$$\begin{aligned}
&\int_0^t (t-s)s^{-1}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds \\
&= \int_0^b (t-s)s^{-1}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds + \int_b^t (t-s)s^{-1}Q\Big(a\sigma^{-1}s^{-1/2}\Big)ds \\
&= O(t) + \Big(1+q(b)\Big)C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p)\int_b^t (t-s)s^{-1}ds \\
&= O(t) + \Big(1+q(b)\Big)C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p)t\log t, \qquad (t\to\infty),
\end{aligned}$$

where $q(b)\to 0$ as $b\to\infty$ and the second step follows from (6.5).

Together with (4.15) and (4.16) in last subsection, we have,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}\bigg[\int_0^t \frac{1\{|x+X_0(s)|\ge a\}}{|x+X_0(s)|^p}ds\bigg]^2 dx \\
&= \big(1+o(1)\big)2\sigma^{-2}C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p)t\log t,
\end{aligned} \tag{4.21}$$

as $t\to\infty$.

The computation of the constant needs a little attention in this regime. We claim that

$$\theta^2\sigma^{-2}C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p) = \rho_4(\theta,\sigma^2) \tag{4.22}$$

By (6.6) with the relation $p = \frac{d+2}{2}$, the left hand side of (4.22) is equal to

$$2^{\frac{d-4}{2}}d\omega_d\Big(\frac{\theta}{\sigma}\Big)^2\left(\frac{\Gamma\Big(\frac{d-p}{2}\Big)}{\Gamma\Big(\frac{p}{2}\Big)}\right)^2 = 2^{\frac{d-4}{2}}d\omega_d\Big(\frac{\theta}{\sigma}\Big)^2\left(\frac{\Gamma\Big(\frac{d-2}{4}\Big)}{\Gamma\Big(\frac{d-2}{4}+1\Big)}\right)^2$$

Therefore, our assertion follows from the relation that

$$\Gamma\Big(\frac{d-2}{4}+1\Big) = \frac{d-2}{4}\Gamma\Big(\frac{d-2}{4}\Big).$$

In view of (4.14), (4.18) and (4.21),

$$\limsup_{t\to\infty}\frac{1}{t\log t}\log\mathbb{E}_0\exp\left\{\bar{\psi}_t(B)\right\}\le\gamma^{-1}\rho_4(\theta,\sigma^2).$$

Letting $\gamma\to 1^-$ on the right hand side, we finish the proof of (4.20). □

It remains to establish the corresponding lower bound. Again, the truncation level $a>0$ is fixed but arbitrary. The challenge is to reverse the inequality by Taylor expansion given in (4.14). Write $K_a(x)=|x|^{-p}1_{\{|x|\ge a\}}$ and

$$v(t,x)=\mathbb{E}\exp\left\{-\theta\int_0^t K_a\big(x+X_0(s)\big)ds\right\},\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

Lemma 4.4

$$\lim_{t\to\infty}\frac{1}{t\log t}\int_{\mathbb{R}^d}\mathbb{E}\psi\left(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\right)dx=\rho_4(\theta,\sigma^2). \tag{4.23}$$

Proof: By the fundamental theorem of calculus, we have

$$\begin{aligned}&\exp\left\{-\theta\int_0^t K_a\big(x+X_0(s)\big)ds\right\}\\&=1-\theta\int_0^t K_a\big(x+X_0(s)\big)\exp\left\{-\theta\int_s^t K_a\big(x+X_0(u)\big)du\right\}ds.\end{aligned}$$

Taking expectation on the both sides, by Markov property, we have

$$v(t,x)=1-\theta\sigma^{-d}\int_0^t\int_{\mathbb{R}^d}p_{t-s}\Big(\frac{x-y}{\sigma}\Big)K_a(y)v(s,y)dyds, \tag{4.24}$$

where

$$p_t(x)=\frac{1}{(2\pi t)^{d/2}}\exp\left\{-\frac{|x|^2}{2t}\right\}.$$

Recall that $\psi(x)=e^{-x}-1+x$. Hence,

$$\begin{aligned}\mathbb{E}\psi\left(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\right)&=v(t,x)-1+\theta\mathbb{E}\int_0^t K_a\big(x+X_0(s)\big)ds\\&=\theta\sigma^{-d}\int_{\mathbb{R}^d}K_a(y)\int_0^t p_{t-s}\Big(\frac{x-y}{\sigma}\Big)\Big(1-v(s,y)\Big)dyds.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathbb{E}\psi\left(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\right)dx\\&=\theta\int_{\mathbb{R}^d}K_a(y)\int_0^t\Big(1-v(s,y)\Big)dyds\\&=\theta^2\sigma^{-d}\int_{\mathbb{R}^d}K_a(y)\int_0^t\left[\int_0^s\int_{\mathbb{R}^d}p_{s-u}\Big(\frac{y-x}{\sigma}\Big)K_a(x)v(u,x)dxdu\right]dyds\\&=\theta^2\sigma^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_a(x)K_a(y)\left[\int_0^t v(u,x)\left(\int_0^{t-u}p_s\Big(\frac{y-x}{\sigma}\Big)ds\right)du\right]dxdy,\end{aligned}$$

where the second step follows from (4.24).

Taking Laplace transform on the both sides, for any $\lambda > 0$

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\Big[\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\Big)dx\Big]dt \\
&=\lambda^{-1}\theta^2\sigma^{-d}\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_a(x)K_a(y)\Big(\int_0^\infty e^{-\lambda t}v(t,x)dt\Big)\Big(\int_0^\infty e^{-\lambda t}p_t\Big(\frac{y-x}{\sigma}\Big)dt\Big)dxdy.
\end{aligned}$$

By the inequality $e^{-x}\geq 1-x$, $(x\geq 0)$,

$$v(t,x)\geq 1-\theta\mathbb{E}\int_0^t K_a(x+X_0(s))ds = 1-\theta\sigma^{-d}\int_{\mathbb{R}^d}\int_0^t K_a(z)p_s\Big(\frac{z-x}{\sigma}\Big)dsdz$$

Hence,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t}v(t,x)dt &\geq \lambda^{-1}\Big\{1-\theta\sigma^{-d}\int_{\mathbb{R}^d}K_a(z)\Big(\int_0^\infty e^{-\lambda t}p_t\Big(\frac{z-x}{\sigma}\Big)dt\Big)dz\Big\} \\
&\geq \lambda^{-1}\Big\{1-\theta\sigma^{-d}\int_{\mathbb{R}^d}K_a(z)\Big(\int_0^\infty p_t\Big(\frac{z-x}{\sigma}\Big)dt\Big)dz\Big\}.
\end{aligned}$$

Notice that $d\geq 3$. It is well-known that

$$\int_0^\infty p_t(x)dt = C_d|x|^{-(d-2)}\qquad x\in\mathbb{R}^d.$$

Summarizing our estimate,

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\Big[\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\Big)dx\Big]dt \\
&\geq \lambda^{-2}\theta^2\sigma^{-d}\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_a(x)K_a(y)\Big(\int_0^\infty e^{-\lambda t}p_t\Big(\frac{y-x}{\sigma}\Big)dt\Big)dxdy \\
&-C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_a(x)K_a(y)K_a(z)\frac{1}{|y-x|^{d-2}}\frac{1}{|z-x|^{d-2}}dxdydz\Big\}.
\end{aligned}\tag{4.25}$$

For the first term on the right hand side, notice that

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_a(x)K_a(y)\Big(\int_0^\infty e^{-\lambda t}p_t\Big(\frac{y-x}{\sigma}\Big)dt\Big)dxdy \\
&=\sigma^{2d}\int_0^\infty e^{-\lambda t}t^{d/2}\Big[\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_a(\sigma t^{1/2}x)K_a(\sigma t^{1/2}y)p_1(y-x)dxdy\Big]dt \\
&=\sigma^{2(d-p)}\int_0^\infty e^{-\lambda t}t^{-1}Q(a\sigma^{-1}t^{-1/2})dt \\
&\sim \sigma^{2(d-p)}C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p)\log\frac{1}{\lambda}
\end{aligned}$$

as $\lambda \to 0^+$, where $Q(b)$ is defined in (6.4) and the last step follows from (6.5) and the fact (Lemma 6.4) that $t^{-1}Q(a\sigma t^{-1/2})$ is bounded for t in a neighborhood of 0.

As for the second term in (4.25), notice that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_a(y)\frac{1}{|y-x|^{d-2}}dy \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|y|^p|y-x|^{d-2}}dy = C_1\frac{1}{|x|^{p-2}}, \qquad x\in\mathbb{R}^d.$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_a(x)K_a(y)K_a(z)\frac{1}{|y-x|^{d-2}}\frac{1}{|z-x|^{d-2}}dxdydz\\ &\le C_2\int_{\mathbb{R}^d} K_a(x)\frac{1}{|x|^{p-2}}\frac{1}{|x|^{p-2}}dx = C_2\int_{|x|\ge a\}}\frac{dx}{|x|^{3p-4}} < \infty,\end{aligned}$$

where the last step follows from the fact that $3p-4>d$ in Regime IV.

By (4.25), with $p=\frac{d+2}{2}$

$$\begin{aligned}&\liminf_{\lambda\to 0^+}\lambda^2\Big(\log\frac{1}{\lambda}\Big)^{-1}\int_0^\infty e^{-\lambda t}\Big[\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\Big)dx\Big]dt\\ &\ge \theta^2\sigma^{-2}C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p) = \rho_4(\theta,\sigma^2).\end{aligned}$$

where the equality comes from (4.22).

On the other hand, from the relation

$$\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\Big)dx \le \frac{\theta^2}{2}\int_{\mathbb{R}^d}\mathbb{E}\Big[\int_0^t K_a\big(x+X_0(s)\big)ds\Big]^2dx$$

and from (4.17) we derive that

$$\limsup_{\lambda\to 0^+}\lambda^2\Big(\log\frac{1}{\lambda}\Big)^{-1}\int_0^\infty e^{-\lambda t}\Big[\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\Big)dx\Big]dt \le \rho_4(\theta,\sigma^2).$$

Consequently,

$$\lim_{\lambda\to 0^+}\lambda^2\Big(\log\frac{1}{\lambda}\Big)^{-1}\int_0^\infty e^{-\lambda t}\Big[\int_{\mathbb{R}^d}\mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)\big)ds\Big)dx\Big]dt = \rho_4(\theta,\sigma^2).$$

(4.23) then follows from Tauberian theorem (see Lemma 2.1.1. of Yakimiv [42], p91). □

Finally, we are ready to consider the lower bounded in this regime.

Proposition 4.5

$$\lim_{t\to\infty}\frac{1}{t\log t}\log\mathbb{E}_0\exp\Big\{\bar{\psi}_t(B)\Big\} = \rho_4(\theta,\sigma^2). \tag{4.26}$$

Proof: Let $a > 1$. We have

$$\begin{aligned}
&\mathbb{E}_0 \exp\big(\bar{\psi}_t(B)\big)\\
&\geq \mathbb{E}_0 \exp\bigg\{\int_{\mathbb{R}^d} \mathbb{E}\psi\Big(\theta\int_0^t K_a\big(x+X_0(s)-B_s\big)ds\Big)dx\Big)\bigg\}\\
&\geq \mathbb{P}_0\Big\{\max_{s\leq t}|B_s|\leq 1\Big\}\exp\bigg\{\int_{\mathbb{R}^d}\mathbb{E}\psi\bigg(\Big(\frac{a}{a+1}\Big)^p\theta\int_0^t K_a\big(x+X_0(s)\big)ds\bigg)dx\bigg\}.
\end{aligned}$$

Replacing θ by $\left(\frac{a}{a+1}\right)^p\theta$ in (4.23),

$$\liminf_{t\to\infty}\frac{1}{t\log t}\log\mathbb{E}_0\exp\Big\{\bar{\psi}_t(B)\Big\}\geq \rho_4\Big(\Big(\frac{a}{a+1}\Big)^p\theta,\sigma^2\Big).$$

Letting $a\to\infty$ on the right hand side and then combining it with (4.20), we obtain (4.26). □

Finally, the conclusion of Theorem 2.1 for Regime IV follows from (3.4) with $K(x) = \theta|x|^{-p}$ and Propositions 4.3 and 4.5.

5 Brownian motion in catalytic medium

We prove Theorem 2.2 in this section. Note that the Assumption (2.1) remains in force in this section. The proof is splitted into three sub-sections according to the value of p.

The following notations are used in this section. For $R > 0$ and $x \in \mathbb{R}^d$, $B(x, R)$ represents the d-dimensional ball with the center x and the radius R. Given an open domain $D \subset \mathbb{R}^d$, $W^{1,2}(D)$ is the Sobolev space over D, defined as the closure of the inner product space of the infinitely differentiable functions compactly supported in D under the Sobolev norm

$$\|g\|_H = \Big\{\|g\|^2_{\mathcal{L}^2(D)} + \|\nabla g\|^2_{\mathcal{L}^2(D)}\Big\}^{1/2}$$

Write

$$\mathcal{F}_d(D) = \Big\{g \in W^{1,2}(D);\ \ \|g\|_{\mathcal{L}^2(D)} = 1\Big\}$$

In particular, $\mathcal{F}_d = \mathcal{F}_d(\mathbb{R}^d)$.

5.1 Sub-critical case $p < 2$

The strategy of the proof is as follows: For the upper bound, a direct use of Pascal principle enables us to assume the particle is motionless. For the lower bound, we restrict ourselves to the event that the particle is almost motionless in the sense of $\big\{\sup_{s\leq t}|B_s|\leq\epsilon\big\}$. Although the probability of this event tends to 0 as $t\to\infty$, its rate is much slower than double exponential.

Given $0 < \gamma < 1$, by convexity of $\Psi(\cdot)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}\Psi\bigg(\theta\int_0^t \frac{ds}{|x+X_0(s)|^p}\bigg)dx \qquad (5.1)\\
&\le (1-\gamma)\int_{\mathbb{R}^d}\mathbb{E}\Psi\bigg(\frac{\theta}{1-\gamma}\int_0^t \frac{1\{|x+X_0(s)|>t^{1/p}\}}{|x+X_0(s)|^p}ds\bigg)dx\\
&+\gamma\int_{\mathbb{R}^d}\mathbb{E}\Psi\bigg(\gamma^{-1}\theta\int_0^t \frac{1\{|x+X_0(s)|\le t^{1/p}\}}{|x+X_0(s)|^p}ds\bigg)dx.
\end{aligned}$$

For the first term, by Jensen inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Psi\bigg(\frac{\theta}{1-\gamma}\int_0^t \frac{1\{|x+X_0(s)|>t^{1/p}\}}{|x+X_0(s)|^p}ds\bigg)dx\\
\le\ & \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}\Psi\bigg(\frac{\theta}{1-\gamma}\frac{t1\{|x+X_0(s)|>t^{1/p}\}}{|x+X_0(s)|^p}ds\bigg)dxds\\
=\ & \int_{\{|x|\ge t^{1/p}\}}\Psi\bigg(\frac{\theta}{1-\gamma}\frac{t}{|x|^p}\bigg)dx\\
=\ & \Big(\frac{\theta t}{1-\gamma}\Big)^{d/p}\int_{\{|x|\ge (1-\gamma)^{1/p}\theta^{-1/p}\}}\Psi\bigg(\frac{1}{|x|^p}\bigg)dx<\infty. \qquad (5.2)
\end{aligned}$$

As for the second term of (5.1), we use the bound

$$\int_0^t \frac{1\{|x+X_0(s)|\le t^{1/p}\}}{|x+X_0(s)|^p}ds \le 1_{\{|x|\le t^{1/p}+\max_{s\le t}|X_0(s)|\}}\sup_{x\in\mathbb{R}^d}\int_0^t\frac{ds}{|x+X_0(s)|^p}.$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mathbb{E}\Psi\bigg(\gamma^{-1}\theta\int_0^t \frac{1\{|x+X_0(s)|\le t^{1/p}\}}{|x+X_0(s)|^p}ds\bigg)dx \qquad (5.3)\\
\le\ & \omega_d\mathbb{E}\bigg[\Big(t^{1/p}+\max_{s\le t}|X_0(s)|\Big)^d\Psi\bigg(\gamma^{-1}\theta\sup_{y\in\mathbb{R}^d}\int_0^t\frac{ds}{|y+X_0(s)|^p}\bigg)\bigg]\\
\le\ & \omega_d\mathbb{E}\bigg[\Big(t^{1/p}+\max_{s\le t}|X_0(s)|\Big)^d\exp\bigg\{\gamma^{-1}\theta\sup_{y\in\mathbb{R}^d}\int_0^t\frac{ds}{|y+X_0(s)|^p}\bigg\}\bigg]\\
\le\ & \omega_d\bigg\{\mathbb{E}\Big(t^{1/p}+\max_{s\le t}|X_0(s)|\Big)^{\frac{d}{1-\gamma}}\bigg\}^{1-\gamma}\bigg(\mathbb{E}\exp\bigg\{\gamma^{-2}\theta\sup_{y\in\mathbb{R}^d}\int_0^t\frac{ds}{|y+X_0(s)|^p}\bigg\}\bigg)^{\gamma}.
\end{aligned}$$

By Theorem 1.3 in [1],

$$\mathbb{E}\exp\bigg\{\gamma^{-2}\theta\sup_{y\in\mathbb{R}^d}\int_0^t\frac{ds}{|y+X_0(s)|^p}\bigg\}<\infty. \qquad (5.4)$$

It follows from (5.1)-(5.4) that

$$\int_{\mathbb{R}^d} \mathbb{E}\Psi\left(\theta\int_0^t \frac{ds}{|x+X_0(s)|^p}\right)dx < \infty.$$

From (3.5) and (3.10), therefore, we obtain (2.6).

Further, according to Theorem 1.3, [1],

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\gamma^{-2}\theta\sup_{y\in\mathbb{R}^d}\int_0^t\frac{ds}{|y+X_0(s)|^p}\right\}\\
&=\sup_{g\in\mathcal{F}_d}\left\{\gamma^{-2}\theta\sigma^{-p}\int_{\mathbb{R}^d}\frac{g^2(x)}{|x|^p}dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\
&=\frac{2-p}{2}p^{\frac{p}{2-p}}\Big(\gamma^{-2}\theta\sigma^{-p}\gamma(d,p)\Big)^{\frac{2}{2-p}},
\end{aligned}$$

where $\mathcal{F}_d=\{g\in W^{1,2}(\mathbb{R}^d);\ \ \|g\|_2=1\}$, and the last step follows from Lemma 7.3, [10].

By (5.1), (3.5) and (3.10) again,

$$\limsup_{t\to\infty}\frac{1}{t}\log\log\mathbb{E}_0\otimes\mathbb{E}\exp\left\{\theta\int_0^t\overline{V}(s,B_s)ds\right\}\le\frac{2-p}{2}p^{\frac{p}{2-p}}\Big(\gamma^{-2}\theta\sigma^{-p}\gamma(d,p)\Big)^{\frac{2}{2-p}}.$$

Letting $r\to 1^-$ on the right hand side leads to the desired upper bound for (2.7).

We now establish the lower bound for (2.7). Given $\epsilon>0$, we estimate as follows

$$\begin{aligned}
&\mathbb{E}_0\exp\left\{\int_{\mathbb{R}^d}\mathbb{E}\Psi\left(\theta\int_0^t\frac{ds}{|x+X_0(s)-B_s|^p}\right)dx\right\} \qquad (5.5)\\
&\ge\mathbb{P}_0\Big\{\max_{s\le t}|B_s|\le\frac{\epsilon}{2}\Big\}\exp\left\{\int_{\{|x|\le\frac{\epsilon}{2}\}}\mathbb{E}\Psi\left(\theta\int_0^t\frac{ds}{\big(|x+X_0(s)|+\frac{\epsilon}{2}\big)^p}\right)dx\right\}\\
&\ge\mathbb{P}_0\Big\{\max_{s\le t}|B_s|\le\frac{\epsilon}{2}\Big\}\exp\left\{\omega_d\Big(\frac{\epsilon}{2}\Big)^d\mathbb{E}\Psi\left(\theta\int_0^t\frac{ds}{\big(\epsilon+|X_0(s)|\big)^p}\right)\right\}.
\end{aligned}$$

By the fact that $\Psi(b)\sim e^b$ as $b\to\infty$,

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Psi\left(\theta\int_0^t\frac{ds}{\big(\epsilon+|X_0(s)|\big)^p}\right)\\
&=\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\int_0^t\frac{ds}{\big(\epsilon+|X_0(s)|\big)^p}\right\}\\
&=\sup_{g\in\mathcal{F}_d}\left\{\theta\int_{\mathbb{R}^d}\frac{g^2(x)}{(\epsilon+\sigma|x|)^p}dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\},
\end{aligned}$$

where the last step follows from a standard treatment of LDP by Feynman-Kac formula (see, e.g., Theorem 1.6, Chapter 4, [9]).

Since $\epsilon > 0$ can be arbitrarily small, we obtain

$$\liminf_{t\to\infty} \frac{1}{t} \log\log \mathbb{E}_0 \exp\left\{ \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)-B_s|^p}\right) dx \right\} \tag{5.6}$$
$$\geq \sup_{g\in\mathcal{F}_d} \left\{\theta\sigma^{-p} \int_{\mathbb{R}^d} \frac{g^2(x)}{|x|^p} dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\}.$$

Thus, the desired lower bound for (2.7) follows from (3.5) and the fact (Lemma 7.3, [10]) that the supremum above is equal to the constant appearing on the right hand side of (2.7). □

5.2 Critical case $p = 2$ and $d = 3$

For this critical case, a much more delicate treatment than that of last subsection is needed.

Proposition 5.1 *If $\theta > \sigma^2/8$, then*

$$\mathbb{E}_0 \exp\left\{ \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)-B_s|^p}\right) dx \right\} = \infty, \qquad (t > 0). \tag{5.7}$$

Proof: Let $t > 0$ be fixed. Given $\epsilon > 0$, by modifying (5.5) slightly,

$$\mathbb{E}_0 \exp\left\{ \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)-B_s|^2}\right) dx \right\} \tag{5.8}$$
$$\geq \mathbb{P}_0\left\{\max_{s\leq t}|B_s| \leq \epsilon\right\} \exp\left\{ \int_{\{|x|\leq 1\}} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{\big(\epsilon+|x+X_0(s)|\big)^2}\right) dx \right\}.$$

Let $0 < \delta < t$ be fixed but arbitrary. By (3.16) in Lemma 3.5, [13]

$$\int_{\{|x|\leq 1\}} \mathbb{E}\exp\left\{\theta \int_0^t \frac{ds}{\big(\epsilon+|x+X_0(s)|\big)^2}\right\} dx$$
$$\geq (2\pi\delta)^{3/2} \exp\left\{ -\delta\epsilon^{-2} + t \sup_{g\in\mathcal{F}_3(B(0,1))} \left(\theta \int_{B(0,1)} \frac{g^2(x)}{(\epsilon+\sigma|x|)^2} dx - \frac{1}{2}\int_{B(0,1)} |\nabla g(x)|^2 dx\right)\right\}.$$

By Lemma 6.2, there is a small $a > 0$ such that

$$\sup_{g\in\mathcal{F}_3(B(0,a\epsilon^{-1}))} \left\{\theta \int_{\{|x|\leq a\epsilon^{-1}\}} \frac{g^2(x)}{(a+\sigma|x|)^2} dx - \frac{1}{2}\int_{\{|x|\leq a\epsilon^{-1}\}} |\nabla g(x)|^2 dx\right\} \geq t^{-1}$$

for sufficiently small $\epsilon > 0$.

By the substitution $g(x) = \left(\frac{\epsilon}{a}\right)^{3/2} f\left(\frac{\epsilon}{a}x\right)$, therefore,

$$\sup_{g\in\mathcal{F}_3(B(0,1))} \left(\theta \int_{B(0,1)} \frac{g^2(x)}{(\epsilon+\sigma|x|)^2} dx - \frac{1}{2}\int_{B(0,1)} |\nabla g(x)|^2 dx\right) \geq t^{-1}\left(\frac{a}{\epsilon}\right)^2.$$

Take $\delta < a^2$. We conclude that there is $\gamma > 0$ such that

$$\int_{\{|x|\le 1\}} \mathbb{E}\exp\left\{\theta \int_0^t \frac{ds}{\big(\epsilon + |x + X_0(s)|\big)^2}\right\}dx \ge \exp\left\{\Big(\frac{\gamma}{\epsilon}\Big)^2\right\}$$

for all small $\epsilon > 0$. By the fact that

$$\begin{aligned}&\int_{\{|x|\le 1\}} \mathbb{E}\Psi\bigg(\theta \int_0^t \frac{ds}{\big(\epsilon + |x + X_0(s)|\big)^2}\bigg\}dx\bigg)\\ &= \int_{\{|x|\le 1\}} \mathbb{E}\exp\left\{\theta \int_0^t \frac{ds}{\big(\epsilon + |x + X_0(s)|\big)^2}\right\}dx - \frac{4}{3}\pi - t\int_{\{|x|\le 1\}} \frac{dx}{\big(\epsilon + |x|\big)^2},\end{aligned}$$

we have

$$\int_{\{|x|\le 1\}} \mathbb{E}\Psi\bigg(\theta \int_0^t \frac{ds}{\big(\epsilon + |x + X_0(s)|\big)^2}\bigg\}dx\bigg) \ge \exp\left\{\Big(\frac{\gamma}{\epsilon}\Big)^2\right\},$$

for a possibly different $\gamma > 0$.

By the classic estimate

$$\mathbb{P}_0\Big\{\max_{s\le t}|B_s| \le \epsilon\Big\} \ge 1 - c_1\exp\Big\{-c_2\epsilon^2\Big\}$$

for some constants $c_1,\ c_2 > 0$, and by (5.8),

$$\begin{aligned}&\mathbb{E}_0\exp\left\{\int_{\mathbb{R}^3} \mathbb{E}\Psi\bigg(\theta\int_0^t \frac{ds}{|x + X_0(s) - B_s|^2}\bigg)dx\right\}\\ &\ge \Big(1 - c_1\exp\Big\{-c_2\epsilon^2\Big\}\Big)\exp\left\{\exp\left\{\Big(\frac{\gamma}{\epsilon}\Big)^2\right\}\right\} \longrightarrow \infty \qquad (\epsilon \to 0^+).\end{aligned}$$

This implies (5.7). □

Proposition 5.2 *If $\theta < \sigma^2/8$, then*

$$\int_{\mathbb{R}^3} \mathbb{E}\Psi\bigg(\theta\int_0^1 \frac{ds}{|x + X_0(s)|^2}\bigg)dx < \infty. \tag{5.9}$$

Proof: Let $0 < \gamma < 1$ be sufficiently close to 1 so that $\gamma^{-1}\theta < \sigma^2/8$. By convexity

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathbb{E}\Psi\bigg(\theta\int_0^1 \frac{ds}{|x + X_0(s)|^2}\bigg)dx\\ &\le (1-\gamma)\int_{\mathbb{R}^3} \mathbb{E}\Psi\bigg(\frac{\theta}{1-\gamma}\int_0^1 \frac{1\{|x + X_0(s)| > 1\}}{|x + X_0(s)|^2}ds\bigg)dx\\ &+ \gamma\int_{\mathbb{R}^3} \mathbb{E}\Psi\bigg(\frac{\theta}{\gamma}\int_0^1 \frac{1\{|x + X_0(s)| \le 1\}}{|x + X_0(s)|^2}ds\bigg)dx.\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}&\int_{\mathbb{R}^3}\Psi\bigg(\frac{\theta}{1-\gamma}\int_0^1\frac{1\{|x+X_0(s)|>1\}}{|x+X_0(s)|^2}ds\bigg)dx\\&\le\int_{\mathbb{R}^3}\int_0^1\Psi\bigg(\frac{\theta}{1-\gamma}\frac{1\{|x+X_0(s)|>1\}}{|x+X_0(s)|^2}\bigg)dxds\\&=\int_{\{|x|\ge1\}}\Psi\Big(\frac{\theta}{1-\gamma}\frac{1}{|x|^2}\Big)dx<\infty.\end{aligned}$$

To prove (5.9), therefore, all we need is to show is that for any $\theta<\sigma^2/8$,

$$\int_{\mathbb{R}^3}\mathbb{E}\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg)dx<\infty. \tag{5.10}$$

Write $\tau_n=\inf\{s\ge0;\ \ |X_0(s)|\ge2^n\}$ $(n=0,1,\cdots)$. We have

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathbb{E}\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg)dx\\&=\int_{\mathbb{R}^3}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_0\ge1\bigg]dx\\&+\sum_{n=0}^\infty\int_{\mathbb{R}^3}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_n<1\le\tau_{n+1}\bigg]dx.\end{aligned}$$

By the fact that $\Psi(0)=0$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_0\ge1\bigg]dx\\&=\int_{\{|x|\le2\}}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_0\ge1\bigg]dx\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_n<1\le\tau_{n+1}\bigg]dx\\&=\int_{\{|x|\le2^{n+1}+1\}}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_n<1\le\tau_{n+1}\bigg]dx\\&\le\int_{\{|x|\le2^{n+2}\}}\mathbb{E}\bigg[\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le1\}}{|x+X_0(s)|^2}ds\bigg);\ \tau_n<1\le\tau_{n+1}\bigg]dx.\end{aligned}$$

By the inequality $\Psi(b)\le e^b$ for $b\ge 0$, therefore,

$$\int_{\mathbb{R}^3}\mathbb{E}\Psi\bigg(\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg)dx \tag{5.11}$$
$$\le\int_{\{|x|\le 2\}}\mathbb{E}\bigg[\exp\bigg\{\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg\};\tau_0\ge 1\bigg]dx$$
$$+\sum_{n=0}^\infty\int_{\{|x|\le 2^{n+2}\}}\mathbb{E}\bigg[\exp\bigg\{\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg\};\tau_n<1\le\tau_{n+1}\bigg]dx.$$

Take $\alpha,\beta>1$ such that $\alpha^{-1}+\beta^{-1}=1$ and that $\alpha\theta<\sigma^2/8$. By Hölder inequality

$$\int_{\{|x|\le 2^{n+2}\}}\mathbb{E}\bigg[\exp\bigg\{\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg\};\tau_n<1\le\tau_{n+1}\bigg]dx$$
$$\le\Big\{\frac{4}{3}\pi 2^{3(n+2)}\mathbb{P}\{\tau_n<1\}\Big\}^{1/\beta}$$
$$\times\bigg\{\int_{\{|x|\le 2^{n+2}\}}\mathbb{E}\bigg[\exp\bigg\{\alpha\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg\};\tau_{n+1}\ge 1\bigg]dx\bigg\}^{1/\alpha}.$$

For each $x\in\mathbb{R}^3$ with $|x|\le 2^{n+2}$, write

$$T_x=\inf\Big\{s\ge 0;\quad |x+X_0(s)|\ge 2^{n+3}\Big\}.$$

Then we have

$$\int_{\{|x|\le 2^{n+2}\}}\mathbb{E}\bigg[\exp\bigg\{\alpha\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg\};\tau_{n+1}\ge 1\bigg]dx$$
$$\le\int_{\{|x|\le 2^{n+3}\}}\mathbb{E}\bigg[\exp\bigg\{\alpha\theta\int_0^1\frac{ds}{|x+X_0(s)|^2}\bigg\};T_x\ge 1\bigg]dx$$
$$\le\frac{4}{3}\pi 2^{3(n+3)}\exp\bigg\{\sup_{g\in\mathcal{F}_d(B(0,2^{n+3}))}\bigg(\alpha\theta\sigma^{-2}\int_{B(0,2^{n+3})}\frac{g^2(x)}{|x|^2}dx-\frac{1}{2}\int_{B(0,2^{n+3})}|\nabla g(x)|^2dx\bigg)\bigg\},$$

where the last step follows from Lemma 4.1, [10].

Notice that $\alpha\theta\sigma^{-2}<1/8$. By (6.3) the g-variation on the right hand side is equal to zero. Hence,

$$\int_{\{|x|\le 2^{n+2}\}}\mathbb{E}\bigg[\exp\bigg\{\theta\int_0^1\frac{1\{|x+X_0(s)|\le 1\}}{|x+X_0(s)|^2}ds\bigg\};\tau_n<1\le\tau_{n+1}\bigg]dx$$
$$\le\frac{4}{3}\pi 2^{3(n+3)}\Big(\mathbb{P}\{\tau_n<1\}\Big)^{1/\beta}\le C_1 2^{3n}\exp\Big\{-C_2 2^{2n}\Big\}.$$

A similar argument shows that the first term on the right hand side of (5.11) is finite. Hence, we have proved (5.10), and therefore (5.9).

For any $t > 0$, by (3.10)

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)-B_s|^2}\right)dx \right\} \qquad (5.12)\\
&\le \exp\left\{ \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)|^2}\right)dx \right\}\\
&= \exp\left\{ t^{3/2} \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^1 \frac{ds}{|x+X_0(s)|^2}\right)dx \right\},
\end{aligned}$$

where the last step follows from Brownian scaling and variable substitution. By (5.9) the right hand side is finite. Combining this bound with (5.7), by (3.5) (with $K(x) = \theta|x|^{-p}$) we have proved (2.8). □

Finally, we are ready to prove the limit (2.9). Using (3.10) and (3.5), the upper bound for (2.9) follows from (5.12) directly. As for the lower bound, notice that

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)-B_s|^2}\right)dx \right\}\\
&= \mathbb{E}_0 \exp\left\{ t^{3/2} \int_{\mathbb{R}^3} \mathbb{E}\Psi\left(\theta \int_0^1 \frac{ds}{|x+X_0(s)-B_s|^2}\right)dx \right\}
\end{aligned}$$

Thus, the lower bound follows from the same argument as in Subsection 4.2. □

5.3 Super-critical case $p > 2$

Let $t > 0$ be fixed. By Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ \int_{\mathbb{R}^d} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X(s)-B_s|^p}\right)dx \right\}\\
&\ge \exp\left\{ \int_{\mathbb{R}^d} \mathbb{E}_0\otimes\mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+X_0(s)-B_s|^p}\right)dx \right\}\\
&= \exp\left\{ \int_{\mathbb{R}^d} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+\sqrt{1+\sigma^{-2}}X_0(s)|^p}\right)dx \right\}.
\end{aligned}$$

Hence, (2.10) follows from (3.5) and the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}\Psi\left(\theta \int_0^t \frac{ds}{|x+\sqrt{1+\sigma^{-2}}X_0(s)|^p}\right)dx\\
&\ge \mathbb{P}\Big\{ \max_{s\le t} |X_0(s)| \le \frac{\epsilon}{\sqrt{1+\sigma^{-2}}}\Big\} \int_{\{|x|\ge 2\epsilon\}} \Psi\Big(\frac{t\theta}{2^p|x|^p}\Big\}dx\\
&\ge \exp\big\{-c_1\epsilon^{-2}\big\}\exp\big\{-c_2\epsilon^{-p}\big\} \longrightarrow \infty, \qquad (\epsilon\to 0^+).
\end{aligned}$$

□

6 Appendix

In this section, we prove some technical results used in the main body of the paper.

6.1 Hardy inequality

Hardy's inequality plays an important role in this paper. Searching in literature, we have found large amount of versions of Hardy's inequality (i.e., [29] and [33]) but the form needed in this paper. For reader's convenience, we state Hardy's inequality for $d = 3$ in the following lemma and provide a short proof.

Lemma 6.1 *For any $f \in W^{1,2}(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \leq 4 \int_{\mathbb{R}^3} |\nabla f(x)|^2 dx. \tag{6.1}$$

Further, the number 4 is the best constant in the sense that for any $\epsilon > 0$ one can find a function $f_\epsilon \in W^{1,2}(\mathbb{R}^3)$ with compact support such that

$$\int_{\mathbb{R}^3} \frac{f_\epsilon^2(x)}{|x|^2} dx > (4-\epsilon) \int_{\mathbb{R}^3} |\nabla f_\epsilon(x)|^2 dx. \tag{6.2}$$

Proof: Write $x = (x_1, x_2, x_3)$. Using integration by parts

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx = \int_{\mathbb{R}^3} x_j \Big[\frac{2x_i}{|x|^4} f^2(x) - \frac{2}{|x|^2} f(x) \frac{\partial f}{\partial x_j} \Big] dx \qquad j = 1, 2, 3.$$

Summing over j on the both sides

$$3 \int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx = 2 \int_{\mathbb{R}^3} \Big[\frac{f^2(x)}{|x|^2} - \frac{\nabla f \cdot x}{|x|^2} f(x) \Big] dx.$$

Thus,

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx = -2 \int_{\mathbb{R}^3} \frac{\nabla f \cdot x}{|x|} \frac{f(x)}{|x|} dx \leq 2 \Big(\int_{\mathbb{R}^3} \frac{|\nabla f \cdot x|^2}{|x|^2} dx \Big)^{1/2} \Big(\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \Big)^{1/2}.$$

Therefore,

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \leq 4 \int_{\mathbb{R}^3} \frac{|\nabla f \cdot x|^2}{|x|^2} dx \leq 4 \int_{\mathbb{R}^3} |\nabla f(x)|^2 dx.$$

To establish (6.2), for each large $M > 0$, we define $g_M \in W^{1,2}(\mathbb{R}^3)$ as following:

$$g_M(x) = \begin{cases} M^{1/2} & 0 \leq |x| \leq M^{-1} \\ |x|^{-1/2} & M^{-1} < |x| \leq M \\ \dfrac{2M - |x|}{M^{3/2}} & M < |x| \leq 2M \\ 0 & |x| > 2M. \end{cases}$$

It is straightforward to exam that g_M is locally supported and

$$\int_{\mathbb{R}^3} \frac{g_M^2(x)}{|x|^2} dx = \left\{ 4 - 28\left(\frac{7}{3} + \frac{1}{2}\log M\right)^{-1} \right\} \int_{\mathbb{R}^3} |\nabla g_M(x)|^2 dx.$$

For each $\epsilon > 0$, take $M > 0$ sufficiently large so

$$28\left(\frac{7}{3} + \frac{1}{2}\log M\right)^{-1} < \epsilon$$

and let $f_\epsilon(x) = g_M(x)$. □

What has been used in this paper is the following version of Hardy's inequality.

Lemma 6.2 *For any $\theta > 0$,*

$$\sup_{g\in\mathcal{F}_3} \left\{ \theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx \right\} = \begin{cases} 0 & \text{if } \theta \le 1/8, \\ \infty & \text{if } \theta > 1/8. \end{cases} \tag{6.3}$$

Proof: By Hardy's inequality, the left hand side of (6.3) is non-positive when $\theta < 1/8$. On the other hand, it is no less than

$$-\frac{1}{2} \inf_{g\in\mathcal{F}_3} \int_{\mathbb{R}^3} |\nabla g(x)|^2 dx$$

which is equal to zero. Thus, for $\theta \le 1/8$,

$$\sup_{g\in\mathcal{F}_3} \left\{ \theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx \right\} = 0.$$

Assume $\theta > 1/8$. By the optimality of Hardy's inequality described in (6.2),

$$H(\theta) \equiv \sup_{g\in\mathcal{F}_3} \left\{ \theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx \right\} > 0.$$

Given $a > 0$, the substitution $g(x) = a^{3/2} f(ax)$ leads to $H(\theta) = a^2 H(\theta)$. So $M(\theta) = \infty$. □

6.2 An auxiliary limit result

Let

$$Q(b) = \int_{\{|x|\ge b\}} \frac{1}{|x|^p} \mathbb{E} \frac{1\{|x+U| \ge b\}}{|x+U|^p} dx \quad (b \ge 0) \tag{6.4}$$

with $U \sim N(0, I_d)$. In this subsection, we give the limiting behaviors of $Q(b)$ as $b \to 0+$ and as $b \to \infty$, respectively.

Lemma 6.3

$$\lim_{b\to 0^+} Q(b) = \mathbb{E}\int_{\mathbb{R}^d} \frac{1}{|x|^p}\frac{1}{|x+U|^p}dx = C(d,p)\mathbb{E}|U|^{-(2p-d)} \tag{6.5}$$
$$= C(d,p)2^{d-p-1}d\omega_d(2\pi)^{-d/2}\Gamma(d-p),$$

where

$$C(d,p) = \pi^{d/2}\frac{\Gamma^2\Big(\dfrac{d-p}{2}\Big)\Gamma\Big(\dfrac{2p-d}{2}\Big)}{\Gamma^2\Big(\dfrac{p}{2}\Big)\Gamma(d-p)}. \tag{6.6}$$

Proof: The first equality follows from the monotone convergence theorem. The second follows from the identity (see p.118, [15] or p.118, (8), [37])

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^p}\frac{1}{|x-z|^p}dx = C(d,p)\frac{1}{|y-z|^{2p-d}} \qquad y,z\in\mathbb{R}^d. \tag{6.7}$$

□

Lemma 6.4 *Under $d/2 < p < d$, we have*

$$\lim_{b\to\infty} b^{2p-d}Q(b) = \frac{d\omega_d}{2p-d}. \tag{6.8}$$

Proof: By variable substitution, we have,

$$\int_{\{|x|\geq b\}} \frac{1}{|x|^p}\frac{1\{|x+U|\geq b\}}{|x+U|^p}dx$$
$$= |U|^{-(2p-d)}\int_{\{|x|\geq b|U|^{-1}\}} \frac{1}{|x|^p}\frac{1\{|x+|U|^{-1}U|\geq b|U|^{-1}\}}{|x+|U|^{-1}U|^p}dx$$
$$= |U|^{-(2p-d)}H\big(b|U|^{-1}\big),$$

where

$$H(b) = \int_{\{|x|\geq b\}} \frac{1}{|x|^p}\frac{1\{|x+x_0|\geq b\}}{|x+x_0|^p}dx,$$

x_0 is a fixed point with $|x_0| = 1$, and the last step follows from the fact that $H(b)$ does not depend on the location of x_0 on the unit sphere. Thus,

$$Q(b) = \mathbb{E}|U|^{-(2p-d)}H\big(b|U|^{-1}\big)$$
$$= \int_{\mathbb{R}^d}\frac{dx}{|x|^p|x+x_0|^p}\bigg[\int_{\mathbb{R}^d}\frac{1\{|y|\geq b|x|^{-1}\}1\{|y|\geq b|x+x_0|^{-1}\}}{|y|^{2p-d}}p_1(y)dy\bigg]dx$$
$$\sim \int_{\{|x|\geq C\}}\frac{dx}{|x|^p|x+x_0|^p}\bigg[\int_{\mathbb{R}^d}\frac{1\{|y|\geq b|x|^{-1}\}1\{|y|\geq b|x+x_0|^{-1}\}}{|y|^{2p-d}}p_1(y)dy\bigg]dx,$$

where $p_1(x)$ is the density of the d-dimensional standard normal distribution, $C > 0$ is a large but fixed constant. By the fact that $C \gg 1 = |x_0|$, for $b \to \infty$, we have

$$
\begin{aligned}
&Q(b) \sim q(C) \int_{\{|x| \geq C\}} \frac{dx}{|x|^{2p}} \left[\int_{\{|y| \geq b|x|^{-1}\}} \frac{1}{|y|^{2p-d}} p_1(y) dy \right] \\
&= q(C) \int_{\mathbb{R}^d} \frac{1}{|y|^{2p-d}} p_1(y) \left[\int_{\{|x| \geq \max\{C, b|y|^{-1}\}\}} \frac{dx}{|x|^{2p}} \right] dy \\
&= q(C) \frac{d\omega_d}{2p-d} \int_{\mathbb{R}^d} \frac{1}{|y|^{2p-d}} p_1(y) \min\left\{ \frac{1}{C^{2p-d}}, \left(\frac{|y|}{b} \right)^{2p-d} \right\} dy \\
&= q(C) \frac{d\omega_d}{2p-d} \frac{1}{C^{2p-d}} \int_{\{|y| \geq C^{-1}b\}} \frac{1}{|y|^{2p-d}} p_1(y) dy \\
&+ q(C) \frac{d\omega_d}{2p-d} \frac{1}{b^{2p-d}} \int_{\{|y| \leq C^{-1}b\}} p_1(y) dy,
\end{aligned}
$$

where $q(C) \to 1$ as $C \to \infty$. The first term on the right hand side is obviously negligible. $\square$

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