# Feynman-Kac representation of parabolic Anderson equations with general Gaussian noise in Stratonovich regime

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**Abstract**

In this paper, we provide the Feynman-Kac representation for the parabolic Anderson equations driven by a general Gaussian noise in the Stratonovich sense. A feature in idea is the argument by sub-additivity in establishing the needed exponential integrability.

Key-words: Feynman-Kac formula, sub-additive process, Brownian motion

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# 1 Introduction

Consider the parabolic Anderson equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + \dot{W}(t,x)u(t,x) & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \\ u(0,x) = u_0(x) & x\in\mathbb{R}^d \end{cases} \tag{1.1}$$

run by a mean zero and possibly generalized time-space Gaussian noise $\dot{W}(x)$ ($(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$) with the covariance function

$$\text{Cov}\big(\dot{W}(t,x),\dot{W}(s,y)\big) = |t-s|^{-\alpha_0}\gamma(x-y) \quad x,y\in\mathbb{R}^d \tag{1.2}$$

where $0<\alpha_0<1$. Throughout, we assume that $\gamma(\cdot)\ge 0$. With maximal generality this paper allowed, $\gamma(\cdot)$ can be a generalized function that is defined as a linear functional on $\mathcal{S}(\mathbb{R}^d)$, the set of all rapidly decreasing functions known as Schwartz space. Since $\gamma(\cdot)$ is non-negative definite as covariance function, by Bochner's theorem there is a unique measure on $\mathbb{R}^d$, known as the spectral measure of $\gamma(\cdot)$, such that

$$\gamma(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\mu(d\xi) \quad x\in\mathbb{R}^d.$$

Further, $\mu(d\xi)$ is tempered in the sense that

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^p\mu(d\xi)<\infty$$

for some $p>0$. In particular, $\mu(d\xi)$ is locally finite.

The possible singularity of the Gaussian noise $\dot{W}(t,x)$ does not make (1.1) a rigorous definition. Mathematically, the parabolic Anderson equation is defined by the integral equation

$$u(t,x) = W(t,x) + \int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-x)u(s,y)W(dsdy) \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \tag{1.3}$$

where $p_t(x)$ is the Brownian semi-group defined as

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}}\exp\Big\{-\frac{1}{2t}|x|^2\Big\} \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

This equation have different meanings, depending how the stochastic integral on the right hand side is defined. The current study focuses on two regimes: when it is defined as Skorokhod integral and when it is defined as Stratonovich integral. In the setting of Stratonovich integral, the stochastic integral on the right hand side of (1.3) is defined by

$$\int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-x)u(s,y)W(dsdy) \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-x)u(s,y)\dot{W}_\epsilon(t,x)dyds$$

in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$-norm, whenever the limit exists, where $\dot{W}_\epsilon(t,x)$ is a smoothed version of $\dot{W}(t,x)$ (see (2.1) below).

In the Skorokhod's regime, it has been proved (Theorem 3.6, [6]) that the equation (1.1) has a solution under the Dalang's condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty. \tag{1.4}$$

In the Stratonovich regime, the results existing in literature are limited to some specific Gaussian noises. In the case when

$$\dot{W}(t, x_1, \cdots, x_d) = \frac{\partial^{d+1} W^H}{\partial t \partial x_1 \cdots \partial x_d}(t, x_1, \cdots, x_d) \tag{1.5}$$

is the formal derivative of a fractional Brownian sheet $W^H(t, x_1, \cdots, x_d)$ with Hurst parameter $(H_0, \cdots, H_d)$ satisfying $H_0, \cdots, H_d > 1/2$, it is proved in Theorem 4.3, [7] that under the condition

$$2H_0 + \sum_{j=1}^d H_j > d+1 \tag{1.6}$$

the parabolic Anderson equation (1.1) is solved by the random field

$$u(t,x) \triangleq \mathbb{E}_x \exp\left\{\int_0^t \dot{W}(t-s, B_s)ds\right\} u_0(B_t) \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{1.7}$$

where $B_s$ is a $d$-dimensional Brownian motion starting at $x$ and independent of $\dot{W}$, "$\mathbb{E}_x$" is the expectation with respect to the Brownian motion, and the time-integral on the right hand side is properly defined by the way of approximation (see (2.1) below).

This result has been extended (Section 6, [4]) to some Gaussian noises with spatial covariance of the homogeneity

$$\gamma(cx) = c^{-\alpha}\gamma(x) \qquad x \in \mathbb{R}^d, \quad c > 0 \tag{1.8}$$

with $0 < \alpha < 2(1-\alpha_0)$.

This work is to solve the parabolic Anderson equation in Stratonovich regime by establishing the representation (1.7) for the Gaussian noise with the general spatial covariance $\gamma(\cdot)$.

**Theorem 1.1** *Assume that $u_0(x)$ is bounded on $\mathbb{R}^d$ and that*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{1-\alpha_0} \mu(d\xi) < \infty. \tag{1.9}$$

*Then the parabolic Anderson equation (1.1) is solved by the well-defined random field $u(t,x)$ given in (1.7). Given an integer $m \geq 1$, further, $u(t,x) \in \mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ with the representation*

$$\mathbb{E}u^m(t,x) = \mathbb{E}_x \exp\left\{\sum_{j,k=1}^m \int_0^t\int_0^t \frac{\gamma(B_j(s) - B_k(r))}{|s-r|^{\alpha_0}} ds dr\right\} \prod_{j=1}^m u_0\big(B_j(t)\big) \tag{1.10}$$

*where $B_1(t), \cdots, B_m(t)$ are independent $d$-dimensional Brownian motions with $B_j(0) = x$, "$\mathbb{E}_x$" is the expectation with respect to the Brownian motions, and the time-Hamiltonians on the right hand side is defined by appropriate approximation (see (2.6) and (2.7) below).*

Consider the special case when $u_0(x) = 1$. By (2.4) and (2.8) below,

$$\mathbb{E} \otimes \mathbb{E}_x \bigg[ \int_0^t \dot{W}(t-s, B_s) ds \bigg]^2 = \mathbb{E}_0 \int_0^t \int_0^t \frac{\gamma(B(s) - B(r))}{|s-r|^{\alpha_0}} ds dr$$
$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\Big\{ -\frac{|\xi|^2}{2} |s-r| \Big\}.$$

One can check (see the computation next to (2.5) below) that the condition (1.9) is equivalent to

$$\mathbb{E} \otimes \mathbb{E}_x \bigg[ \int_0^t \dot{W}(t-s, B_s) ds \bigg]^2 < \infty \quad \text{for some } t > 0 \text{ (or equivalently, for every } t > 0).$$

In view of (1.7), therefore, it is easy to see that the condition (1.9) is necessary for $\mathbb{E}|u(t,x)| < \infty$.

With the homogeneity (1.8), by Lemma 3.10, [3]

$$\int_{\mathbb{R}^d} \Big( \frac{1}{1 + |\xi|^2} \Big)^{1-\alpha_0} \mu(d\xi) = \alpha \mu\big(B(0,1)\big) \int_0^\infty \Big( \frac{1}{1+\rho^2} \Big)^{1-\alpha_0} \rho^{\alpha-1} d\rho.$$

In particular, (1.9) holds if and only if $\alpha < 2(1-\alpha_0)$.

**Corollary 1.2** *In the assumption (1.8) with $0 < \alpha < 2(1-\alpha_0)$, all statements in Theorem 1.1 hold.*

As for the specific case where $\dot{W}(t,x)$ is the fractional Gaussian noise given in (1.5), the homogeneity (1.8 is satisfied with

$$\alpha_0 = 2 - 2H_0 \quad \text{and} \quad \alpha = 2d - 2\sum_{j=1}^d H_j.$$

Consequently, (1.6) is equivalent to "$0 < \alpha < 2(1-\alpha_0)$".

The proof of Theorem 1.1 is given in the next section. It is worth of mentioning a striking fact that the exponential integrability (given in (2.10) below) of the Brownian Hamiltonian

$$\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} ds dr$$

is determined by its local behavior near $t = 0$ (Lemma 2.1), and of mentioning the efficiency of sub-additivity approach in proving this fact.

## 2 Proof of Theorem 1.1

The time-integral in the representation (1.7) is defined as

$$\int_0^t \dot W(t-s,B_s)ds \triangleq \lim_{\epsilon\to 0^+}\int_0^t \dot W_\epsilon(t-s,B_s)ds \quad \text{in } \mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P}_x\otimes\mathbb{P}) \tag{2.1}$$

where $\dot W_\epsilon$ is the point-wisely defined Gaussian field $\dot W_\epsilon(t,x)$ is given as

$$\dot W_\epsilon(t,x) \triangleq \int_{\mathbb{R}^{d+1}} \dot W(u,y)\bigg[(2\pi\epsilon)^{-\frac{d+1}{2}}\exp\Big\{-\frac{(t-u)^2+|x-y|^2}{2\epsilon}\Big\}\bigg]dudy \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

To make it work, we need to show that the limit on the right hand side exists. To this end all we need is to show that the limit

$$\lim_{\epsilon,\epsilon'\to 0^+}\mathbb{E}_x\otimes\mathbb{E}\bigg(\int_0^t \dot W_\epsilon(t-s,B_s)ds\bigg)\bigg(\int_0^t \dot W_{\epsilon'}(t-s,B_s)ds\bigg)$$

exists.

Notice that

$$Cov\big(W_\epsilon(s,x),W_\epsilon(r,y)\big)=\gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(x-y)$$

where

$$\gamma_{0,\epsilon}(u)=\int_{\mathbb{R}}\frac{1}{|v|^{\alpha_0}}\bigg[\frac{1}{\sqrt{2\pi\epsilon}}\exp\Big\{-\frac{(u-v)^2}{2\epsilon}\Big\}\bigg]dv \tag{2.2}$$

$$\gamma_\epsilon(x)=\int_{\mathbb{R}^d}\gamma(y)\bigg[\frac{1}{(2\pi\epsilon)^{d/2}}\exp\Big\{-\frac{|x-y)^2}{2\epsilon}\Big\}\bigg]dy. \tag{2.3}$$

We have

$$\begin{aligned}&\mathbb{E}_x\otimes\mathbb{E}\bigg(\int_0^t \dot W_\epsilon(t-s,B_s)ds\bigg)\bigg(\int_0^t \dot W_{\epsilon'}(t-s,B_s)ds\bigg)\\&=\mathbb{E}_0\int_0^t\!\!\int_0^t \mathbb{E}\dot W_\epsilon(t-s,B_s)\dot W_{\epsilon'}(t-r,B_r)dsdr\\&=\mathbb{E}_0\int_0^t\!\!\int_0^t \gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(B_s-B_r)dsdr.\end{aligned}$$

Notice that for any $\delta>0$, $\gamma_\delta(\cdot)$ has the the spectral measure $e^{-\delta|\xi|^2/2}\mu(d\xi)$. Let $\mu_0(d\lambda)$ be the spectral measure of $|\cdot|^{-\alpha_0}$ (One can easily check that $\mu_0(d\lambda)$ is a constant multiple of $|\lambda|^{-(1-\alpha_0)}d\lambda$). Then $\gamma_{0,\delta}(\cdot)$ has the the spectral measure $e^{-\delta\lambda^2/2}\mu_0(d\lambda)$. By Fourier transform

$$\begin{aligned}&\int_0^t\!\!\int_0^t \gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(B_s-B_r)dsdr\\&=\int_{\mathbb{R}^{d+1}}\exp\Big\{-\frac{\epsilon+\epsilon'}{2}(\lambda^2+|\xi|^2)\Big\}\bigg|\int_0^t\exp\{i\lambda s+i\xi\cdot B_s\}ds\bigg|^2\mu_0(d\lambda)\mu(d\xi).\end{aligned}$$

By dominated convergence theorem, therefore,

$$\lim_{\epsilon,\epsilon'\to 0^+}\mathbb{E}_x\otimes\mathbb{E}\bigg(\int_0^t \dot W_\epsilon(t-s,B_s)ds\bigg)\bigg(\int_0^t \dot W_{\epsilon'}(t-s,B_s)ds\bigg)$$
$$=\int_{\mathbb{R}^{d+1}}\mathbb{E}_0\bigg\vert\int_0^t \exp\big\{i\lambda s+i\xi\cdot B_s\big\}ds\bigg\vert^2\mu_0(d\lambda)\mu(d\xi),$$

provided that

$$\int_{\mathbb{R}^{d+1}}\mathbb{E}_0\bigg\vert\int_0^t \exp\big\{i\lambda s+i\xi\cdot B_s\big\}ds\bigg\vert^2\mu_0(d\lambda)\mu(d\xi)<\infty\quad\forall t>0. \tag{2.4}$$

Here we have used the fact that the integral in (2.4) is independent of the starting point $x$ of the Brownian motion.

Indeed,

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}}\mathbb{E}_0\bigg\vert\int_0^t \exp\big\{i\lambda s+i\xi\cdot B_s\big\}ds\bigg\vert^2\mu_0(d\lambda)\mu(d\xi) \qquad (2.5)\\&=\int_{\mathbb{R}^d}\mu(d\xi)\mathbb{E}_0\int_0^t\!\!\int_0^t|s-r|^{-\alpha_0}\exp\Big\{i\xi\cdot(B_s-B_r)\Big\}dsdr\\&=\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\!\!\int_0^t|s-r|^{-\alpha_0}\exp\Big\{-\frac{|\xi|^2}{2}|s-r|\Big\}dsdr.\end{aligned}$$

Notice that the right hand side is monotonic in $t$. To establish (2.2), all we need to prove that

$$\int_{\mathbb{R}^d}\mu(d\xi)\int_0^\infty dte^{-t}\int_0^t\!\!\int_0^t|s-r|^{-\alpha_0}\exp\Big\{-\frac{|\xi|^2}{2}|s-r|\Big\}dsdr<\infty.$$

In fact,

$$\begin{aligned}&\int_{\mathbb{R}^d}\mu(d\xi)\int_0^\infty dte^{-t}\int_0^t\!\!\int_0^t|s-r|^{-\alpha_0}\exp\Big\{-\frac{|\xi|^2}{2}|s-r|\Big\}dsdr\\&=2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^\infty dte^{-t}\int_0^t\!\!\int_r^t(s-r)^{-\alpha_0}\exp\Big\{-\frac{|\xi|^2}{2}(s-r)\Big\}dsdr\\&=2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^\infty t^{-\alpha_0}\exp\Big\{-\Big(1+\frac{|\xi|^2}{2}\Big)t\Big\}dt\\&=2\bigg(\int_0^\infty t^{-\alpha_0}e^{-t}dt\bigg)\int_{\mathbb{R}^d}\bigg(\frac{1}{1+2^{-1}|\xi|^2}\bigg)^{1-\alpha_0}\mu(d\xi)\end{aligned}$$

where the last step follows from the integration substitution

$$t\mapsto\Big(1+2^{-1}|\xi|^2\Big)^{-1}t.$$

In summary, by the condition (1.9) we have proved (2.4) and therefore have justified the definition in (2.1).

Next, we clarify the time-Hamiltonians in (1.10) by making the definition

$$\int_0^t\int_0^t \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}}dsdr \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)dsdr \ \text{ in } \mathcal{L}(\Omega,\mathcal{A},\mathbb{P}_x) \tag{2.6}$$

$$\int_0^t\int_0^t \frac{\gamma(B_s-\widetilde{B}_r)}{|s-r|^{\alpha_0}}dsdr \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-\widetilde{B}_r)dsdr \ \text{ in } \mathcal{L}(\Omega,\mathcal{A},\mathbb{P}_x) \tag{2.7}$$

for two independent Brownian motions $B_t$ and $\widetilde{B}_t$, where $\gamma_{0,\epsilon}(\cdot)$ and $\gamma_\epsilon(\cdot)$ are given in (2.2) and (2.3), respectively.

Once again, notice that the problem is independent of the starting point $x$ of the Brownian motions and that

$$\begin{aligned}&\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)dsdr\\ &=\int_{\mathbb{R}^{d+1}}\bigg|\int_0^t e^{i\lambda s+i\xi\cdot B_s}ds\bigg|^2\exp\Big\{-\frac{\epsilon}{2}(\lambda^2+|\xi|^2)\Big\}\mu_0(d\lambda)\mu(d\xi)\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-\widetilde{B}_r)dsdr\\ &=\int_{\mathbb{R}^{d+1}}\bigg[\int_0^t e^{i\lambda s+i\xi\cdot B_s}ds\bigg]\bigg[\int_0^t e^{-i\lambda s-i\xi\cdot \widetilde{B}_s}ds\bigg]\exp\Big\{-\frac{\epsilon}{2}(\lambda^2+|\xi|^2)\Big\}\mu_0(d\lambda)\mu(d\xi).\end{aligned}$$

So we have

$$\begin{aligned}&\mathbb{E}_0\bigg|\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)dsdr-\int_0^t\int_0^t \gamma_{0,\epsilon'}(s-r)\gamma_{\epsilon'}(B_s-B_r)dsdr\bigg|\\ &\le\int_{\mathbb{R}^{d+1}}\bigg|\exp\Big\{-\frac{\epsilon}{2}(\lambda^2+|\xi|^2)\Big\}-\exp\Big\{-\frac{\epsilon'}{2}(\lambda^2+|\xi|^2)\Big\}\bigg|\mathbb{E}_0\bigg|\int_0^t e^{i\lambda s+i\xi\cdot B_s}ds\bigg|^2\mu_0(d\lambda)\mu(d\xi)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}_0\bigg|\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-\widetilde{B}_r)dsdr-\int_0^t\int_0^t \gamma_{0,\epsilon'}(s-r)\gamma_{\epsilon'}(B_s-\widetilde{B}_r)dsdr\bigg|\\ &\le\int_{\mathbb{R}^{d+1}}\bigg|\exp\Big\{-\frac{\epsilon}{2}(\lambda^2+|\xi|^2)\Big\}-\exp\Big\{-\frac{\epsilon'}{2}(\lambda^2+|\xi|^2)\Big\}\bigg|\bigg\{\mathbb{E}_0\bigg|\int_0^t e^{i\lambda s+i\xi\cdot B_s}ds\bigg|\bigg\}^2\mu_0(d\lambda)\mu(d\xi).\end{aligned}$$

By (2.4) and dominated convergence, the right hand sides tends to 0 as $\epsilon,\epsilon'\to 0^+$. That is the justification for (2.6) and (2.7).

Further, from above argument we have

$$\int_0^t\int_0^t \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}}dsdr=\int_{\mathbb{R}^{d+1}}\bigg|\int_0^t e^{i\lambda s+i\xi\cdot B_s}ds\bigg|^2\mu_0(d\lambda)\mu(d\xi). \tag{2.8}$$

We now show that the random field $u(t,x)$ in (1.7) is well-defined by proving that

$$\mathbb{E}|u(t,x)| < \infty \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{2.9}$$

We first use the fact that $|u_0(\cdot)| \le C$ for a constant $C > 0$. So we have

$$\mathbb{E}|u(t,x)| \le C\mathbb{E} \otimes \mathbb{E}_x \exp\left\{ \int_0^t \dot{W}(t-s, B_s)ds \right\} = C\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ \int_0^t \dot{W}(t-s, B_s)ds \right\}.$$

From (2.1) and (2.6) we can see that conditioning on the Brownian motion, the random variable

$$\int_0^t \dot{W}(t-s, B_s)ds$$

is a mean zero normal with the variance

$$\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} dsdr.$$

So we have

$$\mathbb{E} \exp\left\{ \int_0^t \dot{W}(t-s, B_s)ds \right\} = \exp\left\{ \frac{1}{2} \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} dsdr \right\} \quad a.s.$$

To established (1.9), therefore, all we need is the exponential integrability

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} dsdr \right\} < \infty \quad \forall \theta, t > 0 \tag{2.10}$$

To this end we first establish the following lemma:

**Lemma 2.1** *Under the condition (1.9),*

$$\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}_0 \int_{\mathbb{R}^{d+1}} \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s} ds \right|^2 \mu_0(d\lambda)\mu(d\xi) = 0 \tag{2.11}$$

**Proof:** From (2.5) and variable substitution

$$\begin{aligned}
&\mathbb{E}_0 \int_{\mathbb{R}^{d+1}} \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s} ds \right|^2 \mu_0(d\lambda)\mu(d\xi) \\
&= \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t} \int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{ -\frac{1}{2}|s-r| \right\} dsdr \\
&= \int_{\{|\xi| \le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t} \int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{ -\frac{1}{2}|s-r| \right\} dsdr \\
&+ \int_{\{|\xi| > t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int_0^{|\xi|^2 t} \int_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{ -\frac{1}{2}|s-r| \right\} dsdr
\end{aligned}$$

For the first term

$$\int_{\{|\xi|\le t^{-1/2}\}}\frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}}\int_0^{|\xi|^2t}\int_0^{|\xi|^2t}\frac{1}{|s-r|^{\alpha_0}}\exp\Big\{-\frac{1}{2}|s-r|\Big\}dsdr$$
$$\le\int_{\{|\xi|\le t^{-1/2}\}}\frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}}\int_0^{|\xi|^2t}\int_0^{|\xi|^2t}\frac{1}{|s-r|^{\alpha_0}}dsdr$$
$$=\frac{2t^{2-\alpha_0}}{(1-\alpha_0)(2-\alpha_0)}\mu\big(B(0,t^{-1/2})\big)$$

According to Kronecker lemma, (1.9) implies that

$$\lim_{t\to 0^+}t^{1-\alpha_0}\mu\big(B(0,t^{-1/2})\big)=0.$$

As for the second term in our decomposition, we use the simple bound

$$\int_{\{|\xi|>t^{-1/2}\}}\frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}}\int_0^{|\xi|^2t}\int_0^{|\xi|^2t}\frac{1}{|s-r|^{\alpha_0}}\exp\Big\{-\frac{1}{2}|s-r|\Big\}dsdr$$
$$\le\Big(\int_0^\infty\int_0^\infty\frac{1}{|s-r|^{\alpha_0}}\exp\Big\{-\frac{1}{2}|s-r|\Big\}dsdr\Big)\int_{\{|\xi|>t^{-1/2}\}}\frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}}$$

and the bound

$$\frac{1}{t}\int_{\{|\xi|>t^{-1/2}\}}\frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}}\le\int_{|\xi|>t^{-1/2}\}}\frac{\mu(d\xi)}{|\xi|^{2(1-\alpha_0)}}\longrightarrow 0\quad (t\to 0^+).$$

□

To establish (2.10), we use the argument by sub-additivity. A stochastic process $Z_t$ $(t\ge 0)$ is said to be sub-additive, if for any $t_1,t_2>0$, there is a random variable $Z'_{t_2}$ such that $Z'_{t_2}\overset{d}{=}Z_{t_1}$, $Z'_{t_2}$ is independent of $\{Z_s;\ s\le t_1\}$ and $Z_{t_1+t_2}\le\mathbb{Z}_{t_1}+Z_{t_2}$ a.s. An interested reader is referred to Section 1.3, [1] for the discussion on this topic. Specifically, a non-negative, non-decreasing and sample-path continuous sub-additive process $Z_t$ with $Z_0=0$ has the property((1.3.7), p.21, [1]) that

$$\mathbb{P}\{Z_t\ge a+b\}\le\mathbb{P}\{Z_t\ge a\}\mathbb{P}\{Z_t\ge b\}\quad \forall a,b,t>0. \tag{2.12}$$

We now exam the sub-additivity for the process

$$Z_t\triangleq\Big(\int_0^t\int_0^t\frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}}dsdr\Big)^{1/2}\qquad t\ge 0.$$

Indeed, by (2.8) and triangle inequality the sub-additivity $Z_{t_1+t_2}\le Z_{t_1}+Z'_{t_2}$ hold with

$$Z'_{t_2}=\Big(\int_{\mathbb{R}^{d+1}}\Big|\int_{t_1}^{t_1+t_2}e^{i\lambda s+i\xi\cdot B_s}ds\Big|^2\mu_0(d\lambda)\mu(d\xi)\Big)^{1/2}$$
$$=\Big(\int_{\mathbb{R}^{d+1}}\Big|\int_0^{t_2}e^{i\lambda s+i\xi\cdot(B_{t_1+s}-B_{t_1})}ds\Big|^2\mu_0(d\lambda)\mu(d\xi)\Big)^{1/2}$$

that satisfies all requirements for sub-addivity. Clearly, $Z_0 = 0$, $Z_t$ is non-decreasing. By (2.8) $Z_t$ is sample-path continuous (More precisely, the relation (2.8) provides a sample-path continuous modification of $Z_t$). So $Z_t$ satisfies (2.12).

For any $\theta > 0$, using (2.12) repeatedly,

$$\mathbb{P}_0\big\{Z_t \ge m\theta^{-1}\sqrt{t}\big\} \le \Big(\mathbb{P}_0\big\{Z_t \ge \theta^{-1}\sqrt{t}\big\}\Big)^m \qquad m = 1, 2, \cdots .$$

By Lemma 2.1, there is a possibly small $t_0 > 0$ such that

$$\sup_{t\le t_0}\mathbb{P}_0\big\{Z_t \ge \theta^{-1}\sqrt{t}\big\} \le e^{-2}.$$

Hence,

$$\mathbb{E}_0 \exp\big\{\theta Z_t/\sqrt{t}\big\} = \int_0^\infty e^b\mathbb{P}_0\big\{Z_t \ge b\theta^{-1}\sqrt{t}\big\}db$$

$$\le e + \sum_{m=1}^\infty e^{m+1}\mathbb{P}_0\big\{Z_t \ge m\theta^{-1}\sqrt{t}\big\} \le e + \sum_{m=0}^\infty e^{m+1}e^{-2m} = \frac{e(2e-1)}{e-1}$$

for all $0 < t \le t_0$. Unfortunately, this is not even close to what requested by (2.10). To improve it, first notice that the above estimation leads to the uniform bound

$$\mathbb{E}_0 Z_t^n \le \frac{e(2e-1)}{e-1}\theta^{-n}n!t^{n/2} \qquad 0 < t < t_0, \quad n = 1, 2, \cdots . \tag{2.13}$$

By sub-additivity, for any $t_1, t_2 > 0$ and integer $n \ge 1$,

$$\mathbb{E}_0 Z_{t_1+t_2}^n \le \mathbb{E}\big[Z_{t_1} + Z'_{t_2}\big]^n = \sum_{l=0}^n \binom{n}{l}\Big\{\mathbb{E}Z_{t_1}^l\Big\}\Big\{\mathbb{E}Z_{t_2}^{n-l}\Big\}.$$

For any $t > 0$ and integer $m \ge 1$, repeating the above inequality we have

$$\mathbb{E}Z_t^n \le \sum_{l_1+\cdots+l_m=n}\frac{n!}{l_1!\cdots l_m!}\prod_{k=1}^m \mathbb{E}Z_{t/m}^{l_k} = \sum_{l_1+\cdots+l_m=n}\frac{n!}{l_1!\cdots l_m!}\prod_{k=1}^m \mathbb{E}Z_{t/m}^{l_k}.$$

Taking $m = n$ and $t \le t_0$, by (2.13)

$$\mathbb{E}Z_t^n \le \sum_{l_1+\cdots+l_n=n}\frac{n!}{l_1!\cdots l_n!}\prod_{k=1}^n \frac{e(2e-1)}{e-1}\theta^{-l_j}l_j!\Big(\frac{t}{n}\Big)^{l_j/2}$$

$$= \Big(\frac{\theta^{-1}e(2e-1)}{e-1}\Big)^n n!n^{-n/2}t^{n/2}\sum_{l_1+\cdots+l_n=n} 1.$$

A simple combinatorial argument gives

$$\sum_{l_1+\cdots+l_n=n} 1 = \binom{2n-1}{n} \le 4^n.$$

Thus, we obtain the following improved version of (2.13):

$$\mathbb{E}_0 Z_t^n \le \Big(\frac{4\theta^{-1}e(2e-1)}{e-1}\Big)^n \sqrt{n!}t^{n/2} \quad 0<t\le t_0 \;\; n=1,2,\cdots.$$

Replacing $n$ by $2n$,

$$\mathbb{E}_0 Z_t^{2n} \le \Big(\frac{4\theta^{-1}e(2e-1)}{e-1}\Big)^{2n} \sqrt{(2n)!}t^n \le \Big(\frac{4\sqrt{2}\theta^{-1}e(2e-1)}{e-1}\Big)^{2n} n!t^n$$

for any $0<t\le t_0$ and $n=1,2,\cdots$. Consequently, by Taylor expansion

$$\sup_{0<t\le t_0} \mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8e(2e-1)}\right)^2 \frac{Z_t^2}{t}\right\} < \infty. \tag{2.14}$$

In addition, one can check that the process

$$S_t \triangleq \frac{Z_t^2}{t} = \frac{1}{t}\int_0^t\int_0^t \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}} dsdr \quad t>0$$

is sub-additive. Indeed, by (2.8) and Jensen's inequality one can establish the sub-aditivity $S_{t_1+t_2} \le S_t + S'_{t_2}$ where

$$\begin{aligned} S'_{t_2} &= \frac{1}{t_2}\int_{\mathbb{R}^{d+1}} \left|\int_{t_1}^{t_1+t_2} e^{i\lambda s + i\xi\cdot B_s} ds\right|^2 \mu_0(d\lambda)\mu(d\xi) \\ &= \frac{1}{t_2}\int_{\mathbb{R}^{d+1}} \left|\int_0^{t_2} e^{i\lambda s + i\xi\cdot (B_{t_1+s}-B_{t_1})} ds\right|^2 \mu_0(d\lambda)\mu(d\xi) \end{aligned}$$

satisfies all requests in the definition of sub-additivity[1]. Therefore,

$$\mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8e(2e-1)}\right)^2 S_{t_1+t_2}\right\} \le \mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8e(2e-1)}\right)^2 S_{t_1}\right\}\mathbb{E}_0 \exp\left\{\left(\frac{(e-1)\theta}{8e(2e-1)}\right)^2 S_{t_2}\right\}$$

for any $0<t_1,t_2<t_0$. By (2.14), the right hand side is finite. In this way, (2.14) can be extended to all $t>0$:

$$\mathbb{E}\exp\left\{\left(\frac{(e-1)\theta}{8e(2e-1)}\right)^2 S_t\right\} < \infty \quad \forall t>0.$$

In particular, take $t=1$ and notice that $\theta>0$ is arbitrary. We therefore reaches the conclusion

$$\mathbb{E}_0 \exp\left\{\theta\int_0^1\int_0^1 \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}} dsdr\right\} < \infty \quad \forall \theta>0.$$

This can be further extended to (2.10) since (through (2.8)) for any $t_1,t_2>0$

$$\int_0^{t_1+t_2}\int_0^{t_1+t_2} \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}} dsdr \le 2\int_0^{t_1}\int_0^{t_1} \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}} dsdr + 2\int_0^{t_2}\int_0^{t_2} \frac{\gamma(\widetilde{B}_s-\widetilde{B}_r)}{|s-r|^{\alpha_0}} dsdr$$

[1] We do not have (2.12) this time for lack of monotonicity and for not being defined at 0

with $\widetilde{B}(s) = B_{t_1+s} - B_{t_1}$ being a Brownian motion independent of $\{B_s;\ s \le t_1\}$.

More than (2.9), we now show that for any integer $m \ge 1$, $u(t,x) \in \mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ with the representation (1.10).

Indeed, conditioning on the Brownian motions, the random variable

$$\sum_{j=1}^m \int_0^t \dot{W}\big(t-s, B_j(s)\big)ds$$

is a mean-zero normal random variable with the variance

$$\sum_{j,k=1}^m \mathbb{E}\bigg[\int_0^t \dot{W}\big(t-s, B_j(s)\big)ds\bigg]\ \bigg[\int_0^t \dot{W}\big(t-s, B_k(s)\big)ds\bigg]$$

For any $\epsilon > 0$, on the other hand,

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^t \dot{W}_\epsilon\big(t-s, B_j(s)\big)ds\bigg]\bigg[\int_0^t \dot{W}_\epsilon\big(t-s, B_k(s)\big)ds\bigg]\\ &= \int_0^t\int_0^t \gamma_{0,2\epsilon}(s-r)\gamma_{2\epsilon}\big(B_j(s) - B_k(r)\big)dsdr.\end{aligned}$$

By (2.1), (2.6) and (2.7), therefore,

$$\mathbb{E}\bigg[\int_0^t \dot{W}\big(t-s, B_j(s)\big)ds\bigg]\bigg[\int_0^t \dot{W}\big(t-s, B_k(s)\big)ds\bigg] = \int_0^t\int_0^t \frac{\gamma\big(B_j(s) - B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr. \quad (2.15)$$

In summary,

$$\mathbb{E}\exp\bigg\{\sum_{j=1}^m \int_0^t \dot{W}\big(t-s, B_j(s)\big)ds\bigg\} = \exp\bigg\{\frac{1}{2}\sum_{j,k=1}^m \int_0^t\int_0^t \frac{\gamma\big(B_j(s) - B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr\bigg\}.$$

On the other hand, from (1.7)

$$u^m(t,x) = \mathbb{E}_x \exp\bigg\{\sum_{j=1}^m \int_0^t \dot{W}\big(t-s, B_j(s)\big)ds\bigg\}\prod_{j=1}^m u_0\big(B_j(t)\big)$$

By Fubini theorem,

$$\begin{aligned}\mathbb{E}u^m(t,x) &= \mathbb{E}_x\Bigg(\mathbb{E}\exp\bigg\{\sum_{j=1}^m \int_0^t \dot{W}\big(t-s, B_j(s)\big)ds\bigg\}\Bigg)\prod_{j=1}^m u_0\big(B_j(t)\big)\\ &= \mathbb{E}_x \exp\bigg\{\frac{1}{2}\sum_{j,k=1}^m \int_0^t\int_0^t \frac{\gamma\big(B_j(s) - B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr\bigg\}\prod_{j=1}^m u_0\big(B_j(t)\big).\end{aligned}$$

This is (1.10). The inegrability issue arising from the right hand side is resolved by the boundedness of $u_0(\cdot)$, the relation (from (2.15) that

$$\begin{aligned}&\int_0^t\int_0^t \frac{\gamma\big(B_j(s)-B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr \qquad (2.16)\\ &\le \frac{1}{2}\int_0^t\int_0^t \frac{\gamma\big(B_j(s)-B_j(r)\big)}{|s-r|^{\alpha_0}}dsdr+\frac{1}{2}\int_0^t\int_0^t \frac{\gamma\big(B_k(s)-B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr \qquad j\neq k\end{aligned}$$

and (2.10).

We finally come to the step to show that the random field $u(t,x)$ in (1.7) solves the parabolic Anderson equation (1.1). Here we essentially follow the idea appearing in [7]. Consider the smoothed version of (1.1):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x)=\dfrac{1}{2}\Delta u(t,x)+\dot{W}_\epsilon(t,x)u(t,x) & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\\ u(0,x)=u_0(x) & x\in\mathbb{R}^d\end{cases} \tag{2.17}$$

where $\dot{W}_\epsilon(t,x)$ is given in (2.1). Conditioning on $\dot{W}_\epsilon$ and applying Feynman-Kac formula ([5]) of the deterministic potential we obtain a path-wise solution $u_\epsilon(t,x)$ to (2.17) given as

$$u_\epsilon(t,x)\triangleq \mathbb{E}_x\exp\left\{\int_0^t \dot{W}_\epsilon(t-s,B_s)ds\right\}u_0(B_t)\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

To approximate the system (1.1) by (2.17) all we need is to show, according to the argument performed in ([7]), that

$$\lim_{\epsilon\to 0^+}u_\epsilon(t,x)=u(t,x)\quad \text{ in } \mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P}).$$

To this end, it suffices to establish

$$\lim_{\epsilon,\epsilon\to 0^+}\mathbb{E}u_\epsilon(t,x)u_{\epsilon'}(t,x)=\mathbb{E}u^2(t,x)\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d. \tag{2.18}$$

Let $B_1(s)$ and $B_2(s)$ be independent Brownian motions starting at $x$. By Fubini theorem

$$\begin{aligned}&\mathbb{E}u_\epsilon(t,x)u_{\epsilon'}(t,x)\\ &=\mathbb{E}_x\left(\mathbb{E}\exp\left\{\int_0^t \dot{W}_\epsilon\big(t-s,B_1(s)\big)ds+\int_0^t \dot{W}_{\epsilon'}\big(t-s,B_2(s)\big)ds\right\}\right)\prod_{j=1}^2 u_0\big(B_j(t)\big)\\ &=\mathbb{E}_x\exp\left\{\frac{1}{2}\sum_{j,k=1}^2\int_0^t\int_0^t \gamma_{0,\epsilon_{j,k}}(s-r)\gamma_{\epsilon_{j,k}}\big(B_j(s)-B_k(r)\big)dsdr\right\}\prod_{j=1}^2 u_0\big(B_j(t)\big)\end{aligned}$$

where $\epsilon_{1,1}=2\epsilon$, $\epsilon_{2,2}=2\epsilon'$ and $\epsilon_{1,2}=\epsilon_{2,1}=\epsilon+\epsilon'$.

By (1.10) with $m=2$, (2.18) is equivalent to

$$\lim_{\epsilon,\epsilon'\to 0^+}\mathbb{E}_x\exp\left\{\frac{1}{2}\sum_{j,k=1}^2\int_0^t\int_0^t\gamma_{0,\epsilon_{j,k}}(s-r)\gamma_{\epsilon_{j,k}}\big(B_j(s)-B_k(r)\big)dsdr\right\}\prod_{j=1}^2 u_0\big(B_j(t)\big) \tag{2.19}$$
$$=\mathbb{E}_x\exp\left\{\frac{1}{2}\sum_{j,k=1}^2\int_0^t\int_0^t\frac{\gamma\big(B_j(s)-B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr\right\}\prod_{j=1}^2 u_0\big(B_j(t)\big).$$

In view of (2.6) and (2.7), in the following we construct a dominating and integrable random variable that makes (2.19) happen.

First notice that

$$\int_0^t\int_0^t\gamma_{0,\epsilon_{1,2}}(s-r)\gamma_{\epsilon_{1,2}}\big(B_1(s)-B_2(r)\big)dsdr$$
$$\le\frac{1}{2}\sum_{j=1}^2\int_0^t\int_0^t\gamma_{0,\epsilon_{j,j}}(s-r)\gamma_{\epsilon_{j,j}}\big(B_j(s)-B_j(r)\big)dsdr.$$

By the boundedness of $u_0(\cdot)$,

$$\left|\exp\left\{\frac{1}{2}\sum_{j,k=1}^2\int_0^t\int_0^t\frac{\gamma\big(B_j(s)-B_k(r)\big)}{|s-r|^{\alpha_0}}dsdr\right\}\prod_{j=1}^2 u_0\big(B_j(t)\big)\right|$$
$$\le C\exp\left\{\sum_{j=1}^2\int_0^t\int_0^t\gamma_{0,\epsilon_{j,j}}(s-r)\gamma_{\epsilon_{j,j}}\big(B_j(s)-B_j(r)\big)dsdr\right\}$$
$$\le C\exp\left\{\sum_{j=1}^2\int_0^t\int_0^t\frac{\gamma\big(B_j(s)-B_j(r)\big)}{|s-r|^{\alpha_0}}dsdr\right\}$$

where the last step follows from the observation that

$$\int_0^t\int_0^t\gamma_{0,\epsilon_{j,j}}(s-r)\gamma_{\epsilon_{j,j}}\big(B_j(s)-B_j(r)\big)dsdr$$
$$=\int_{\mathbb{R}^{d+1}}\left|\int_0^t e^{i\lambda s+i\xi\cdot B_j(s)}ds\right|^2\exp\left\{-\frac{\epsilon_{j,j}}{2}\Big(\lambda^2+|\xi|^2\Big)\right\}\mu_0(d\lambda)\mu(d\xi)$$
$$\le\int_{\mathbb{R}^{d+1}}\left|\int_0^t e^{i\lambda s+i\xi\cdot B_j(s)}ds\right|^2\mu_0(d\lambda)\mu(d\xi)$$
$$=\int_0^t\int_0^t\frac{\gamma\big(B_j(s)-B_j(r)\big)}{|s-r|^{\alpha_0}}dsdr\qquad j=1,2.$$

By (2.10) and dominated convergence, we have proved (2.19). □

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