

FREE ENERGY IN A MEAN FIELD OF BROWNIAN PARTICLES

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ABSTRACT. We compute the limit of the free energy

$$\frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \quad (N \rightarrow \infty)$$

of the mean field generated by the independent Brownian particles $\{B_j(s)\}$ interacting through the non-negative definite function $\gamma(\cdot)$. Our main theorem is relevant to the high moment asymptotics for the parabolic Anderson models with Gaussian noise that is white in time, white or colored in space. Our approach makes a novel connection to the celebrated Donsker-Varadhan's large deviation principle for the i.i.d. random variables in infinite dimensional spaces. As an application of our main theorem, we provide a probabilistic treatment to the Hartree's theory on the asymptotics for the ground state energy of bosonic quantum system.

1. Introduction. Mean field theory considers the behavior of the stochastic system consisting of large number of small particles interacting to each other. In this paper, the independent d -dimensional Brownian motions $\{B_1(s) \cdots, B_N(s)\}$ represent the locations of these particles at the time s , and the function $N^{-1}\gamma(x - y)$ measures the pairwise interactions among the Brownian particles. The long term behavior of the system is the result of the balance between two typical phenomena in the mean field regime: increasing number of the particles (i.e., $N \rightarrow \infty$) and uniform negligibility of individual contribution (indicated by the multiple $1/N$ in the interaction function). The quantity

$$\frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds$$

stands for the integral potential of the system due to the interaction of the Brownian particles up to the time t_N . In this work, $t_N \rightarrow \infty$ as $N \rightarrow \infty$. Our goal is to study

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the asymptotics for the partition function

$$\mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\}. \quad (1)$$

In the case when $d = 2$, a slight different quantity

$$\mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^1 N^{2\beta} \gamma \left(N^\beta (B_j(s) - B_k(s)) \right) ds \right\} \quad (2)$$

corresponds to the N -body system with Schrödinger Hamiltonian

$$H_N^{BEC} = \frac{1}{2} \sum_{1 \leq j < k \leq N} \Delta_j + \frac{1}{N} \sum_{1 \leq j < k \leq N} N^{2\beta} \gamma(N^\beta(x_j - x_k)) \quad (3)$$

with the parameter $\beta \in (0, 1]$ that appears in the investigation of Bose-Einstein condensation ([15]).

Our work is also motivated by the recent investigations ([7], [8] and [4]) of the spatial asymptotics

$$\max_{|x| \leq R} u(t, x) \quad R \rightarrow \infty$$

for the parabolic Anderson equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta u(t, x) + V(t, x)u(t, x), \\ u(0, x) = 1, \end{cases} \quad (4)$$

where $V(t, x)$ is a Gaussian noise which is white in time, white or colored in space, i.e.,

$$\text{Cov} \left(V(s, x), V(t, y) \right) = \delta_0(s - t) \gamma(x - y) \quad (s, x), (t, y) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

In these works, the most substantial step is the investigation of the high moment asymptotics for $\mathbb{E}u(t, x)^N$ as $N \rightarrow \infty$. Under proper positive homogeneity assumption on $\gamma(\cdot)$, the problem is relevant to the investigation proposed in this paper due to the moment representation (Theorem 5.3, [13] and Theorem 3.1, [6])

$$\mathbb{E}u(t, x)^N = \mathbb{E} \exp \left\{ \sum_{1 \leq j < k \leq N} \int_0^t \gamma(B_j(s) - B_k(s)) ds \right\}. \quad (5)$$

In this work we consider a more general phenomena beyond the setting of positive homogeneity. In addition, we shall work with a larger (than those considered in [4]) class of the covariance functions $\gamma(x)$ which do not have to be, for instance, pointwise defined, non-negative or vanishing at infinity. Indeed, the issues such as the singularity of $\gamma(x)$ arising from some practical needs pose new challenges. In addition, we point out that the approach given in [4] is no-longer working in the case when $\gamma(\cdot)$ switches signs. A practically interesting example is when

$$\gamma(x) = C_H \int_{-\infty}^{\infty} e^{i\xi x} |\xi|^{1-2H} d\xi, \quad x \in \mathbb{R} \quad (6)$$

where $C_H > 0$ is a constant, which corresponds to the parabolic Anderson equation (4) with the Gaussian noise $V(t, x)$ being white in time, fractional (with the Hurst parameter H) in space. Recently, it is pointed out ([12]) that the parabolic Anderson equation (4) is solvable with the moment representation (5) as $d = 1$ and $1/4 < H < 1/2$. On the other hand, the fact that $1 - 2H > 0$ indicates that $\gamma(x)$ is not

defined at any point $x \in \mathbb{R}$. Later we shall show that $\gamma(\cdot)$ is sign-switching in a suitable sense. This example provides a way to measure our capability in dealing with the issue of singularity and sign-alternativity.

To include the cases like (6), $\gamma(x)$ is allowed to be generalized function defined as a linear functional $\gamma: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{R}$ symbolically given as

$$\int_{\mathbb{R}^d} \gamma(x)\varphi(x)dx \equiv \langle \gamma, \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing and infinitely smooth functions.

The quadratic form is defined as

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y)\varphi(x)\psi(y)dxdy \equiv \langle \gamma, \varphi * \tilde{\psi} \rangle, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d),$$

where $\tilde{\psi}(x) = \psi(-x)$. Finally, $\gamma(\cdot)$ is said to be non-negative definite if

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y)\varphi(x)\varphi(y)dxdy \geq 0 \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{7}$$

In connection to the parabolic Anderson model given in (4), the covariance function is non-negative definite. Throughout, we assume non-negative definite condition (7) on $\gamma(\cdot)$.

According to Bochner representation, there is a positive and symmetric measure $\mu(d\lambda)$ on \mathbb{R}^d such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y)\varphi(x)\psi(y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(\xi)\overline{\mathcal{F}(\psi)(\xi)}\mu(d\xi), \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{8}$$

where

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x}\varphi(x)dx \tag{9}$$

is the Fourier transform of $\varphi(\cdot)$. Further, $\mu(d\xi)$ is tempered in the sense that

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^p}\mu(d\xi) < \infty \tag{10}$$

for some $p > 0$. Bochner representation can be written symbolically as

$$\gamma(x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x}\mu(d\xi). \tag{11}$$

Noticing that the difference of two independent Brownian motions is a constant multiple of a Brownian motion. To make sense of the exponential moment given in (1), it is required that the time integral

$$\int_0^t \gamma(B(s))ds$$

be properly defined and exponentially integrable. When $\mu(d\xi)$ is a finite measure, $\gamma(x)$ is pointwise-defined, bounded and continuous. The above time integral is nothing more than an ordinary Riemann integral and the exponential integrability follows from the boundedness of $\gamma(x)$.

The problem is highly non-trivial in the general setting. Given $\epsilon > 0$, the function

$$\gamma_\epsilon(x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \exp \left\{ -\frac{\epsilon}{2}|\xi|^2 \right\} \mu(d\xi) \tag{12}$$

is non-negative definite with the finite spectral measure (see (10))

$$\mu_\epsilon(d\xi) = \exp \left\{ -\frac{\epsilon}{2}|\xi|^2 \right\} \mu(d\xi).$$

We assume that for every $t > 0$

$$\left\{ \begin{aligned} \int_0^t \gamma(B(s)) ds &\stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow \infty} \int_0^t \gamma_\epsilon(B(s)) ds \quad \text{exist in } \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}), \\ \mathbb{E} \exp \left\{ \theta \int_0^t \gamma(B(s)) ds \right\} &< \infty, \quad \text{for every } \theta > 0. \end{aligned} \right. \tag{13}$$

One can see that the function $\gamma(x)$ in (13) is replicable by $\gamma(ax)$ for any constant $a > 0$ as

$$\int_0^t \gamma(aB(s)) ds \stackrel{d}{=} a^{-1} \int_0^{at} \gamma(B(s)) ds.$$

In particular, the Brownian motion $B(s)$ in (13) is replicable by $B_j(s) - B_k(s)$. Finally, the exponential moment written in (1) is well-defined and finite under the assumption (13).

Theorem 1.1. *Under non-negative definite condition (7) and the assumption (13) on $\gamma(\cdot)$,*

$$\lim_{N \rightarrow \infty} \frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} = \frac{1}{2} \mathcal{E}_H, \tag{14}$$

where

$$\mathcal{E}_H = \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x - y) g^2(x) g^2(y) dx dy - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}, \tag{15}$$

which is well-defined and finite with

$$\mathcal{F}_d = \left\{ g \in \mathcal{W}^{1,2}(\mathbb{R}^d); \quad \|g\|_2 = 1 \right\}. \tag{16}$$

We now comment on the condition (13) by considering the following three classes of $\gamma(x)$ that are encompassed by (13).

The first class consists of the constant multiples of all characteristic functions $\gamma(x)$ on \mathbb{R}^d that correspond to symmetric probability distributions $\mu(d\xi)$. As mentioned, (13) becomes automatic when $\mu(d\xi)$ is finite. In particular, Theorem 1.1 holds for every characteristic function $\gamma(x)$ on \mathbb{R}^d . In view of the example

$$\gamma(x) = \frac{\sin x}{x}, \quad x \in \mathbb{R},$$

we see that $\gamma(x)$ is allowed to pick positive and negative values. In addition, one can make $\gamma(x)$ periodic (in particular, $\gamma(x)$ does not vanishing at ∞) by considering the distribution $\mu(d\xi)$ supported on the lattice \mathbb{Z}^d .

The second class consists of all non-negative and non-negative definite $\gamma(x)$. In the case when $\gamma(\cdot)$ is not defined point-wise, “ $\gamma(\cdot) \geq 0$ ” means $\gamma_\epsilon(\cdot) \geq 0$ for sufficiently small $\epsilon > 0$. It is well-known ([9]) that for any non-negative definite $\gamma(\cdot) \geq 0$, the assumption (13) is equivalent to the Dalang’s condition

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty. \tag{17}$$

This class includes some practically interesting cases such as $\gamma(x) = \delta_0(x)$ for $d = 1$ where the time integral is $\int_0^t \delta_0(B(s))ds$ is the Brownian local time; $\gamma(x) = |x|^{-\alpha}$ for $0 < \alpha < \max\{2, d\}$, where the time integral $\int_0^t |B(s)|^{-\alpha}ds$ stands for the Riesz potential ([2]); and the covariance function $\gamma(x) = C_H \prod_{j=1}^d |x_j|^{2H_j-2}$ of the space-time noise $\dot{W}(s, x)$ with the Hurst index $H = (1/2, H_1, \dots, H_d)$ satisfying $1/2 < H_j < 1$ for $1 \leq j \leq d$.

The third class contains the non-negative definite functions $\gamma(\cdot)$ that have infinite spectral measure and are allowed to take negative values. Dalang’s condition is no longer sufficient without assuming $\gamma(\cdot) \geq 0$. A good example is given in (6) where $\xi(d\xi) = |\xi|^{1-2H}d\xi$. It has been pointed out recently ([12]) that (13) holds for $1/4 < H < 1/2$. On the other hand, it is easy to see that the Dalang’s condition (17) holds for any $0 < H < 1$. Therefore, (17) alone is not sufficient if “ $H > 1/4$ ” is necessary for (6).

To show the necessity of “ $H > 1/4$ ”, we start from an easy-to-check identity

$$\begin{aligned} \mathbb{E} \left[\int_0^\tau \gamma(B(s))ds \right]^2 &= 2C_H^2 \int_{\mathbb{R} \times \mathbb{R}} \frac{|\xi|^{1-2H}}{1 + 2^{-1}|\xi|^2} \frac{|\eta|^{1-2H}}{1 + 2^{-1}|\xi + \eta|^2} d\xi d\eta \\ &= 2C_H^2 \int_{\mathbb{R} \times \mathbb{R}} \frac{|\xi|^{1-2H}}{1 + 2^{-1}|\xi|^2} \frac{|\eta - \xi|^{1-2H}}{1 + 2^{-1}|\eta|^2} d\xi d\eta, \end{aligned}$$

where τ is an independent exponential time with $\mathbb{E}\tau = 1$. The integral on the right hand side is bounded from below by

$$\left(\frac{1}{2}\right)^{1-2H} \int_{\{|\xi| \geq 2\}} \int_{\{|\eta| \leq 1\}} \frac{|\xi|^{2-4H}}{1 + 2^{-1}|\xi|^2} \frac{1}{1 + 2^{-1}|\eta|^2} d\xi d\eta,$$

which is finite only when $H > 1/4$. On the other hand, by Brownian scaling

$$\mathbb{E} \left[\int_0^\tau \gamma(B(s))ds \right]^2 = t^{-(1+2H)} \left(\mathbb{E}\tau^{1+2H} \right) \mathbb{E} \left[\int_0^1 \gamma(B(s))ds \right]^2.$$

Summarizing our argument, “ $H > 1/4$ ” is necessary for (6).

Finding a condition for (13) that is “uniformly right” for the third class appears to be a hard problem beyond the scope of the current paper.

We now discuss the links to the high moment asymptotics of the parabolic Anderson equation and to the model of the Bose-Einstein condensation given in (2). By Brownian scaling, (14) can be rewritten as

$$\lim_{N \rightarrow \infty} \frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{t_N}{N} \sum_{1 \leq j < k \leq N} \int_0^t \gamma \left(\sqrt{t_N} (B_j(s) - B_k(s)) \right) ds \right\} = \frac{t}{2} \mathcal{E}_H. \tag{18}$$

for any $t > 0$

In the special case when $\gamma(\cdot)$ satisfies the homogeneity $\gamma(Cx) = |C|^{-\alpha} \gamma(x)$ ($x \in \mathbb{R}^d$ and $C \in \mathbb{R}$) for some $0 < \alpha < 2$, taking $t_N = N^{\frac{2}{2-\alpha}}$ in (18) we have, in view of the moment representation (5), the high moment asymptotics

$$\lim_{N \rightarrow \infty} N^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E} u(t, x)^N = \frac{t}{2} \mathcal{E}_H \tag{19}$$

for the parabolic Anderson equation (4). In the special case of (6), $\alpha = 2 - 2H$. By variable rescaling $\mathcal{E}_H = C_H^{\alpha/(2-\alpha)} \mathcal{E}'_H$, where the variation \mathcal{E}'_H is generated by the interaction function $\gamma(\cdot)$ in (6) with $C_H = 1$. We therefore have

Corollary 1. *When $1/4 < H < 1/2$ in the setting of (6), for any $t > 0$*

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{-\frac{1+H}{H}} \log \mathbb{E}u(t, x)^N \\ &= \frac{t}{2} C_H^{\frac{1}{H}} \sup_{g \in \mathcal{F}_1} \left\{ \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i\xi x} g^2(x) dx \right|^2 |\xi|^{1-2H} d\xi - \int_{\mathbb{R}} |g'(x)|^2 dx \right\}. \end{aligned} \tag{20}$$

Here we point out that (19) is achieved in [4] under the extra assumptions that $\alpha < d$ (The equality is allowed in the case $d = 1$ and $\gamma(\cdot) = \delta_0(\cdot)$) and $\gamma(\cdot) \geq 0$. These extra assumptions are not required by Theorem 1.1. Corollary 1 provides a concrete example of the improvement where $\gamma(\cdot)$ switches sign (as analyzed above) and $\alpha = 2 - 2H > 1 = d$.

In case $d = 2$, substituting $t_N = N^{2\beta}$ into (18) for some $\beta > 0$, we obtain the following asymptotic law for the model of the Bose-Einstein condensation given in (2):

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N^{1+2\beta}} \log \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^1 N^{2\beta} \gamma \left(N^\beta (B_j(s) - B_k(s)) \right) ds \right\} \\ &= \frac{1}{2} \mathcal{E}_H. \end{aligned} \tag{21}$$

The proof of Theorem 1.1 consists two steps: The first step is carried out in Section 2 where we prove Theorem 1.1 in the special case when the spectral measure $\mu(d\xi)$ is finite. The main idea in this step is linearization and tangent approximation. A fascinating feature of our treatment is its relevance to the famous Donsker-Varadhan large deviation principle for the i.i.d. random variables with values in infinite dimensional spaces. The general setting is treated in step 2 that is given in Section 3, where $\gamma(\cdot)$ is approximated by $\gamma_\epsilon(\cdot)$ defined in (11). After completing this work mathematically, we became aware of the literature on bosonic quantum system, the very recent development [16] on Hartree’s theory and their relevance to the main topic of our paper. We therefore add Section 4 to address this link.

2. When the measure $\mu(d\xi)$ is finite. In this case, everything stated in Theorem 1.1 can be directly defined. In particular, the fact that $\gamma(x)$ is uniformly bounded implies that $\mathcal{E}_H < \infty$. Notice that

$$\sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds = \frac{1}{2} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds - N\mu(\mathbb{R}^d).$$

Theorem 1.1 can be restated as

$$\lim_{N \rightarrow \infty} \frac{1}{N t_N} \log \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} = \frac{1}{2} \mathcal{E}_H. \tag{22}$$

2.1. Lower bound for (22). By integral substitution

$$\frac{1}{N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds = N t_N \int_0^1 \int_{\mathbb{R}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{i\xi \cdot B_j(t_N s)} \right|^2 \mu(d\xi) ds.$$

Let $\mathcal{H} \subset \mathcal{L}^2([0, 1] \times \mathbb{R}^d; ds \otimes \mu)$ be the subspace consisting of the functions with $f(s, -\lambda) = \overline{f(s, \lambda)}$. Then \mathcal{H} is a real Hilbert space. Here we point out that in order for \mathcal{H} to be a real Hilbert space, the functions in \mathcal{H} do not have to be real valued. What matters is that for any real number c_1, c_2 and $h_1, h_2 \in \mathcal{H}$, $c_1 h_1 + c_2 h_2 \in \mathcal{H}$ and that the linear functional

$$\langle h_1, h_2 \rangle = \int_0^1 \int_{\mathbb{R}^d} h_1(s, \xi) \overline{h_2(s, \xi)} \mu(d\xi) ds = \int_0^1 \int_{\mathbb{R}^d} h_1(s, \xi) h_2(s, -\xi) \mu(d\xi) ds$$

takes real values. All those hold thanks to the symmetry of $\mu(d\xi)$.

Let $f \in \mathcal{H}$ be a fixed bounded function. By the fact that $\|h\|^2 \geq -\|f\|^2 + 2\langle f, h \rangle$ for all $h \in \mathcal{H}$, we have that

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{i\xi \cdot B_j(t_N s)} \right|^2 \mu(d\xi) ds \\ & \geq -\|f\|^2 + \frac{2}{N} \sum_{j=1}^N \int_0^1 \bar{f}(s, B_j(t_N s)) ds \\ & = -\|f\|^2 + \frac{2}{N t_N} \sum_{j=1}^N \int_0^{t_N} \bar{f}\left(\frac{s}{t_N}, B_j(t_N s)\right) ds \end{aligned}$$

where

$$\bar{f}(s, x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(s, -\xi) \mu(d\xi) \quad 0 \leq s \leq 1, \quad x \in \mathbb{R}^d.$$

By independence

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \\ & \geq \left(\exp \left\{ -\frac{1}{2} \|f\|^2 t_N \right\} \mathbb{E} \exp \left\{ \int_0^{t_N} \bar{f}\left(\frac{s}{t_N}, B(s)\right) ds \right\} \right)^N. \end{aligned}$$

By Proposition 3.1 in [5],

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{t_N} \log \mathbb{E} \exp \left\{ \int_0^{t_N} \bar{f}\left(\frac{s}{t_N}, B(s)\right) ds \right\} \\ & = \sup_{g \in \mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d} \bar{f}(s, x) g^2(s, x) dx ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \\ & = \sup_{g \in \mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(s, -\xi) \left[\int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right] \mu(d\xi) ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \end{aligned}$$

where

$$\mathcal{A}_d = \left\{ g(s, x); \quad g(s, \cdot) \in \mathcal{F}_d \quad \forall 0 \leq s \leq 1 \right\}. \tag{23}$$

Thus,

$$\liminf_{N \rightarrow \infty} \frac{1}{N t_N} \log \mathbb{E} \exp \left\{ \frac{1}{N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \tag{24}$$

$$\begin{aligned} &\geq \sup_{g \in \mathcal{A}_d} \left\{ -\frac{1}{2} \|f\|^2 + \int_0^1 \int_{\mathbb{R}^d} f(s, -\xi) \left[\int_{\mathbb{R}^d} e^{i\lambda \cdot x} g^2(s, x) dx \right] \mu(d\xi) ds \right. \\ &\quad \left. - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\}. \end{aligned}$$

Notice that the relation $\|h\|^2 \geq -\|f\|^2 + 2\langle f, h \rangle$ becomes an equality when $h = f$. Hence, for any dense sub-space \mathcal{H}_0 of \mathcal{H}_0

$$\sup_{f \in \mathcal{H}_0} \left\{ -\|f\|^2 + 2\langle f, h \rangle \right\} = \|h\|^2 \quad h \in \mathcal{H}. \quad (25)$$

We call the identity approximation by tangent planes.

Let \mathcal{H}_0 be the space of the bounded functions in \mathcal{H} . Taking supremum over $f \in \mathcal{H}_0$ on the right hand side of (24), it becomes

$$\begin{aligned} &\sup_{g \in \mathcal{A}_d} \left\{ \sup_{f \in \mathcal{H}_0} \left(-\frac{1}{2} \|f\|^2 + \int_0^1 \int_{\mathbb{R}^d} f(s, -\xi) \left[\int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right] \mu(d\xi) ds \right) \right. \\ &\quad \left. - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \\ &= \frac{1}{2} \sup_{g \in \mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d} \left| e^{i\xi \cdot x} g^2(s, x) \right|^2 \mu(d\xi) ds - \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \\ &= \frac{1}{2} \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d} \left| e^{i\xi \cdot x} g^2(x) \right|^2 \mu(d\xi) - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} = \frac{1}{2} \mathcal{E}_H. \end{aligned}$$

Summarizing our estimates, we obtain the lower bound for (22):

$$\liminf_{N \rightarrow \infty} \frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \geq \frac{1}{2} \mathcal{E}_H. \quad (26)$$

2.2. Upper bound for (22). Let $t > 0$ be fixed but large. For the sake of simplification we assume that t_N/t always remains to be an integer. By Markov property,

$$\begin{aligned} &\mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \\ &\leq \left(\sup_{\tilde{b}} \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot b_j} e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \right)^{t_N/t}, \quad (27) \end{aligned}$$

where the supremum is taken over $\tilde{b} = (b_1, \dots, b_N) \in (\mathbb{R}^d)^N$.

We now claim that for any $\tilde{b} = (b_1, \dots, b_N) \in (\mathbb{R}^d)^N$, and integer $n \geq 1$,

$$\begin{aligned} &\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot b_j} e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right]^n \\ &\leq \mathbb{E} \left[\int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right]^n. \quad (28) \end{aligned}$$

Indeed,

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot b_j} e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right]^n \\ &= \int_{[0,t]^N} \int_{(\mathbb{R}^d)^N} \mathbb{E} \prod_{l=1}^n \left| \sum_{j=1}^N e^{i\xi_l \cdot b_j} e^{i\xi_l \cdot B_j(s_l)} \right|^2 \mu(d\xi_1) \cdots \mu(d\xi_N) ds_1 \cdots ds_N. \end{aligned}$$

Write

$$\begin{aligned} & \prod_{l=1}^n \left| \sum_{j=1}^N e^{i\xi_l \cdot b_j} e^{i\xi_l \cdot B_j(s_l)} \right|^2 \\ &= \prod_{l=1}^n \left(\sum_{j=1}^N e^{i\xi_l \cdot b_j} e^{i\xi_l \cdot B_j(s_l)} \right) \left(\sum_{j=1}^N e^{-i\xi_l \cdot b_j} e^{-i\xi_l \cdot B_j(s_l)} \right) \\ &= \sum_{j_1, \dots, j_n=1}^N C(j_1, \dots, j_n) \exp \left\{ i \sum_{l=1}^n \alpha_l \cdot B_{j_l}(s_{j_l}) \right\}, \end{aligned}$$

where $C(j_1, \dots, j_n)$ are deterministic complex numbers with norm 1, and $\alpha_l \in \mathbb{R}^d$ are deterministic such that

$$\sum_{j_1, \dots, j_n=1}^N \exp \left\{ i \sum_{l=1}^n \alpha_l \cdot B_{j_l}(s_{j_l}) \right\} = \left| \prod_{l=1}^n \sum_{j=1}^N e^{i\xi_l \cdot B_j(s_l)} \right|^2.$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot b_j} e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right]^n \\ &= \int_{[0,t]^N} \int_{(\mathbb{R}^d)^N} \left[\sum_{j_1, \dots, j_n=1}^N C(j_1, \dots, j_n) \mathbb{E} \exp \left\{ i \sum_{l=1}^n \alpha_l \cdot B_{j_l}(s_{j_l}) \right\} \right] \\ & \times \mu(d\xi_1) \cdots \mu(d\xi_N) ds_1 \cdots ds_N \\ &\leq \int_{[0,t]^N} \int_{(\mathbb{R}^d)^N} \left[\sum_{j_1, \dots, j_n=1}^N \mathbb{E} \exp \left\{ i \sum_{l=1}^n \alpha_l \cdot B_{j_l}(s_{j_l}) \right\} \right] \mu(d\xi_1) \cdots \mu(d\xi_N) ds_1 \cdots ds_N \\ &= \int_{[0,t]^N} \int_{(\mathbb{R}^d)^N} \mathbb{E} \left(\prod_{l=1}^n \left| \sum_{j=1}^N e^{i\xi_l \cdot B_j(s_l)} \right|^2 \right) \mu(d\xi_1) \cdots \mu(d\xi_N) ds_1 \cdots ds_N \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right]^n \end{aligned}$$

where the inequality follows from the fact that

$$\mathbb{E} \exp \left\{ i \sum_{l=1}^n \alpha_l \cdot B_{j_l}(s_{j_l}) \right\} > 0.$$

Thus, we have proved (28). From (28), and by Taylor expansion, for any $\tilde{b} = (b_1, \dots, b_N) \in (\mathbb{R}^d)^N$,

$$\mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot b_j} e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\}$$

$$\leq \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\}.$$

Then, in view of (27), we have

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \\ & \leq \left(\mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \right)^{t_N/t}. \end{aligned} \quad (29)$$

Recall the following Donsker-Varadhan's large deviation principle (Theorem 5.3, [11]): Let E be a real separable Banach space with E^* as its topological dual. Let $X = \{X_k\}_{k \geq 1}$ be a sequence of i.i.d. random variables taking values in E such that

$$\mathbb{E} \exp \left\{ \theta \|X\| \right\} < \infty \quad \forall \theta > 0.$$

Then for any close set $F \subset E$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P} \left\{ \frac{1}{N} \sum_{j=1}^N X_j \in F \right\} \leq - \inf_{x \in F} \Lambda^*(x)$$

and for any open set $G \subset E$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P} \left\{ \frac{1}{N} \sum_{j=1}^N X_j \in G \right\} \geq - \inf_{x \in G} \Lambda^*(x)$$

where the rate function is given as the convex conjugate

$$\Lambda^*(x) = \sup_{f \in E^*} \left\{ \langle f, x \rangle - \Lambda(f) \right\} \quad x \in E$$

of the convex functional

$$\Lambda(f) = \log \mathbb{E} \exp \left\{ \langle f, X \rangle \right\} \quad f \in E^*.$$

This result appears to be an infinite dimensional extension of Cramer's large deviation (Theorem 2.2.3, p.27, [10]). We refer an interested reader also to [1] for an elegant proof of the Donsker-Varadhan large deviation principle.

Recall that $\mathcal{H} \subset \mathcal{L}^2([0, t] \times \mathbb{R}^d; ds \otimes \mu)$ is the subspace consists of the functions with $f(s, -\xi) = \overline{f(s, \xi)}$ a.e.. Also, note that \mathcal{H} is a real Hilbert space. To apply Donsker-Varadhan large deviation principle to the space $E = \mathcal{H}$, we define the i.i.d. \mathcal{H} -valued random variables $\{X_k\}_{k \geq 1}$ as

$$X_k(s, \xi) = e^{i\xi \cdot B_k(s)} \quad (s, \xi) \in [0, t] \times \mathbb{R}^d.$$

By the fact that $\|X\| \leq t\mu(\mathbb{R}^d)$, the condition of Donsker-Varadhan large deviation principle holds. Further, by Varadhan integral lemma (Theorem 4.3.1, [10]),

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^t \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \\ & = \sup_{h \in \mathcal{H}} \left\{ \frac{1}{2} \|h\|^2 - \Lambda^*(h) \right\}. \end{aligned} \quad (30)$$

Some delicate steps are needed in handling the variation on the right hand side. To this end we first claim that $\Lambda^*(h) = \infty$ for any $h \in \mathcal{H}$ with $\|h\|_\infty > 1$, where

$$\|h\|_\infty = \sup \left\{ c \geq 0; \int ds \otimes \mu \left\{ (s, \xi); |h(s, \xi)| \geq c \right\} > 0 \right\}.$$

Indeed, applying Hahn-Banach theorem there is a $\epsilon > 0$ and $f \in \mathcal{L}([0, t] \times \mathbb{R}^d)$ such that

$$\int_0^t \int_{\mathbb{R}^d} |f(s, \xi)| \mu(d\xi) ds = 1,$$

and $\langle f, h \rangle > 1 + \epsilon$. In addition, we may make $f \in \mathcal{H}$. In particular,

$$\langle f, X \rangle = \int_0^t \int_{\mathbb{R}^d} e^{i\xi \cdot B(s)} f(s, -\xi) \mu(d\xi) ds \leq \int_0^t \int_{\mathbb{R}^d} |f(s, \xi)| \mu(d\xi) ds = 1.$$

Hence, for any $C > 0$

$$\Lambda(Cf) = \log \mathbb{E} \exp \left\{ C \langle f, X \rangle \right\} \leq C.$$

Thus,

$$\Lambda^*(h) \geq C \langle f, h \rangle - \Lambda(Cf) \geq (1 + \epsilon)C - C = \epsilon C$$

which leads to $\Lambda^*(h) = \infty$ as C can be arbitrarily large.

Thus,

$$\sup_{h \in \mathcal{H}} \left\{ \frac{1}{2} \|h\|^2 - \Lambda^*(h) \right\} = \sup_{\substack{\|h\|_\infty \leq 1 \\ h \in \mathcal{H}}} \left\{ \frac{1}{2} \|h\|^2 - \Lambda^*(h) \right\}. \tag{31}$$

Let

$$\mathcal{N}_t = \left\{ f \in \mathcal{H}; \|f\|_\infty \leq 1 \text{ and } f(s, \xi) \text{ is continuous on } [0, t] \times \mathbb{R}^d \right\}.$$

An obvious modification of (25) leads to

$$\|h\|^2 = \sup_{f \in \mathcal{N}_t} \left\{ -\|f\|^2 + 2 \langle f, h \rangle \right\} \quad \|h\|_\infty \leq 1, \quad h \in \mathcal{H}.$$

Hence,

$$\begin{aligned} \sup_{\substack{\|h\|_\infty \leq 1 \\ h \in \mathcal{H}}} \left\{ \frac{1}{2} \|h\|^2 - \Lambda^*(h) \right\} &= \sup_{\substack{\|h\|_\infty \leq 1 \\ h \in \mathcal{H}}} \left\{ \frac{1}{2} \sup_{f \in \mathcal{N}_t} \left\{ -\|f\|^2 + 2 \langle f, h \rangle \right\} - \Lambda^*(h) \right\} \\ &= \sup_{f \in \mathcal{N}_t} \left\{ -\frac{1}{2} \|f\|^2 + \sup_{\substack{\|h\|_\infty \leq 1 \\ h \in \mathcal{H}}} \left\{ \langle f, h \rangle - \Lambda^*(h) \right\} \right\}. \end{aligned}$$

By duality lemma (Lemma 4.5.8, p.152, [10])

$$\sup_{\substack{\|h\|_\infty \leq 1 \\ h \in \mathcal{H}}} \left\{ \langle f, h \rangle - \Lambda^*(h) \right\} \leq \sup_{h \in \mathcal{H}} \left\{ \langle f, h \rangle - \Lambda^*(h) \right\} = \Lambda(f).$$

Therefore, it follows that

$$\begin{aligned} \sup_{\substack{\|h\|_\infty \leq 1 \\ h \in \mathcal{H}}} \left\{ \frac{1}{2} \|h\|^2 - \Lambda^*(h) \right\} &\leq \sup_{f \in \mathcal{N}_t} \left\{ -\frac{1}{2} \|f\|^2 + \log \mathbb{E} \exp \left\{ \langle f, X \rangle \right\} \right\} \\ &= \sup_{f \in \mathcal{N}_t} \left\{ -\frac{1}{2} \|f\|^2 + \log \mathbb{E} \exp \left\{ \int_0^t \bar{f}(s, B(s)) ds \right\} \right\} \end{aligned}$$

where

$$\bar{f}(s, x) = \int_{\mathbb{R}^d} f(s, -\xi) e^{i\xi \cdot x} \mu(d\xi).$$

Combining (29), (30) and (31), we obtain,

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{1}{2N} \int_0^{t_N} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N e^{i\xi \cdot B_j(s)} \right|^2 \mu(d\xi) ds \right\} \\ & \leq \frac{1}{t} \log \sup_{f \in \mathcal{N}_t} \mathbb{E} \exp \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \bar{f}(s, B(s)) ds \right\}. \end{aligned}$$

Thus, all we need is to prove that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \sup_{f \in \mathcal{N}_t} \mathbb{E} \exp \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \bar{f}(s, B(s)) ds \right\} \leq \frac{1}{2} \mathcal{E}_H. \tag{32}$$

Define $\tau_t = \inf\{s \geq 0; |B(s)| \geq t^2\}$. Then,

$$\begin{aligned} & \mathbb{E} \exp \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \bar{f}(s, B(s)) ds \right\} \tag{33} \\ & = \mathbb{E} \left[\exp \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \bar{f}(s, B(s)) ds \right\}; \tau_t \geq t \right] \\ & \quad + \mathbb{E} \left[\exp \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \bar{f}(s, B(s)) ds \right\}; \tau_t < t \right]. \end{aligned}$$

Notice that $|\bar{f}(s, x)| \leq \mu(\mathbb{R}^d)$. The second term on the right hand side is bounded by

$$\exp \left\{ t\mu(\mathbb{R}^d) \right\} \mathbb{P} \left\{ \max_{s \leq 1} |B(s)| \geq t^{3/2} \right\}$$

which is negligible.

As for the first term, first notice that

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \int_0^t \bar{f}(s, B(s)) ds \right\}; \tau_t \geq t \right] \\ & \leq \exp \left\{ \mu(\mathbb{R}^d) \right\} \mathbb{E} \left[\exp \left\{ \int_1^t \bar{f}(s, B(s)) ds \right\}; \tau_t \geq t \right] \\ & = \exp \left\{ \mu(\mathbb{R}^d) \right\} \int_{B(0, t^2)} \tilde{p}_1(x) \mathbb{E}_x \left[\exp \left\{ \int_0^{t-1} \bar{f}(1+s, B(s)) ds \right\}; \tau_t \geq t \right] dx, \end{aligned}$$

where $\tilde{p}_1(x)$ is the density of the measure $\nu(A) = \mathbb{P}\{B(1) \in A; \tau_t \geq 1\}$.

Let $p_1(x)$ be the density of $B(1)$ and notice that $\tilde{p}_1(x) \leq p_1(x) \leq (2\pi)^{-d/2}$. Hence, the right hand side is no greater than

$$\begin{aligned} & (2\pi)^{-d} \exp \left\{ \mu(\mathbb{R}^d) \right\} \int_{B(0, t^2)} \mathbb{E}_x \left[\exp \left\{ \int_0^{t-1} \bar{f}(1+s, B(s)) ds \right\}; \tau_t \geq t \right] dx \\ & \leq (2\pi)^{-d} |B(0, t^2)| \exp \left\{ \mu(\mathbb{R}^d) \right\} \exp \left\{ \int_0^{t-1} \lambda(\bar{f}(1+s, \cdot)) ds \right\}, \end{aligned}$$

where

$$\lambda(\bar{f}(1+s, \cdot)) = \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d} \bar{f}(1+s, x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\},$$

and the last step follows from Lemma 7.1 in [4].

Further, notice that

$$\begin{aligned} & \int_0^{t-1} \lambda(\bar{f}(1+s, \cdot)) ds \leq \int_0^t \lambda(\bar{f}(s, \cdot)) ds \\ & = \sup_{g \in \mathcal{A}_d(t)} \left\{ \int_0^t \int_{\mathbb{R}^d} f(s, -\xi) \left[\int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right] \mu(d\xi) ds \right. \\ & \quad \left. - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\}, \end{aligned}$$

where $\mathcal{A}_d(t) = \left\{ g(s, x); g(s, \cdot) \in \mathcal{F}_d \text{ for every } 0 \leq s \leq t \right\}$.

Summarizing our computation, we obtain the bound

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \int_0^t \bar{f}(s, B(s)) ds \right\}; \tau_t \geq t \right] \\ & \leq Ct^{2d} \exp \left(\sup_{g \in \mathcal{A}_d(t)} \left\{ \int_0^t \int_{\mathbb{R}^d} f(s, -\xi) \left[\int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right] \mu(d\xi) ds \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \right) \end{aligned}$$

uniformly over $f \in \mathcal{N}_t$. By the relation $-\|f\|^2 + 2\langle f, h \rangle \leq \|h\|_2^2$,

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \bar{f}(s, B(s)) ds \right\}; \tau_t \geq t \right] \\ & \leq Ct^{2d} \exp \left(\sup_{g \in \mathcal{A}_d(t)} \left\{ -\frac{1}{2} \|f\|^2 + \int_0^t \int_{\mathbb{R}^d} f(s, -\xi) \left[\int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right] \mu(d\xi) ds \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \right) \\ & \leq Ct^{2d} \exp \left(\frac{1}{2} \sup_{g \in \mathcal{A}_d(t)} \left\{ \int_0^t \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right|^2 \mu(d\xi) ds \right. \right. \\ & \quad \left. \left. - \int_0^t \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \right). \end{aligned}$$

By the fact that

$$\begin{aligned} & \sup_{g \in \mathcal{A}_d(t)} \left\{ \int_0^t \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(s, x) dx \right|^2 \mu(d\xi) ds - \int_0^t \int_{\mathbb{R}^d} |\nabla_x g(s, x)|^2 dx ds \right\} \\ & \leq \int_0^t \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(x) dx \right|^2 \mu(d\xi) - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} ds \\ & = t\mathcal{E}_H, \end{aligned}$$

we therefore reach the bound (32). □

3. When the measure $\mu(d\xi)$ is infinite. The first thing we need to show is that \mathcal{E}_H is well defined and finite under our assumption. We specifically point out that the integrals

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y)g^2(x)g^2(y)dx dy \quad g \in \mathcal{F}_d$$

have to be properly defined as $g^2(x)$ is not necessarily in $\mathcal{S}(\mathbb{R}^d)$.

With the method used in proving (29), one can show that for any $t_1, t_2 > 0$

$$\begin{aligned} & \mathbb{E} \exp \left\{ \theta \int_0^{t_1+t_2} \gamma(B(s)) ds \right\} \\ & \leq \mathbb{E} \exp \left\{ \theta \int_0^{t_1} \gamma(B(s)) ds \right\} \mathbb{E} \exp \left\{ \theta \int_0^{t_2} \gamma(B(s)) ds \right\}. \end{aligned} \tag{34}$$

Consequently, the limit

$$\Lambda \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ \theta \int_0^t \gamma(B(s)) ds \right\}$$

exists and finite.

Recall that $\gamma_\epsilon(x)$ is the non-negative definite function defined in (12). Given $\epsilon > \epsilon' > 0$, notice that $\gamma_{\epsilon'}(\cdot) - \gamma_\epsilon(\cdot)$ is non-negative definite with the spectral measure

$$\left(\exp \left\{ -\frac{\epsilon'}{2} |\xi|^2 \right\} - \exp \left\{ -\frac{\epsilon}{2} |\xi|^2 \right\} \right) \mu(d\xi).$$

For any given $x \in \mathbb{R}^d$, $t > 0$ and the integer $n \geq 1$, similar to (28)

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \left\{ \gamma_{\epsilon'}(x+B(s)) - \gamma_\epsilon(x+B(s)) \right\} ds \right]^n \\ & \leq \mathbb{E} \left[\int_0^t \left\{ \gamma_{\epsilon'}(B(s)) - \gamma_\epsilon(B(s)) \right\} ds \right]^n. \end{aligned} \tag{35}$$

By (13), this implies the moment convergence

$$\int_0^t \gamma(B(s)-x) ds \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow \infty} \int_0^t \gamma_\epsilon(B(s)-x) ds \quad x \in \mathbb{R}^d.$$

Further, the moment comparison similar to (28) also leads to

$$\mathbb{E} \exp \left\{ \theta \int_0^t \gamma(x+B(s)) ds \right\} \leq \mathbb{E} \exp \left\{ \theta \int_0^t \gamma(B(s)) ds \right\} < \infty, \tag{36}$$

for every $x \in \mathbb{R}^d$ and $\theta > 0$.

By Theorem 4.1.6, p.99, [3], for any $\epsilon > 0$

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ \theta \int_0^t \gamma_\epsilon(B(s)) ds \right\} \\ & = \sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d} \gamma_\epsilon(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

By Jensen inequality and (36), on the other hand,

$$\begin{aligned} \mathbb{E} \exp \left\{ \theta \int_0^t \gamma_\epsilon(B(s)) ds \right\} & \leq \int_{\mathbb{R}^d} p_\epsilon(x) \mathbb{E} \exp \left\{ \theta \int_0^t \gamma(x+B(s)) ds \right\} dx \\ & \leq \mathbb{E} \exp \left\{ \theta \int_0^t \gamma(B(s)) ds \right\}, \end{aligned}$$

where $p_\epsilon(x)$ is the density of $B(\epsilon)$. Consequently,

$$\sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d} \gamma_\epsilon(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} \leq \Lambda. \tag{37}$$

Noticing that for any $g \in \mathcal{F}_d$

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\epsilon(x-y) g^2(x) g^2(y) dx dy &= \int_{\mathbb{R}^d} g^2(y) \left[\int_{\mathbb{R}^d} \gamma_\epsilon(x-y) g^2(x) dx \right] dy \\ &\leq \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \gamma_\epsilon(x-y) g^2(x) dx = \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \gamma_\epsilon(x) g^2(x+y) dx. \end{aligned}$$

Consequently, for any $\theta > 0$ and $\epsilon > 0$

$$\begin{aligned} &\sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\epsilon(x-y) g^2(x) g^2(y) dx dy - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} \\ &\leq \sup_{g \in \mathcal{F}_d} \left\{ \theta \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \gamma_\epsilon(x) g^2(x+y) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} \\ &= \sup_{y \in \mathbb{R}^d} \sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d} \gamma_\epsilon(x) g^2(x+y) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} \\ &= \sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d} \gamma_\epsilon(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} \leq \Lambda, \end{aligned}$$

where the third step follows from translation invariance.

Thus, by the relation

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\lambda \cdot x} g^2(x) dx \right|^2 \exp \left\{ -\frac{\epsilon}{2} |\xi|^2 \right\} \mu(d\xi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\epsilon(x-y) g^2(x) g^2(y) dx dy$$

and monotonic convergence, the integral

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dx dy \equiv \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\lambda \cdot x} g^2(x) dx \right|^2 \mu(d\xi)$$

is well-defined and finite for every $g \in \mathcal{F}_d$. Further,

$$\sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dx dy - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} \leq \Lambda < \infty.$$

In particular, \mathcal{E}_H is well-defined and finite.

To establish (14), first notice that for any fixed $\epsilon > 0$, $\gamma_\epsilon(x)$ is non-negative definite with the spectral measure $\mu_\epsilon(d\xi) = e^{-\epsilon|\xi|^2/2} \mu(d\xi)$ that is finite by (10). By what have been proved,

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{N t_N} \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma_\epsilon(B_j(s) - B_k(s)) ds \right\} \\ &= \frac{1}{2} \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\epsilon(x-y) g^2(x) g^2(y) dx dy - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}. \end{aligned} \tag{38}$$

For any integer $n \geq 1$

$$\begin{aligned} & \mathbb{E} \left[\sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma_\epsilon(B_j(s) - B_k(s)) ds \right]^n \\ &= \int_{[0, t_N]^n} \int_{(\mathbb{R}^d)^N} d\mu(d\xi_1) \cdots \mu(d\xi_n) ds_1 \cdots ds_n \exp \left\{ -\frac{\epsilon}{2} \sum_{l=1}^n |\xi_l|^2 \right\} \\ & \times \mathbb{E} \prod_{l=1}^n \sum_{1 \leq j < k \leq N} \exp \left\{ i\xi_l \cdot (B_j(s_l) - B_k(s_l)) \right\}. \end{aligned}$$

By the fact that

$$\mathbb{E} \prod_{l=1}^n \sum_{1 \leq j < k \leq N} \exp \left\{ i\xi_l \cdot (B_j(s_l) - B_k(s_l)) \right\} > 0$$

we obtain that

$$\begin{aligned} & \mathbb{E} \left[\sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma_\epsilon(B_j(s) - B_k(s)) ds \right]^n \tag{39} \\ & \leq \int_{[0, t_N]^n} \int_{(\mathbb{R}^d)^N} d\mu(d\xi_1) \cdots \mu(d\xi_n) ds_1 \cdots ds_n \mathbb{E} \prod_{l=1}^n \sum_{1 \leq j < k \leq N} \exp \left\{ i\xi_l \cdot (B_j(s_l) - B_k(s_l)) \right\} \\ & = \mathbb{E} \left[\sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right]^n. \end{aligned}$$

Therefore, it follows from Taylor expansion that

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \\ & \geq \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma_\epsilon(B_j(s) - B_k(s)) ds \right\}. \end{aligned}$$

By (38)

$$\begin{aligned} & \liminf_{N \rightarrow \infty} \frac{1}{N t_N} \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \\ & \geq \frac{1}{2} \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\epsilon(x - y) g^2(x) g^2(y) dx dy - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

Letting $\epsilon \rightarrow 0^+$ on the right hand side leads to

$$\liminf_{N \rightarrow \infty} \frac{1}{N t_N} \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \geq \frac{1}{2} \mathcal{E}_H. \tag{40}$$

On the other hand, set $\zeta_\epsilon(x) = \gamma(x) - \gamma_\epsilon(x)$. We claim that

$$\lim_{\epsilon \rightarrow 0^+} \limsup_{N \rightarrow \infty} \frac{1}{N t_N} \log \mathbb{E} \exp \left\{ \frac{\theta}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds \right\} = 0 \tag{41}$$

for any $\theta > 0$

By Jensen’s inequality

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{\theta}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds \right\} \\ & \geq \exp \left\{ \frac{\theta}{N} \mathbb{E} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds \right\} \geq 1, \end{aligned}$$

where the last step follows from (39) with $n = 1$.

Thus, all we need is to show the upper bound of (41). Write

$$\sum_{1 \leq j < k \leq N} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds = \frac{1}{2} \sum_{j=1}^N \sum_{k: k \neq j} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds.$$

By Hölder’s inequality,

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{\theta}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds \right\} \\ & \leq \prod_{j=1}^N \left(\mathbb{E} \exp \left\{ \frac{\theta}{2} \sum_{k: k \neq j} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds \right\} \right)^{1/N} \\ & = \mathbb{E} \exp \left\{ \frac{\theta}{2} \sum_{k=2}^N \int_0^{t_N} \zeta_\epsilon(B_1(s) - B_k(s)) ds \right\}. \end{aligned}$$

Write

$$\begin{aligned} & \sum_{k=2}^N \int_0^{t_N} \zeta_\epsilon(B_1(s) - B_k(s)) ds \\ & = \int_{\mathbb{R}^d} \left[\int_0^{t_N} e^{i\xi \cdot B_1(s)} \sum_{k=2}^N \exp \left\{ -i \sum_{k=2}^N \xi \cdot B_k(s) \right\} ds \right] \nu_\epsilon(d\xi) \end{aligned}$$

where

$$\nu_\epsilon(d\xi) = \left(1 - \exp \left\{ -\frac{\epsilon}{2} |\xi|^2 \right\} \right) \mu(d\xi).$$

For any $n \geq 1$, by independence between B_1 and $\{B_2, \dots, B_N\}$

$$\begin{aligned} & \mathbb{E} \left[\sum_{k=2}^N \int_0^{t_N} \zeta_\epsilon(B_1(s) - B_k(s)) ds \right]^n \\ & = \int_{(\mathbb{R}^d)^N} \nu_\epsilon(d\xi_1) \cdots \nu_\epsilon(d\xi_N) \int_{[0, t_N]^N} \mathbb{E} \exp \left\{ i \sum_{l=1}^n \xi_l \cdot B_1(s_l) \right\} \\ & \quad \times \mathbb{E} \left(\prod_{l=1}^n \sum_{k=2}^N \exp \left\{ -i \xi_l \cdot B_k(s_l) \right\} \right) ds_1 \cdots ds_n \\ & \leq \int_{(\mathbb{R}^d)^N} \nu_\epsilon(d\xi_1) \cdots \nu_\epsilon(d\xi_N) \int_{[0, t_N]^N} \mathbb{E} \left(\prod_{l=1}^n \sum_{k=2}^N \exp \left\{ i \xi_l \cdot B_k(s_l) \right\} \right) ds_1 \cdots ds_n \\ & = \mathbb{E} \left[\sum_{k=2}^N \int_0^{t_N} \zeta_\epsilon(B_k(s)) ds \right]^n, \end{aligned}$$

where the inequality follows from the fact that

$$\mathbb{E} \left(\prod_{l=1}^n \sum_{k=2}^N \exp \left\{ -i \xi_l \cdot B_k(s_l) \right\} \right) > 0 \quad \text{and} \quad 0 < \mathbb{E} \exp \left\{ i \sum_{l=1}^n \xi_l \cdot B_1(s_l) \right\} \leq 1.$$

Hence, by Taylor expansion

$$\begin{aligned} \mathbb{E} \exp \left\{ \frac{\theta}{2} \sum_{k=2}^N \int_0^{t_N} \zeta_\epsilon(B_1(s) - B_k(s)) ds \right\} &\leq \mathbb{E} \exp \left\{ \frac{\theta}{2} \sum_{k=2}^N \int_0^{t_N} \zeta_\epsilon(B_k(s)) ds \right\} \\ &= \left(\mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^{t_N} \zeta_\epsilon(B(s)) ds \right\} \right)^{N-1}, \end{aligned}$$

where the last step follows from the independence of the Brownian motions.

To prove (41), we need only to show that

$$\lim_{\epsilon \rightarrow 0^+} \limsup_{N \rightarrow \infty} \frac{1}{t_N} \log \mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^{t_N} \zeta_\epsilon(B(s)) ds \right\} \leq 0. \tag{42}$$

To simplify our notation, we may assume t_N goes to infinity along the integers. By Markov property,

$$\mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^{t_N} \zeta_\epsilon(B(s)) ds \right\} \leq \left(\sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^1 \zeta_\epsilon(x + B(s)) ds \right\} \right)^{t_N}.$$

On the other hand, letting $\epsilon' \rightarrow 0^+$ in (35),

$$\mathbb{E} \left[\int_0^1 \zeta_{\epsilon'}(x + B(s)) ds \right]^n \leq \mathbb{E} \left[\int_0^1 \zeta_{\epsilon'}(B(s)) ds \right]^n$$

for every integer $n \geq 1$. Then, by Taylor expansion

$$\mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^1 \zeta_\epsilon(x + B(s)) ds \right\} \leq \mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^1 \zeta_\epsilon(B(s)) ds \right\}$$

for every $x \in \mathbb{R}^d$. Therefore,

$$\mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^{t_N} \zeta_\epsilon(B(s)) ds \right\} \leq \left(\mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^1 \zeta_\epsilon(B(s)) ds \right\} \right)^{t_N}.$$

Thus, (42) follows from the obvious fact that

$$\lim_{\epsilon \rightarrow 0^+} \mathbb{E} \exp \left\{ \frac{\theta}{2} \int_0^1 \zeta_\epsilon(B(s)) ds \right\} = 1$$

which is clearly a consequence of the assumption (13).

We now prove the upper bound for (14). Let $p, q > 1$ are conjugate numbers. By Hölder inequality

$$\begin{aligned} &\mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \\ &\leq \left(\mathbb{E} \exp \left\{ \frac{p}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma_\epsilon(B_j(s) - B_k(s)) ds \right\} \right)^{1/p} \end{aligned}$$

$$\times \left(\mathbb{E} \exp \left\{ \frac{q}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \zeta_\epsilon(B_j(s) - B_k(s)) ds \right\} \right)^{1/q}.$$

By (38) (with $\gamma_\epsilon(\cdot)$ being replaced by $p\gamma_\epsilon(\cdot)$), (41) (with $\theta = q$) and the fact that

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\epsilon(x - y) g^2(x) g^2(y) dx dy &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(x) dx \right|^2 e^{-\epsilon|\xi|^2/2} \mu(d\xi) \\ &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\xi \cdot x} g^2(x) dx \right|^2 \mu(d\xi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x - y) g^2(x) g^2(y) dx dy \end{aligned}$$

for every $g \in \mathcal{F}_d$, we conclude that

$$\begin{aligned} &\limsup_{N \rightarrow \infty} \frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \\ &\leq \frac{1}{2p} \sup_{g \in \mathcal{F}_d} \left\{ p \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x - y) g^2(x) g^2(y) dx dy - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx ds \right\}. \end{aligned}$$

Letting $p \rightarrow 1^+$ on the right hand side leads to the upper bound

$$\limsup_{N \rightarrow \infty} \frac{1}{Nt_N} \log \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \leq \frac{1}{2} \mathcal{E}_H. \tag{43}$$

Finally, Theorem 1.1 with its full generality follows from (40) and (43). □

4. Link to bosonic quantum system. After completion of this project, we became aware of the subject on bosonic quantum system or more specifically, a very recent development [16] on Hartree’s theory.

Limited to our setting and in our notation, Hartree’s theory supports the statements such that

$$\lim_{N \rightarrow \infty} \frac{\mathcal{E}(N)}{N} = \frac{1}{2} \mathcal{E}_H, \tag{44}$$

where \mathcal{E}_H is given in (15) and is known as Hartree energy in the literature of quantum mechanics, and ($\tilde{x} = (x_1, \dots, x_N)$)

$$\mathcal{E}(N) = \sup_{g \in \mathcal{F}_{Nd}} \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_{\mathbb{R}^{Nd}} \gamma(x_j - x_k) g^2(\tilde{x}) d\tilde{x} - \frac{1}{2} \int_{\mathbb{R}^{Nd}} |\nabla g(\tilde{x})|^2 d\tilde{x} \right\} \tag{45}$$

is called the ground state energy which appears as the principal eigenvalue of the N -body problem ([14]) with the Shrödinger Hamiltonian

$$H_N = \frac{1}{2} \sum_{j=1}^N \Delta_j + \frac{1}{N} \sum_{1 \leq j < k \leq N} \gamma(x_j - x_k). \tag{46}$$

Here \mathcal{F}_{Nd} is given in (16) with d being replaced by Nd . In quantum mechanics, H_N formulates N -body problem (see, e.g, [14]).

The statement (44) claims that the ground state of the non-linear Shrödinger operator (also called Hartree operator)

$$H = \Delta + (\gamma * g^2)g$$

is approximated by the ground states of linear Shrödinger operators given in (45). The reason behind (44) is simple: Replacing \mathcal{F}_{Nd} by the sub-class of the functions of the form

$$\tilde{g}(x_1, \dots, x_N) = \prod_{j=1}^N g(x_j) \quad g \in \mathcal{F}_d \tag{47}$$

we have

$$\begin{aligned} & \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_{(\mathbb{R}^d)^N} \gamma(x_j - x_k) \tilde{g}^2(\tilde{x}) d\tilde{x} - \frac{1}{2} \int_{(\mathbb{R}^d)^N} |\nabla_N \tilde{g}(\tilde{x})|^2 d\tilde{x} \\ &= \frac{N-1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dx dy - \frac{N}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx. \end{aligned}$$

This leads to the lower bound for (44):

$$\liminf_{N \rightarrow \infty} \frac{\mathcal{E}(N)}{N} \geq \frac{1}{2} \mathcal{E}_H. \tag{48}$$

The above analysis shows that (44) rests on the fact that the maximizer for the variation in (45) is approximated in a suitable sense by the functions in the form given in (47) as $N \rightarrow \infty$.

Most of the earlier results (to which we refer the references cited in [16]) were for the ground state energy

$$\begin{aligned} \tilde{\mathcal{E}}^V(N) &= \inf_{g \in \mathcal{F}_{Nd}} \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_{\mathbb{R}^{Nd}} \gamma(x_j - x_k) g^2(\tilde{x}) d\tilde{x} \right. \\ &\quad \left. + \sum_{j=1}^N \int_{\mathbb{R}^{Nd}} V(x_j) g^2(\tilde{x}) d\tilde{x} + \frac{1}{2} \int_{\mathbb{R}^{Nd}} |\nabla g(\tilde{x})|^2 d\tilde{x} \right\} \end{aligned} \tag{49}$$

with the conclusion that

$$\lim_{N \rightarrow \infty} \frac{\tilde{\mathcal{E}}^V(N)}{N} = \frac{1}{2} \tilde{\mathcal{E}}_H^V, \tag{50}$$

where

$$\begin{aligned} \tilde{\mathcal{E}}_H^V &= \inf_{g \in \mathcal{F}_{Nd}} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dx dy + 2 \int_{\mathbb{R}^d} V(x) g^2(x) dx \right. \\ &\quad \left. + \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

In the case when $\gamma(\cdot) \equiv 0$, it is easy to see that the functions given in (47) maximize the variation $\tilde{\mathcal{E}}_H^V(N)$, which partially explains why we have (50). On the other hand, (50) does not give a clear picture on the role played by the function $\gamma(\cdot)$ in this “factorization” dynamics. One might take $V \equiv 0$ for observing the behavior of $\gamma(\cdot)$. In the case when $\lim_{|x| \rightarrow \infty} \gamma(x) = 0$, $\tilde{\mathcal{E}}_H^0 = 0$ and $\tilde{\mathcal{E}}^0(N) = 0$ for all $N \geq 1$, which is not necessarily the consequence of factorization, as one can make $\tilde{\mathcal{E}}^0(N) = 0$ by choosing “flat” functions $g(\tilde{x})$.

Therefore, it makes sense even only for the sake of understanding how the pair interaction function $\gamma(\cdot)$ response to factorization, to switch the sign of $\gamma(\cdot)$ in the variation $\tilde{\mathcal{E}}^0(N)$ so that it becomes the problem given in (45) where the two terms in the variation compete against each other and Hartree’s theory takes the form given in (44). On the other hand, note that, as remarked by Lewin, Nam and Rougerie

in [16, p. 579], “The validity of Hartree’s theory in this simple case ¹ is already a nontrivial problem and does not seem to have been proven before”. It appears that the paper [16] is the first work where Hartree’s theory mathematically includes the form given (44).

As an application of Theorem 1.1, we provide a probabilistic treatment to Hartree’s theory.

Theorem 4.1. *Under the same condition as Theorem 1.1, (44) holds.*

It should be pointed out that results obtained by Lewin, Nam and Rougerie [16] takes a form more general than (44). As for the assumptions, both the condition (13) and the condition posted in [16] are sufficient but “nearly necessary” for

$$\sup_{g \in \mathcal{F}_d} \left\{ \theta \int_{\mathbb{R}^d} \gamma(x)g^2(x)dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\} < \infty \quad \theta > 0$$

which ensures the finiteness of $\mathcal{E}(N)$. Our goal here is not to establish the most general form of Hartree’s theory but to show a probabilistic relevance to Hartree’s theory.

Proof of Theorem 4.1. In view of (48), all we need is to establish the upper bound

$$\limsup_{N \rightarrow \infty} \frac{\mathcal{E}(N)}{N} \leq \frac{1}{2} \mathcal{E}_H. \tag{51}$$

The idea is Feynman-Kac formula. To this end we work on the (Nd) -dimensional Brownian motion

$$\tilde{B}(s) = \left(B_1(s), \dots, B_N(s) \right) \quad s \geq 0.$$

We introduce the notation $\tilde{x} = (x_1, \dots, x_N)$ for $x_1, \dots, x_N \in \mathbb{R}^d$. We use “ $\mathbb{E}_{\tilde{x}}$ ” for the expectation associated with the Brownian motion $\tilde{B}(s)$ with $\tilde{B}(0) = \tilde{x}$. For fixed N , the transform

$$T_t g(\tilde{x}) = \mathbb{E}_{\tilde{x}} \left[\exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^t \gamma(B_j(s) - B_k(s)) ds \right\} g(\tilde{B}(t)) \right], \quad \tilde{x} \in \mathbb{R}^{Nd}$$

defines a semi-group of continuous linear operators on $\mathcal{L}^2(\mathbb{R}^{Nd})$ in the sense that $T_{s+t} = T_t \circ T_s$. Further, the semi-group $\{T_t; t \geq 0\}$ takes the Schrödinger operator H_N (given in (46)) as its infinitesimal generator and is formally written as $T_t = e^{tH_N}$. Here we mention the fact that H_N is initially a symmetric linear operator and can be extended into a self-adjointed operator by Friedrichs’s extension.

Let $\epsilon > 0$ be fixed and notice the fact that $\mathcal{E}(N) = \sup_{g \in \mathcal{F}_d} \langle g, H_N g \rangle$. For any $N \geq 1$, one can find $g_N \in \mathcal{F}_{Nd}$ such that g_N is locally supported, and $\langle g_N, H_N g_N \rangle > \mathcal{E}(N) - N\epsilon$. Let $R_N > 0$ be the radius of the (Nd) -dimensional ball which supports g_N and let $t_N \rightarrow \infty$ ($N \rightarrow \infty$) with sufficient increasing rate so that

$$\lim_{N \rightarrow \infty} \frac{1}{N t_N} \log \left(\|g_N\|_{\infty}^2 \omega_{Nd} R_N^{Nd} \right) = 0 \tag{52}$$

where ω_{Nd} is the volume of the (Nd) -dimensional unit ball.

¹refer to the case of finite spectrum measure $\mu(d\xi)$

By the moment comparison similar to (28), one can show that for any $\tilde{x} \in \mathbb{R}^{Nd}$

$$\begin{aligned} & \mathbb{E}_{\tilde{x}} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \\ & \leq \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \tag{53} \\ & \geq |B(0, R_N)|^{-1} \int_{B(0, R_N)} \mathbb{E}_{\tilde{x}} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \\ & \geq \|g_N\|_{\infty}^{-2} |B(0, R_N)|^{-1} \int_{\mathbb{R}^d} g_N(\tilde{x}) \\ & \quad \times \mathbb{E}_{\tilde{x}} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} g_N(B_{t_N}) \Big] d\tilde{x} \\ & = \|g_N\|_{\infty}^{-2} (\omega_{Nd} R_N^{Nd})^{-1} \langle g_N, T_{t_N} g_N \rangle. \end{aligned}$$

By the spectral representation

$$\langle g_N, T_{t_N} g_N \rangle = \int_{-\infty}^{\infty} \exp\{t_N \lambda\} \mu_{g_N}(d\lambda)$$

where the measure $\mu_{g_N}(d\lambda)$ satisfies $\mu_{g_N}(\mathbb{R}) = \|g_N\|_2^2 = 1$. In other words, μ_{g_N} is a probability measure on \mathbb{R} . By Jensen's inequality

$$\begin{aligned} \langle g_N, T_{t_N} g_N \rangle & \geq \exp \left\{ t_N \int_{-\infty}^{\infty} \lambda \mu_{g_N}(d\lambda) \right\} \\ & = \exp \left\{ t_N \langle g_N, H_N g_N \rangle \right\} \geq \exp \left\{ t_N (\mathcal{E}(N) - N\epsilon) \right\}. \end{aligned}$$

Combining this with (52) and (53),

$$\limsup_{N \rightarrow \infty} \frac{1}{N t_N} \log \mathbb{E} \exp \left\{ \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_0^{t_N} \gamma(B_j(s) - B_k(s)) ds \right\} \geq \limsup_{N \rightarrow \infty} \frac{\mathcal{E}(N)}{N} - \epsilon.$$

In view of Theorem 1.1, letting $\epsilon \rightarrow 0^+$ on the right hand side leads to (51). \square

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