Homework solution #1 Math523 Fall 2016

1.9. Partition generated σ-algebra.

(a). $\sigma(\mathcal{C}) = \{\phi, A, B, C, A\cup B, B\cup C, A\cup C, E\}$. One can see the part (b) for the proof, as the above class is given as the class of the countable unions made from A, B, C.

(b). Here I have a correction to make: $\sigma(\mathcal{C})$ is the class of the countable unions of the partition sets **together** with ϕ. .

Let $\mathcal{E}$ be the class of the countable unions of the sets from $\mathcal{E}$ together with ϕ. All we need is to show $\sigma(\mathcal{C}) = \mathcal{E}$.

First, $\mathcal{C}\subset\mathcal{E}$ since for any $A\in\mathcal{C}$, A can be viewed as a countable union of the repeating sequence $A, A, \cdots$—so $A\in\mathcal{E}$.

We claim that $\mathcal{E}$ is a σ-algebra. Write the partition sets as $\mathcal{C} = \{A_\alpha;\ \alpha\in I\}$. By assumption, the index set I is finite or countable.

First, $\phi\in\mathcal{E}$ and $E = \cup_{\alpha\in I}A_\alpha \in\mathcal{E}$.

Let $A\in\mathcal{E}$. Then A is either ϕ or $\cup_{\alpha\in J}A_\alpha$ for some non-empty $J\subset I$. Thus, A^c is either E or $\cup_{\alpha\in I\setminus J}A_\alpha$ (Here is the place we use the fact that $\{A_\alpha\}$ is a partition of E). Thus, $A^c\in\mathcal{E}$ in any case.

A countable union $\cup_n B_n$ of the sets $B_n\in\mathcal{E}$ is either ϕ if $B_n = \phi$ for all n, or a countable (or finite) union of the sets in $\mathcal{C}$ if $B_n\neq\phi$ for some n. In any case, $\cup_n B_n\in\mathcal{E}$.

Conclusion: $\mathcal{E}$ is a σ-algebra.

By definition, $\sigma(\mathcal{C})$ is the smallest σ-algebra that contains $\mathcal{C}$. Consequently, $\sigma(\mathcal{C})\subset\mathcal{E}$.

On the other hand, $\phi\in\sigma(\mathcal{C})$ by the definition of σ-algebra. The fact that $\mathcal{C}\subset\sigma(\mathcal{E})$ implies that any countable union of the partition sets is in $\sigma(\mathcal{E})$. Therefore, $\mathcal{E}\subset\sigma(\mathcal{C})$.

(c). The fundamental difference between (b) and (c) is: The partition by the singletons are not countable! In this set-up, $\mathcal{C}$ is the class of all singletons and we show that $\sigma(\mathcal{C})$ is the class consisting of the finite or countable sets and the sets whose complementaries are finite or countable. First, let $\mathcal{E}$ be this class. Clearly, $\mathcal{C}\subset\mathcal{E}$. It is not hard to show $\mathcal{E}$ is a σ-algebra. Therefore, $\sigma(\mathcal{C})\subset\mathcal{E}$. On the other hand, $\mathcal{C}\subset\sigma(\mathcal{C})$ implies that all finite or countable unions of the singletons are in $\sigma(\mathcal{C})$. So are their complementaries. Thus, $\mathcal{E}\subset\sigma(\mathcal{C})$.

2.20. To show that the inverse $f^{-1}(\mathcal{F})$ of a σ-algebra $\mathcal{F}$ is a σ-algebra, the key fact is that $f^{-1}(B^c) = f^{-1}(B)^c$ and $f^{-1}(\cup_n B_n) = \cup_n f^{-1}(B_n)$.

First, f: $E\longrightarrow F$ is measurable with respect to $f^{-1}(\mathcal{F})$. Indeed, for any $A\in\mathcal{F}$ there is $f^{-1}(A)\in f^{-1}\mathcal{F})$.

Second, $f^{-1}(\mathcal{F})$ is the smallest σ-algebra making f measurable: Let $\mathcal{E}$ be another σ-algebra on E such that f is $\mathcal{E}$-measurable. For any $A\in f^{-1}(\mathcal{F})$ there is a $B\in\mathcal{F}$ such that $A = f^{-1}(B)$. This implies that $A\in\mathcal{E}$. We have proved that $f^{-1}(\mathcal{F})\subset\mathcal{E}$.

2.32 By monotonicity, $f^{-1}(-\infty, y] = \{x;\ f(x) \leq y\}$ is an (possibly empty) interval. So it is in $\mathcal{B}_{\mathbf{R}}$.