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PRECISE MOMENT ASYMPTOTICS FOR THE STOCHASTIC HEAT EQUATION OF A TIME-DERIVATIVE GAUSSIAN NOISE*

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Abstract This article establishes the precise asymptotics

$$\mathbb{E}u^m(t,x) \quad (t\to\infty \quad \text{or} \quad m\to\infty)$$

for the stochastic heat equation

$$\frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\Delta u(t,x)+u(t,x)\frac{\partial W}{\partial t}(t,x)$$

with the time-derivative Gaussian noise $\frac{\partial W}{\partial t}(t,x)$ that is fractional in time and homogeneous in space.

Key words Stochastic heat equation; time-derivative Gaussian noise; Brownian motion; Feynman-Kac representation; Schilder's large deviation

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1 Introduction

Moment asymptotics for solutions to stochastic partial differential equations are known as the problem of intermittency that has been studied extensively in the past two decades [1, 8]. In this work, we investigate the asymptotics problem

$$\mathbb{E}u^m(t,x) \qquad (t\to\infty \ \ \text{or} \ \ m\to\infty)$$

for the stochastic heat equation

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x)=\dfrac{1}{2}\Delta u(t,x)+u(t,x)\dfrac{\partial W}{\partial t}(t,x) & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\\ u(0,x)=u_0(x)\end{cases}\tag{1.1}$$

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with the Gaussian noise $\frac{\partial}{\partial t}W(t,x)$ that is formally given as the time derivative of the mean-zero Gaussian field $\{W(t,x);\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\}$ with the covariance function

$$\mathrm{Cov}\Big(W(t,x),W(s,y)\Big)=\frac{1}{2}(t^{2H_0}+s^{2H_0}-|t-s|^{2H_0})\Gamma(x,y)\qquad (t,x),(s,y)\in\mathbb{R}^+\times\mathbb{R}^d,$$

where the time Hurst parameter $H_0\in(0,1)$ and we assume that the space covariance function $\Gamma(x,y)$ is locally bounded and has the homogeneity in the sense that

$$\begin{cases}\Gamma(Cx,Cx)=|C|^{2H}\Gamma(x,x)\\ \Gamma(x,x)+\Gamma(y,y)-2\Gamma(x,y)=\Gamma(x-y,x-y)\end{cases}\tag{1.2}$$

for any $x,y\in\mathbb{R}^d$ and $C\in\mathbb{R}$, where the constant $H\in(0,1)$. Assumption (1.2) can be restated as

$$\begin{cases}\big\{W(c_0t,cx);\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\big\}\overset{d}{=}\big\{c_0^{H_0}c^HW(t,x);\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\big\}\\ W(t,x)-W(s,y)\overset{d}{=}W(t-s,x-y)\qquad (c_0,c>0,\ (t,x),(s,y)\in\mathbb{R}^+\times\mathbb{R}^d).\end{cases}\tag{1.3}$$

For simplicity, we assume that bounded initial condition is as follows:

$$0<\inf_{x\in\mathbb{R}^d}u_0(x)\le\sup_{x\in\mathbb{R}^d}u_0(x)<\infty.\tag{1.4}$$

Mathematically, $\frac{\partial}{\partial t}W(t,x)$ is defined as a generalized centered Gaussian field with

$$\mathrm{Cov}\Big(\frac{\partial}{\partial t}W(t,x),\frac{\partial}{\partial s}W(s,y)\Big)=\gamma_0(t-s)\Gamma(x,y)\qquad (t,x),(s,y)\in\mathbb{R}^+\times\mathbb{R}^d.\tag{1.5}$$

Here, the time-covariance $\gamma_0(t-s)$ is morally considered as the derivative

$$\frac{\partial^2}{\partial t\partial s}\Big\{\frac{1}{2}(t^{2H_0}+s^{2H_0}-|t-s|^{2H_0})\Big\}.$$

In particular,

$$\gamma_0(t-s)=\begin{cases}H_0(2H_0-1)|t-s|^{-(2-2H_0)} & H_0>1/2\\ \delta_0(t-s) & H_0=1/2.\end{cases}\tag{1.6}$$

The function $\gamma_0(\cdot)$ defined in (1.6) is qualified as covariance function as it is non-negative definite. Indeed, it can be shown that

$$\gamma_0(u)=\frac{\Gamma(2H_0+1)\sin(\pi H_0)}{2\pi}\int_{\mathbb{R}}e^{\mathrm{i}\lambda u}|\lambda|^{1-2H_0}\mathrm{d}\lambda\qquad u\in\mathbb{R}.\tag{1.7}$$

When $H_0<1/2$, the function $|\cdot|^{-(2-2H_0)}$ is no longer non-negative definite and is not qualified for being a covariance function. As consequence, the covariance function $\gamma_0(\cdot)$ can not be legally defined by (1.6) when $H_0<1/2$. As $H_0<1/2$, the function $\gamma_0(\cdot)$ is defined as a generalized function given in (1.7). It should be emphasized that $\gamma_0(\cdot)$ is not defined point-wise when $H_0<1/2$.

Under suitable conditions (such as the one assumed in our main theorem), the solution to (1.1) yields the following Feynman-Kac formula:

$$u(t,x)=\mathbb{E}_x\bigg[u_0(B_t)\exp\bigg\{\int_0^tW(\mathrm{d}s,B_{t-s})\bigg\}\bigg]\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\tag{1.8}$$

where $\{B_t;\ t\ge 0\}$ is a d-dimensional Brownian motion independent of $\{W(t,x);\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\}$ with $B_0=x$, and "$\mathbb{E}_x$" stands for the Brownian expectation. We point to [5] and [6] for existing literature.