Asymptotics of negative exponential moments for annealed Brownian motion in a renormalized Poisson potential

Xia Chen*, Alexey M. Kulik†

April 6, 2011

Abstract

In [14], a method of renormalization was proposed for constructing some more physically realistic random potentials in a Poisson cloud. This paper is devoted to the detailed analysis of the asymptotic behavior of the annealed negative exponential moments for the Brownian motion in a renormalized Poisson potential. The main results of the paper are applied to studying the Lifshitz tails asymptotics of the integrated density of states for random Schrödinger operators with their potential terms represented by renormalized Poisson potentials.

Key-words: renormalization, Poisson field, Brownian motion in a renormalized Poisson potential, parabolic Anderson model, random Schrödinger operator, integrated density of states, Lifshitz tails asymptotics.

AMS subject classification (2010): 60J45, 60J65, 60K37, 60K37, 60G55.

1 Introduction

This paper is motivated by the model of Brownian motion in Poisson potential, which describes how a Brownian particle survives from being trapped by the Poisson obstacles. We recall briefly the general set-up of that model, referring the reader to the book by Sznitman [66] for a systematic representation, to [47] for a survey, and to [6], [7], [33], [57] for specific topics and for recent development on this subject.

Let $\omega(dx)$ be a Poisson field in $\mathbb{R}^d$ with intensity measure $\nu\, dx$, and let B be an independent Brownian motion in $\mathbb{R}^d$. Throughout, $\mathbb{P}$ and $\mathbb{E}$ denote the probability law and the expectation, respectively, generated by the Poisson field $\omega(dx)$, while $\mathbb{P}_x$ and $\mathbb{E}_x$ denote the

*Research partially supported by NSF grant #DMS-0704024.

†Research partially supported by Ukrainian Nat. Acad. of Sciences, grant #0107U002029.

probability law and the expectation, respectively, generated by the Brownian motion B with $B_0 = x$. For a properly chosen (say, continuous and compactly supported) non-negative function K on $\mathbb{R}^d$ (known as a shape function), define the respective random function (known as a Poisson potential)

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy), \tag{1.1}$$

which heuristically represents the net force at $x \in \mathbb{R}^d$ generated by the Poisson obstacles. The model of Brownian motion in a Poisson potential is defined in two different settings. In the quenched setting, the set-up is conditioned on the random environment created by the Poisson obstacles, and the model is described in the terms of the Gibbs measure $\mu_{t,\omega}$ defined by

$$\frac{d\mu_{t,\omega}}{d\mathbb{P}_0} = \frac{1}{Z_{t,\omega}} \exp\left\{ -\int_0^t V(B_{\kappa s})ds \right\}, \quad Z_{t,\omega} = \mathbb{E}_0 \exp\left\{ -\int_0^t V(B_{\kappa s})ds \right\}. \tag{1.2}$$

Here κ is a positive parameter, responsible for the time scaling $s \mapsto \kappa s$, introduced here for further references convenience. In the annealed setting, the model averages on both the Brownian motion and the environment, and respective Gibbs measure μ_t is defined by

$$\frac{d\mu_t}{d(\mathbb{P}\otimes\mathbb{P}_0)} = \frac{1}{Z_t} \exp\left\{ -\int_0^t V(B_{\kappa s})ds \right\}, \quad Z_t = \mathbb{E}\otimes\mathbb{E}_0 \exp\left\{ -\int_0^t V(B_{\kappa s})ds \right\}. \tag{1.3}$$

Heuristically, the integral

$$\int_0^t V(B_{\kappa s})ds \tag{1.4}$$

measures the total net attraction to which the Brownian particle is subject up to the time t, and henceforth, under the law $\mu_{t,\omega}$ or μ_t, the Brownian paths heavily impacted by the Poisson obstacles are penalized and become less likely.

In the Sznitman's model of "soft obstacles", the shape function K is assumed to be locally bounded and compactly supported. However, these limitations may appear to be too restrictive in certain cases. Important particular choice of a shape function, physically motivated by the Newton's law of universal attraction, is

$$K(x) = \theta|x|^{-p}, \qquad x \in \mathbb{R}^d, \tag{1.5}$$

which clearly is both locally unbounded and supported by whole $\mathbb{R}^d$. This discrepancy is not just a formal one, and brings serious problems. For instance, under the choice (1.5), the integral (1.1) blows up at every $x \in \mathbb{R}^d$ when $p \leq d$.

To resolve such a discrepancy, in a recent paper ([14]) it was proposed to consider, apart with a Poisson potential (1.1), a renormalized Poisson potential

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(y-x)[\omega(dy) - \nu dy]. \tag{1.6}$$

Assume for a while that K is locally bounded and compactly supported. Then

$$\begin{aligned}\overline{V}(x) &= \int_{\mathbb{R}^d} K(y-x)[\omega(dy) - \nu dy] = \int_{\mathbb{R}^d} K(y-x)\omega(dy) - \nu \int_{\mathbb{R}^d} K(y-x)\, dy \\ &= V(x) - \nu \int_{\mathbb{R}^d} K(y)\, dy,\end{aligned}$$

that is $\overline{V} - V =$const. Consequently, replacing V by $\overline{V}$ in (1.2) and (1.3) does not change the measures $\mu_{t,\omega}$ and μ_t because both the exponents therein and the normalizers $Z_{t,\omega}$ and Z_t are multiplied by the same constant $e^{t\mathbb{E}V(0)}$ (this is where the word "renormalization" comes from). On the other hand, for unbounded and not locally supported K, the renormalized potential (1.6) may be well defined while the potential (1.1) blows up. The most important example here is the shape function (1.5) under the assumption $d/2 < p < d$. In that case $\overline{V}$ is well defined, as well as the Gibbs measures

$$\frac{d\overline{\mu}_{t,\omega}}{d\mathbb{P}_0} = \frac{1}{\overline{Z}_{t,\omega}} \exp\Big\{ - \int_0^t \overline{V}(B_{\kappa s}) ds\Big\}, \quad \overline{Z}_{t,\omega} = \mathbb{E}_0 \exp\Big\{ - \int_0^t \overline{V}(B_{\kappa s}) ds\Big\}, \tag{1.7}$$

$$\frac{d\overline{\mu}_{t}}{d(\mathbb{P}\otimes\mathbb{P}_0)} = \frac{1}{\overline{Z}_{t}} \exp\Big\{ - \int_0^t \overline{V}(B_{\kappa s}) ds\Big\}, \quad \overline{Z}_{t} = \mathbb{E}\otimes\mathbb{E}_0 \exp\Big\{ - \int_0^t \overline{V}(B_{\kappa s}) ds\Big\}, \tag{1.8}$$

see Corollary 1.3 [14]. We use separate notation $\overline{\mu}_{t,\omega}, \overline{\mu}_t$ because the Gibbs measures (1.2) and (1.3) are not well defined now.

The above exposition shows that, using the notion of the renormalized Poisson potential, one can extend the class of the shape functions significantly. Note that, in general, the domain of definition for (1.6) does not include the one for (1.1). For instance, for the shape function (1.5) the potential V and the renormalized potential $\overline{V}$ are well defined under the mutually excluding assumptions $p > d$ and $d/2 < p < d$, respectively. This, in particular, does not give one a possibility to define respective Gibbs measures in a uniform way. This inconvenience is resolved in the terms of the Poisson potential V^h, partially renormalized at the level h, see Chapter 6 [14]. By definition,

$$V^h(x) = \int_{\mathbb{R}^d} \Big(K(y-x) - h\Big)_+ \omega(dy) + \int_{\mathbb{R}^d} \Big(K(y-x)\wedge h\Big)[\omega(dy) - \nu dy], \tag{1.9}$$

where $h \in [0, \infty]$ is a renormalization level. Clearly, $V^0 = V, V^\infty = \overline{V}$. It is known (see Chapter 6 [14]) that V^h is well defined for every $h \in (0, +\infty)$ as soon as $V^{h'}$ is well defined for some $h' \in [0, +\infty]$, and in that case there exists a constant $C_{K,h,h'}$ such that $V^h - V^{h'} = \nu C_{K,h,h'}$. This makes it possible to define the respective Gibbs measures in a uniform way, replacing V in (1.2), (1.3) by V^h with (any) $h \in (0, +\infty)$. In addition, such a definition extends the class of shape functions: for K given by (1.5), V^h with $h \in (0, +\infty)$ is well defined for $p > d/2$.

The main objective of this paper is to study the asymptotic behavior, as $t \to +\infty$, of the annealed exponential moments

$$\mathbb{E}\otimes\mathbb{E}_0 \exp\Big\{ - \frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s}) ds\Big\}. \tag{1.10}$$

This problem is clearly relevant with the model discussed above: in the particular case $\kappa = 1, \alpha_t \equiv 1$, this is just the natural question about the limit behavior of the normalizer $\overline{Z}_t$ in the formula (1.3) for the annealed Gibbs measure. In the quenched setting, similar problem was studied in the recent paper ([13]). In some cases, we also consider (1.10) with a renormalized Poisson potential $\overline{V}$ replaced by either a Poisson potential V or a partially renormalized potential V^h with $h \in (0, +\infty)$.

The function α_t in (1.10) appears, on one hand, because of our further intent to study in further publications the a.s. behavior

$$\int_0^t \overline{V}(B_{\kappa s})ds, \quad t \to \infty.$$

On the other hand, this function can be naturally included into the initial model. One can think about making penalty (1.4) to be additionally dependent on the length of the time interval by dividing the total net attraction for the Brownian particle by some scaling parameter. Because of this interpretation, further on we call the function α_t a "scale".

Let us discuss two other mathematically related problems, studied extensively both in mathematical and in physical literature. The first one is known as the continuous parabolic Anderson model

$$\begin{cases} \partial_t u(t,x) = \kappa \Delta u(t,x) \pm Q(x) u(t,x), \\ \\ u(0,x) = 1, \quad x \in \mathbb{R}^d. \end{cases} \tag{1.11}$$

This problem appears in the context of chemical kinetics and population dynamics. Its name goes back to the work by Anderson [4] on entrapment of electrons in crystals with impurities. In the existing literature, the random field Q is usually chosen as the Poisson potential V, with the shape function K assumed to be bounded (and often locally supported), so that the potential function (1.1) can be defined. A localized shape is analogous to the usual set-up in the discrete parabolic Anderson model, where the potential $\{Q(x); x \in \mathbb{Z}^d\}$ is an i.i.d. sequence; we refer the reader to the monograph [10] by Carmona and Molchanov for the overview and background of this subject.

On the other hand, there are practical needs for considering the shape functions of the type (1.5), which means that the environment has both a long range dependency and extreme force surges at the locations of the Poison obstacles. To that end we consider (1.11) with a renormalized Poisson potential $\overline{V}$ instead of Q. Note that, in that case, the field Q represents fluctuations of the environment along its "mean field value" rather than the environment itself, although this "mean field value" may be infinite.

It is well known that (1.11) is solved by the following Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x \exp\left\{ \pm \int_0^t Q(B_{2\kappa s}) ds \right\} \tag{1.12}$$

when Q is Hölder continuous and satisfies proper growth bounds. When $Q = \overline{V}$ with K from (1.5), local unboundedness of K induces local irregularity of Q (Proposition 2.9 in [14]), which does not allow one to expect that the function (1.12) solves (1.11) in the strong sense.

However, it is known (Proposition 1.2 and Proposition 1.6 in [14]) that under appropriate conditions the function (1.12) solves (1.11) in the mild sense. It is a local unboundedness of K again, that brings a serious asymmetry to the model, making essentially different the cases "+" and "−" of the sign in the right hand sides of (1.11) and (1.12). For the sign "−", the random field (1.12) is well defined and integrable for $d/2 < p < d$ (Theorem 1.1 in [14]). For the sign "+", the random field (1.12) is not integrable for any p. On the other hand, the random field (1.12) is well defined for $d/2 < p < \min(2, d)$ (Theorem 1.4 and Theorem 1.5 in [14]).

In view of (1.12), our main problem relates immediately to the asymptotic behavior of the moments of the solution to the parabolic Anderson problem (1.11) with the sign "−". Here we cite [8], [10], [11], [17], [18], [27], [29], [30], [31], [32], [65] as a partial list of the publications that deal with various asymptotic topics related to the parabolic Anderson model.

Another problem related to our main one is the so called *Lifshitz tails* asymptotic behavior of the *integrated density of states* function N of a random Schrödinger operator of the type

$$H = -\frac{\kappa}{2}\Delta + Q. \tag{1.13}$$

This function, written IDS in the sequel, is a deterministic spectral mean-field characteristic of H. Under quite general assumptions on the random potential Q, it is well defined as

$$N(\lambda) = \lim_{U\uparrow\mathbb{R}^d} \frac{1}{|U|}\sum_k \mathbf{1}_{\lambda_{k,U}\le\lambda},$$

where $\{\lambda_{k,U}\}$ is the set of eigenvalues for the operator H in a cube U with the Dirichlet boundary conditions, $|U|$ denotes the Lebesgue measure of U in $\mathbb{R}^d$, and the limit pass is made w.r.t. a sequence of cubes which has same center and extends to the whole $\mathbb{R}^d$. The classic references for the definition of the IDS function are [54] and [38], see also a brief exposition in Sections 2 and 5.1 below.

Heuristically, the bottom (that is, the left-hand side) λ_0 of the spectrum of H mainly describes the low-temperature dynamics for a system defined by the Hamiltonian (1.13). This motivates the problem of asymptotic behavior of $\log N(\lambda), \lambda \searrow \lambda_0$, studied extensively in the literature. The name of the problem goes back to the papers by Lifshitz [49], [50], we also give [22], [23], [28], [35], [37], [38], [40], [39], [41], [42], [43], [44], [45], [46], [48], [52], [53], [54], [55], [56], [63], [64], [66] as a partial list of references on the subject.

Connection between the Lifshitz tails asymptotics for the IDS function N and the problem discussed above is provided by the representation for the Laplace transform of N:

$$\int_{\mathbb{R}} e^{-\lambda t} dN(\lambda) = (2\pi\kappa t)^{-\frac{d}{2}} \mathbb{E}\otimes\mathbb{E}_{0,0}^{\kappa t} \exp\left[-\int_0^t Q(B_{\kappa s})\, ds\right], \quad t \ge 0.$$

Here $\mathbb{E}_{0,0}^{\kappa t}$ denotes the distribution of the Brownian bridge, i.e. the Brownian motion conditioned by $B_{\kappa t} = 0$. Our estimates for (1.10) appear to be process insensitive to some extent, and remain true with $\mathbb{E}_0$ in (1.10) replaced by $\mathbb{E}_{0,0}^{\kappa t}$. This, via appropriate Tauberian theorem, provides information on Lifshitz tail asymptotics for the respective IDS function

N. Note that, in this case, the asymptotic behavior of the $\log N(\lambda)$ as $\lambda \to -\infty$ should be studied because the bottom of the spectrum is equal $\lambda_0 = -\infty$, unlike the (usual) Poisson case where $\lambda_0 = 0$. This difference is caused by the renormalization procedure, which brings the negative part to the potential.

We now outline the rest of the paper. The main results about negative exponential moments for annealed Brownian motion in a renormalized Poisson potential are collected in Theorem 2.1. They are formulated for the shape function defined by (1.5). Depending on p in this definition, we separate three cases

$$\alpha_t = o(t^{\frac{d+2-p}{d+2}}), \quad t \to \infty; \tag{1.14}$$

$$t^{\frac{d+2-p}{d+2}} = o(\alpha_t), \quad t \to \infty; \tag{1.15}$$

$$\alpha_t \sim \alpha t^{\frac{d+2-p}{d+2}} \quad \text{with some} \quad \alpha > 0, \quad t \to \infty, \tag{1.16}$$

calling them a "light scale", a "heavy scale", and a "critical" case, respectively. There is a close analogy between our "light" vs. "heavy" scale classification for a renormalized Poisson potential, and the well known "classic" vs. "quantum" regime classification for a (usual) Poisson potential; see detailed discussion in Section 2.

In all three cases listed above, our approach relies on the identity

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\, ds\right] = \mathbb{E}_0 \exp\left[\nu \int_{\mathbb{R}^d} \psi\left(\frac{1}{\alpha_t}\xi(t,x)\right)\, dx\right] \tag{1.17}$$

with

$$\psi(u) = e^{-u} - 1 + u, \tag{1.18}$$

$$\xi(t,x) = \int_0^t K(B_{\kappa s} - x)\, ds; \tag{1.19}$$

see Proposition 2.7 and Proposition 3.1 in [14].

Further analysis of the Wiener integral in the r.h.s. of (1.17) in the light scale case is quite straightforward. First, the upper bound follows from Jensen's inequality and is "universal" in the sense that the Brownian motion B therein can be replaced by an arbitrary process. Then we choose a ball in the Wiener space, which simultaneously is "sufficiently heavy" in probability and "sufficiently small" in size. This smallness allows one to transform the integral in the r.h.s. of (1.17) into

$$\nu \int_{\mathbb{R}^d} \psi\left(\frac{1}{\alpha_t}\int_0^t K(-x)\, ds\right) dx = \nu \int_{\mathbb{R}^d} \psi\left(\frac{t}{\alpha_t}K(-x)\right) dx,$$

which after a straightforward transformation gives a lower bound that coincides with the universal upper bound obtained before.

We call this approach the "small heavy ball method". It is quite flexible, and by means of this method we also give a complete description of the light scale asymptotic behavior for a Poisson potential V and a partially renormalized Poisson potential V^h (Theorem 2.4). This method differs from the functional methods, typical in the field, which go back to the paper [55] by Pastur. It gives a new and transparent principle explaining the transition from quantum to classical regime; note that the phenomenology of such a transition is a problem discussed in the literature intensively, see Section 3.5, [41] for a detailed overview. In the context of the small heavy ball method, we can identify the classic regime with the situation where a sufficient amount of Brownian paths stay in a suitable neighborhood. So the relation $\overline{V}(B_{\kappa t}) \approx \overline{V}(0)$ donimates in this regime.

In the quantum regime, i.e. in the critical and the heavy scale cases, the contribution of Brownian paths can not be neglected. In this situation, the key role in our analysis of the Wiener integral in the r.h.s. of (1.17) is played by a large deviations result (Theorem 4.1) formulated and proved in Section 4. In the same section, by means of appropriate rescaling procedure, the asymptotics of the Wiener integral in the r.h.s. of (1.17) in the quantum regime is obtained. In the heavy scale case, this asymptotics appears to be closely related to the large deviations asymptotics for a Brownian motion in a Wiener sheet potential, studied in ([15]); we discuss this relation in Section 4.4.

Finally, we discuss an application of the main results of the paper to the Lifshitz tails asymptotics of the integrated density of states functions for random Schrödinger operators, with their potential terms represented by either renormalized Poisson potential or partially renormalized Poisson potential.

2 Main results

Throughout the paper, ω_d denotes the volume of the d-dimensional unit ball. We denote

$$\mathcal{F}_d = \left\{ g \in W_2^1(\mathbb{R}^d) : \quad \int_{\mathbb{R}^d} g^2(x)dx = 1 \right\};$$

where $W_2^1(\mathbb{R}^d)$ is used for the Sobolev space of functions that belong to $\mathcal{L}_2$ together with their first order derivatives. We also denote

$$\varphi(u) = 1 - e^{-u}, \quad \Xi(u,v) = \psi(u) - e^{-u}\varphi(v) = e^{-u-v} - 1 + u, \quad u, v \in \mathbb{R}$$

(ψ is introduced in (1.18)). Clearly, the functions ψ, $-\varphi$, and Ξ are convex; this simple observation is crucial for the most constructions below.

Our main results about the asymptotics of negative exponential moments for annealed Brownian motion in a renormalized Poisson potential are represented by the following theorem.

Theorem 2.1 *Let $p \in (d/2, d)$.*

I. In the "light scale" case,

$$\begin{aligned} \lim_{t\to\infty} \left(\frac{\alpha_t}{t}\right)^{d/p} \log \mathbb{E} \otimes \mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t} \int_0^t \overline{V}(B_{\kappa s})\, ds \right] &= \nu \int_{\mathbb{R}^d} \psi(\theta |x|^{-p})\, dx \\ &= \nu \omega_d \theta^{d/p} \left(\frac{p}{d-p}\right) \Gamma\left(\frac{2p-d}{p}\right) = -\nu\omega_d\theta^{d/p}\Gamma\left(\frac{p-d}{p}\right). \end{aligned} \tag{2.1}$$

II. In the "critical" case,

$$\begin{aligned}\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\,ds\right] \\ = \sup_{g\in\mathcal{F}_d}\left\{\nu\int_{\mathbb{R}^d}\psi\left(\frac{\theta}{\alpha}\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}.\end{aligned} \tag{2.2}$$

III. In the "heavy scale" case, under additional assumption $p < (d+2)/2$,

$$\begin{aligned}\lim_{t\to\infty} \alpha_t^{\frac{4}{d+2-2p}} t^{-\frac{d+4-2p}{d+2-2p}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\,ds\right] \\ = \sup_{g\in\mathcal{F}_d}\left\{\frac{\nu\theta^2}{2}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)^2 dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}.\end{aligned} \tag{2.3}$$

Remark 2.2 *The additional assumption $p < (d+2)/2$ in statement III is exactly the condition for $\xi(t,x)$ to be square integrable (see [15]), and henceforth for respective central limit theorem to hold true, see Proposition 4.4 and discussion in Section 4.4 below.*

Let us discuss this theorem in comparison with the following, well known in the field, results for annealed Brownian motion in a Poisson potential.

Theorem 2.3 *Let K be bounded and satisfy*

$$K(x) \sim \theta|x|^{-p}, \quad |x|\to\infty \tag{2.4}$$

with $p > d$.

I. ([55]) If $p\in(d,d+2)$,

$$\lim_{t\to\infty} t^{-d/p}\log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\int_0^t V(B_{\kappa s})\,ds\right] = -\nu\omega_d\theta^{d/p}\Gamma\left(\frac{p-d}{p}\right). \tag{2.5}$$

II. ([53]) If $p = d+2$,

$$\begin{aligned}\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\int_0^t V(B_{\kappa s})\,ds\right] \\ = -\inf_{g\in\mathcal{F}_d}\left\{\nu\int_{\mathbb{R}^d}\varphi\left(\theta\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx + \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}.\end{aligned} \tag{2.6}$$

III. ([20]) If $p > d+2$,

$$\begin{aligned}\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\int_0^t V(B_{\kappa s})\,ds\right] \\ = -\inf_{g\in\mathcal{F}_d}\left\{\nu\int_{\mathbb{R}^d}\mathbf{1}_{g(x)>0}\,dx + \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}.\end{aligned} \tag{2.7}$$

It is an effect, discovered by L.Pastur in [55], that the asymptotic behavior of the Brownian motion in a Poisson potential is essentially different in the cases $p > d+2$ and $p \in (d, d+2)$, called frequently "light tailed" and "heavy tailed", respectively. This difference was discussed intensively in the literature, especially in the connection with the asymptotic behavior of respective IDS function. The main asymptotic term in (2.5) is completely determined by the potential, and does not involve κ, i.e. the "intensity" of the Brownian motion. On the other hand, (2.7) depends on κ but not on the shape function K. Since K and κ, heuristically, are related to "regular" and "chaotic" parts of the dynamics, an alternative terminology "classic regime" ($p > d+2$) and "quantum regime" ($p \in (d, d+2)$) is frequently used.

Theorem 2.1 shows that the dichotomy "classic vs. quantum regimes" is still in force for the model with a renormalized Poisson potential, with conditions on the shape function K to be either heavy or light tailed replaced by conditions on the scale α_t to be respectively light or heavy. Note that, for $\alpha_t \equiv 1$, (1.14) and (1.15) transform exactly to $p < d+2$ and $p > d+2$, respectively. In the classic regime, an analogy between a Poisson potential and a renormalized Poisson potential is very close: for $\alpha_t \equiv 1$, (2.1) and (2.5) coincide completely. However, in the quantum regime the right hand side in (2.3), although being principally different from (2.1), is both scale dependent (i.e. involves α_t) and shape dependent (i.e. involves p).

It is a natural question whether Theorem 2.1 can be extended to other types of potentials, like a Poisson potential V or a partially renormalized Poisson potential V^h. We strongly believe that such an extension is possible in a whole generality; however, we can not give such an extension in the quantum regime (i.e. critical and heavy scaled cases) so far, because we do not have an analogue of Theorem 4.1 for functions v which are convex, but are not increasing (like $-\varphi$ and Ξ). Such a generalization is a subject for further research.

In the classic regime (i.e. light scale case), such an extension can be made efficiently. Moreover, in this case the assumptions on the shape function K can be made very mild: instead of (1.5), we assume (2.4) with $p > d/2$ and, when $p < d$,

$$\int_{\mathbb{R}^d} \psi(K(x))\, dx < +\infty, \tag{2.8}$$

which is just the assumption for $\overline{V}$ to be well defined.

Theorem 2.4 *Let the shape function K satisfy (2.4) and scale function α_t satisfy (1.14).*

I. Statement I of Theorem 2.1 holds true assuming K satisfies (2.8).

II. For $p > d$,

$$\begin{aligned}\lim_{t\to\infty} \left(\frac{\alpha_t}{t}\right)^{d/p} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t V(B_{\kappa s})\, ds\right] \\ = -\nu \int_{\mathbb{R}^d} \varphi(\theta|x|^{-p})\, dx = -\nu\omega_d \theta^{d/p}\Gamma\left(\frac{p-d}{p}\right).\end{aligned} \tag{2.9}$$

III. For $p = d$ and $h > 0$,

$$\begin{aligned}\log\mathbb{E}\otimes\mathbb{E}_0 \exp&\left[-\frac{1}{\alpha_t}\int_0^t V^h(B_{\kappa s})\,ds\right]\\ &= \nu\left[\int_{\mathbb{R}^d}\Big(\min(K(y),h)-\alpha_t/t\Big)_+ dy + \omega_d\theta\,\mathrm{Eu}\right]\left(\frac{t}{\alpha_t}\right)+o\left(\frac{t}{\alpha_t}\right), \quad t\to\infty,\end{aligned} \tag{2.10}$$

where $\mathrm{Eu} = -\Gamma'(1) = 0,57721\ldots$ *is the Euler constant. In particular, when K has the form (1.5),*

$$\begin{aligned}\log\mathbb{E}\otimes\mathbb{E}_0 \exp&\left[-\frac{1}{\alpha_t}\int_0^t V^h(B_{\kappa s})\,ds\right]\\ &= \nu\omega_d\theta\left[\log\left(\frac{t}{\alpha_t}\right)+\log h+\mathrm{Eu}\right]\left(\frac{t}{\alpha_t}\right)+o\left(\frac{t}{\alpha_t}\right), \quad t\to\infty.\end{aligned} \tag{2.11}$$

The following theorem shows that statements of Theorem 2.1 and Theorem 2.4 are process insensitive, to some extent.

Theorem 2.5 *Relations (4.4), (2.1) – (2.3), (2.9), (2.10), and (2.11) hold true with $\mathbb{E}_0$ replaced by $\mathbb{E}^{\kappa t}_{0,0}$, i.e. the expectation w.r.t. the law of the Brownian bridge.*

This theorem makes it possible to investigate the Lifshitz tails asymptotics for the integrated density of states of the random Schrödinger operators with (partially) renormalized Poisson potentials. Let us outline the construction of respective objects.

For a given random field $Q(x), x \in \mathbb{R}^d$ and a cube $U \subset \mathbb{R}^d$, denote by H^Q_U the random Schrödinger operator in U with the potential Q and the Dirichlet boundary conditions:

$$H^Q_U f = -\frac{\kappa}{2}\Delta f + Qf, \quad f|_{\partial U} = 0. \tag{2.12}$$

When the field Q is assumed to have locally bounded realizations, the operator H^Q_U is a.s. well defined as an operator on $L_2(U, dx)$ and is self-adjoint. In addition, respective semigroup $R^Q_{t,U} = e^{-tH^Q_U}, t \ge 0$ has a Feynman-Kac representation ([66], p.13):

$$R^Q_{t,U} f(x) = \mathbb{E}^t_x\left(\exp\left[-\int_0^t Q(B_{\kappa s})\,ds\right]\chi_{U,t}(B_\cdot)f(B_t)\right), \quad x \in U, \quad t \ge 0, \tag{2.13}$$

where

$$\chi_{U,t}(B_\cdot) = \mathbf{1}_{B_{\kappa s}\in U, s\in[0,t]}.$$

For general Q, we define H^Q_U by the following limit procedure. Consider truncations $Q_N = (|Q| \wedge N)\mathrm{sgn}\,Q$. Under appropriate assumptions on Q, for almost every its realization operators $R^{Q_N}_{t,U}$ converge strongly for every $t \ge 0$ as $N \to \infty$. In that case, H^Q_U is defined as the generator of the limit semigroup $R^Q_{t,U}, t \ge 0$. Assuming the spectrum of H^Q_U to be discrete (we verify this assumption below), we denote this spectrum $\{\lambda^Q_{k,U}\}$ and define the function

$$N^Q_U(\lambda) = \frac{1}{|U|}\sum_k \mathbf{1}_{\lambda^Q_{k,U}\le\lambda}, \quad \lambda \in \mathbb{R}. \tag{2.14}$$

Proposition 2.6 *Let the shape function K be such that, for some $g > 0$ the following conditions hold:*

(i) $K^g(x) = (K(x) - g)_+$ is compactly supported;

(ii) $K_g = \min(K(x), g)$ is Lipschitz continuous and belongs to the Sobolev space $W_2^1(\mathbb{R}^d)$.

Consider either a partially renormalized potential $Q = V^h$ with $h \in (0, \infty)$, or a renormalized potential $Q = \overline{V}$, in the latter case assuming additionally (2.8).

Then

(a) for a.s. realization of the potential Q and every cube U, the described above procedure well defines both the random Schrödinger operator H_U^Q and respective function N_U^Q;

(b) there exists an integrated density of states *N^Q; that is, a deterministic monotonous function such that*

$$N^Q(\lambda) = \lim_{U \uparrow \mathbb{R}^d} N_U^Q(\lambda)$$

a.s. for every point of continuity of N^Q. Respective Laplace transform has the representation

$$\int_{\mathbb{R}} e^{-\lambda t} dN^Q(\lambda) = (2\pi\kappa t)^{-\frac{d}{2}} \mathbb{E} \otimes \mathbb{E}_{0,0}^{\kappa t} \exp\left[- \int_0^t Q(B_{\kappa s})\, ds \right], \quad t \geq 0. \tag{2.15}$$

Note that, in the proof of Proposition 2.6 (Section 5.1 below), most difficulties are concerned with the statement (a), because of local irregularity of the potential Q (Proposition 2.9 in [14]).

As a corollary of Theorem 2.5 and representation (2.15), we deduce the following Lifshitz tails asymptotics for random Schrödinger operators with random potentials $\overline{V}$ and V^h.

Theorem 2.7 *Let K satisfy (2.4).*

I. For $p \in (d/2, d)$, assuming additionally (2.8), we have in limit $\lambda \to -\infty$

$$\log N^{\overline{V}}(\lambda) = -\left[\nu\omega_d \Gamma\left(\frac{2p-d}{p} \right) \right]^{-\frac{p}{d-p}} \left(\frac{\theta(d-p)}{d} \right)^{\frac{d}{d-p}} \Big(-\lambda \Big)^{\frac{d}{d-p}} \Big(1 + o(1) \Big). \tag{2.16}$$

II. For $p = d$ and $h \in (0, \infty)$, we have in limit $\lambda \to -\infty$

$$\begin{aligned} \log N^{V^h}(\lambda) &= -\nu\omega_d\theta \exp\left[-\frac{\lambda}{\nu\omega_d\theta} - \log h - \mathrm{Eu} - 1 \right] \Big(1 + o(1) \Big) \\ &= -\frac{\nu\omega_d\theta}{h} \exp\left[-\frac{\lambda}{\nu\omega_d\theta} - \mathrm{Eu} - 1 \right] \Big(1 + o(1) \Big). \end{aligned} \tag{2.17}$$

Theorem 2.7 involves the asymptotic results for exponential moments (Theorem 2.5) only in a partial form, for the trivial scale function $\alpha_t \equiv 1$. This observation naturally motivates the following extension of the definition of the IDS function and respective generalization of Theorem 2.7.

Consider the family of random Schrödinger operators

$$H_\gamma = -\frac{\kappa}{2}\Delta + \gamma Q, \quad \gamma > 0. \tag{2.18}$$

Assuming every potential $Q_\gamma = \gamma Q$ being such that respective IDS function N^{Q_γ} is well defined, denote $N^Q(\lambda,\gamma) = N^{Q_\gamma}(\lambda)$. We call the family

$$N^Q(\lambda,\gamma), \quad \lambda \in \mathbb{R}, \gamma > 0$$

the *integrated density of states field* of the family of random Schrödinger operators (2.18). In the Theorem 2.8 below, we describe the asymptotic behavior of this field for random Schrödinger operators with a renormalized Poisson potential. Let us anticipate this theorem by a brief discussion.

Three statements of Theorem 2.8 below relate directly to our light scale, heavy scale, and critical cases respectively. This means that the integrated density of states field for random Schrödinger operators with a renormalized Poisson potential may demonstrate asymptotic behavior typical either to the classic or to the quantum regime, while for the integrated density of states function only the classic regime is available.

Next, observe that $\frac{d+2-p}{2} > \frac{d+4-2p}{4}$. Hence conditions, that $(-\lambda)^{\frac{d+4-2p}{4}}/\gamma \to \infty$ and $(-\lambda)^{\frac{d+2-p}{2}}/\gamma$ is bounded, yield $\lambda \to 0-$. Therefore, the quantum regime for the integrated density of states field requires that λ and γ tend to 0 in an adjusted way (statement II of Theorem 2.8 below). On the contrary, conditions of the statement I of the same theorem allow $\lambda \to -\infty$ (in that case γ may tend to ∞), $\lambda \to 0-$, or λ to stay bounded away both from 0 and $-\infty$ (in these two cases $\gamma \to 0+$ necessarily). This is the reason that two conditions $(-\lambda)^{\frac{p}{d}}/\gamma \to \infty$ and $(-\lambda)^{\frac{d+2-p}{2}}/\gamma \to \infty$ are imposed in this case: when $\lambda \to -\infty$, the first one includes the second one, but when $\lambda \to 0-$ the inclusion is opposite.

Theorem 2.8 *Let K be of the form (1.5) with $p \in (d/2, d)$.*

I. When $(-\lambda)^{\frac{p}{d}}/\gamma \to \infty$ and $(-\lambda)^{\frac{d+2-p}{2}}/\gamma \to \infty$,

$$\log N^{\overline{V}}(\lambda,\gamma) = -\left[\nu\omega_d\Gamma\left(\frac{2p-d}{p}\right)\right]^{-\frac{p}{d-p}}\left(\frac{\theta(d-p)}{d}\right)^{\frac{d}{d-p}}\Big(-\lambda\Big)^{\frac{d}{d-p}}\gamma^{-\frac{p}{d-p}}\Big(1+o(1)\Big). \tag{2.19}$$

II. When $(-\lambda)^{\frac{d+4-2p}{4}}/\gamma \to \infty$ and $(-\lambda)^{\frac{d+2-p}{2}}/\gamma \to 0$, under additional assumption $p < (d+2)/2$

$$\log N^{\overline{V}}(\lambda,\gamma) = -\left(\frac{2\mathsf{C}_2}{d+2-2p}\right)^{-\frac{d+2-2p}{2}}\left(\frac{2}{d+4-2p}\right)^{\frac{d+4-2p}{2}}\Big(-\lambda\Big)^{\frac{d+4-2p}{2}}\gamma^{-2}\Big(1+o(1)\Big), \tag{2.20}$$

where C_2 denotes the constant in the r.h.s of (2.3).

III. When $\lambda \to 0-$ and $(-\lambda)^{\frac{d+2-p}{2}}/\gamma$ is bounded away both from 0 and from ∞,

$$\log N^{\overline{V}}(\lambda,\gamma) = -\left(\frac{(d-p)\mathsf{C}_\psi}{p}\right)^{-\frac{p}{d-p}}\left(\frac{(d-p)}{d}\right)^{\frac{d}{d-p}}\Big(-\lambda\Big)^{\frac{d}{d-p}}\gamma^{-\frac{p}{d-p}}\Big(1+o(1)\Big), \tag{2.21}$$

where C_ψ denotes the constant in the right hand side of (2.2) with $\alpha = 1$.

Note that, under the assumptions of Theorem 2.8, the right hand sides of (2.19), (2.20), (2.21) tend to $-\infty$. So Theorem 2.8 controls the exponential decay of the IDS field, similarly to Theorem 2.7. What may look non-typical in this theorem when compared with other references in the field, is that some part of the statements are formulated when $\lambda \to 0-$. This in general reflects the fact that for $\gamma \to 0+$ the negative part of the spectrum becomes negligible. Theorem 2.8, in particular, quantifies such a negligibility.

3 Classic regime

In this section, we prove Theorem 2.4, which includes statement I of Theorem 2.1 as a partial case. For a given $h > 0$, denote

$$\xi^h(t,x) = \int_0^t \Big(K(y-x) - h\Big)_+ ds, \quad \xi_h(t,x) = \int_0^t \Big(K(y-x) \wedge h\Big)\, ds.$$

Similarly to (1.17), we have

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t V(B_{\kappa s})\, ds\right] = \mathbb{E}_0 \exp\left[-\int_{\mathbb{R}^d} \varphi\left(\frac{1}{\alpha_t}\xi(t,x)\right) dx\right], \tag{3.1}$$

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t V^h(B_{\kappa s})\, ds\right] = \mathbb{E}_0 \exp\left[\int_{\mathbb{R}^d} \Xi\left(\frac{1}{\alpha_t}\xi_h(t,x), \frac{1}{\alpha_t}\xi^h(t,x)\right) dx\right]. \tag{3.2}$$

The first relation is provided by Proposition 2.7 and Proposition 3.1 in [14], the proof for the second one is completely analogous and is omitted.

In what follows, we analyse the Wiener integrals in the r.h. sides of (1.17) and (3.1). However, (3.2) appears not to be well designed for an immediate analysis, which motivates the following auxiliary construction. Instead of V^h, we consider a partially renormalized Poisson potential with the properly chosen renormalization level, dependent on t. Let $g > 0$ and $h_t = g\alpha_t/t$. Then, assuming $p = d$, (2.4) and (1.14), we will prove that

$$\begin{aligned}\lim_{t\to\infty} \left(\frac{\alpha_t}{t}\right) \log \mathbb{E} \otimes \mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t V^{h_t}(B_{\kappa s})\, ds\right] \\ = \nu \int_{\mathbb{R}^d} \Xi\Big((\theta|x|^{-d}) \wedge g, (\theta|x|^{-d} - g)_+\Big)\, dx = \nu\omega_d\theta\Big[\log g + \mathrm{Eu}\Big].\end{aligned} \tag{3.3}$$

Note that, by Proposition 6.1 in [14],

$$V^{h'}(x) - V^h(x) = \nu \int_{\mathbb{R}^d} \Big(\min(K(y), h) - h'\Big)_+ dy \tag{3.4}$$

for any $h \geq h'$ such that $V^h, V^{h'}$ is well defined. Henceforth, changing a renormalization level just multiplies respective exponential moment by an explicit constant. Therefore (2.10) is provided by (3.3).

In Section 3.1 and Section 3.2, we prove respectively upper and lower bounds in (2.1), (2.9), and (3.3) with the constants represented in an integral form. Calculation of the integrals is postponed to Section 3.3.

3.1 Proof of the upper bound.

For any convex function ϱ, by the Jensen inequality we have

$$\varrho\left(\frac{1}{\alpha_t}\xi(t,x)\right) = \varrho\left(\int_0^t \left(\frac{1}{\alpha_t}K(B_{\kappa s}-x)\right) ds\right) \leq \frac{1}{t}\int_0^t \varrho\left(\frac{t}{\alpha_t}K(B_{\kappa s}-x)\right) ds.$$

Denote $\lambda_t = (t/\alpha_t)^{1/p}$, $K(x,\lambda) = \lambda^p K(\lambda x)$. By the inequality above, one has the following estimate with non-random right hand side:

$$\begin{aligned}\int_{\mathbb{R}^d} \varrho\left(\frac{1}{\alpha_t}\xi(t,x)\right) dx &\leq \frac{1}{t}\int_{\mathbb{R}^d}\int_0^t \varrho\left(\frac{t}{\alpha_t}K(B_{\kappa s}-x)\right) dsdx \\ &= \int_{\mathbb{R}^d} \varrho\left(\frac{t}{\alpha_t}K(-x)\right) dx = \left(\frac{t}{\alpha_t}\right)^{d/p}\int_{\mathbb{R}^d}\varrho\left(K(x,\lambda_t)\right) dx.\end{aligned} \tag{3.5}$$

Assumption (1.14) yields $\lambda_t \to \infty$. Therefore, in order to prove the upper bound either in (2.1) or in (2.9), it is sufficient to apply (3.5) to either ψ or $-\varphi$, and then prove respectively

$$\int_{\mathbb{R}^d}\psi\left(K(x,\lambda)\right) dx \to \int_{\mathbb{R}^d}\psi\left(K(x)\right) dx \quad \text{or} \quad \int_{\mathbb{R}^d}\varphi\left(K(x,\lambda)\right) dx \to \int_{\mathbb{R}^d}\varphi\left(K(x)\right) dx, \tag{3.6}$$

$\lambda \to \infty$. By assumption (2.4), for every $\varepsilon > 0$ there exists λ^ε such that

$$K(x,\lambda)|x|^p \in [\theta-\varepsilon, \theta+\varepsilon], \quad |x| > \varepsilon, \lambda > \lambda^\varepsilon.$$

When $p > d$, this easily provides

$$\lim_{\lambda\to\infty}\int_{|x|>\varepsilon}\varphi\left(K(x,\lambda)\right) dx = \int_{|x|>\varepsilon}\varphi\left(K(x)\right) dx, \quad \varepsilon > 0. \tag{3.7}$$

Since φ is bounded on $\mathbb{R}^+$, (3.7) provides the second relation in (3.6).

When $p \in (d/2, d)$, similar argument leads to the relation analogous to (3.7) with φ replaced by ψ. Consequently, with condition (2.8) in mind, it remains to prove

$$\lim_{\varepsilon\to 0}\limsup_{\lambda\to\infty}\int_{|x|\leq\varepsilon}\psi\left(K(x,\lambda)\right) dx = 0.$$

To that end, we choose r_1, θ_1 such that $K(x) \leq \theta_1 |x|^{-p}$, $|x| > r$, and write for λ large enough

$$\begin{aligned}\int_{|x|\leq\varepsilon}\psi\left(K(x,\lambda)\right) dx &= \left[\int_{|x|\leq r_1/\lambda} + \int_{r_1/\lambda<|x|\leq\varepsilon}\right]\psi\left(K(x,\lambda)\right) dx \\ &\leq \lambda^{-d}\int_{|x|\leq r_1}\psi(\lambda^p K(x))\, dx + \int_{|x|\leq\varepsilon}\psi\left(\theta_1|x|^{-p}\right) dx.\end{aligned}$$

Recall that $p < d$, $\psi(u)$ is dominated by u, and K is locally integrable under condition (2.8). Then the first term in the above sum is negligible when $\lambda \to \infty$. This proves (3.7) and completes the proof.

Similarly, for $p=d$ from the Jensen's inequality for the convex function $\Xi:\mathbb{R}^2\to\mathbb{R}$ we have

$$\Xi\left(\frac{1}{\alpha_t}\xi_h(t,x),\frac{1}{\alpha_t}\xi^h(t,x)\right)\le\frac{1}{t}\int_0^t\Xi\left(\frac{t}{\alpha_t}(K(B_{\kappa s}-x)\wedge h),\frac{t}{\alpha_t}(K(B_{\kappa s}-x)-h)_+\right)ds,$$

and consequently, for $h_t=g\alpha_t/t$

$$\begin{aligned}\int_{\mathbb{R}^d}\Xi\left(\frac{1}{\alpha_t}\xi_{h_t}(t,x),\frac{1}{\alpha_t}\xi^{h_t}(t,x)\right)dx&\le\int_{\mathbb{R}^d}\Xi\left(\frac{t}{\alpha_t}(K(-x)\wedge h_t),\frac{t}{\alpha_t}(K(-x)-h_t)_+\right)dx\\&=\left(\frac{t}{\alpha_t}\right)^{d/p}\int_{\mathbb{R}^d}\Xi\Big(K(x,\lambda_t)\wedge g,(K(x,\lambda_t)-g)_+\Big)\,dx.\end{aligned}$$

Similarly to (3.6), one can prove

$$\int_{\mathbb{R}^d}\Xi\Big(K(x,\lambda)\wedge g,(K(x,\lambda)-g)_+\Big)\,dx\to\int_{\mathbb{R}^d}\Xi\Big(K(x,\lambda)\wedge g,(K(x,\lambda)-g)_+\Big)\,dx,\quad\lambda\to\infty,$$

which provides the upper bound in (3.3). $\square$

3.2 Proof of the lower bound.

For a fixed $\varepsilon>0$, take R fixed but large enough, so that

$$(\theta-\varepsilon)|x|^{-p}\le K(x)\le(\theta+\varepsilon)|x|^{-p},\quad|x|\ge R.\tag{3.8}$$

Take $\beta>0$ and consider the set

$$A_{t,\beta}=\left\{\sup_{s\le t}|B_{\kappa s}|\le\beta\lambda_t\right\},$$

keeping the notation $\lambda_t=(t/\alpha_t)^{1/p}$. By the scaling property and the well known small balls probability asymptotics for the Brownian motion we have, for t large enough,

$$\log\mathbb{P}_0(A_{t,\beta})\ge-ct(\beta\lambda_t)^{-2}$$

with some constant $c>0$. Therefore condition $\alpha_t=o(t^{\frac{d+2-p}{d+2}})$ yields

$$\left(\frac{\alpha_t}{t}\right)^{d/p}\log\mathbb{P}_0(A_{t,\beta})\to0,\quad t\to+\infty.\tag{3.9}$$

Take $\gamma>2\beta$. On the set $A_{t,\beta}$, one has

$$|B_{\kappa s}-x|\ge\beta\left(\frac{t}{\alpha_t}\right)^{\frac{1}{p}},\quad s\in[0,t],\quad|x|\ge\gamma\lambda_t.$$

Then, for t large enough to provide $\beta\lambda_t>R$, we have

$$(\theta-\varepsilon)|B_{\kappa s}-x|^{-p}\le K(B_{\kappa s}-x)\le(\theta+\varepsilon)|B_{\kappa s}-x|^{-p},\quad s\in[0,t],\quad|x|\ge\gamma\lambda_t.$$

Therefore, a two-sided estimate

$$(\theta-\varepsilon)\left(1+\frac{\beta}{\gamma}\right)^{-p}|x|^{-p}\le K(B_{\kappa s}-x)\le(\theta+\varepsilon)\left(1-\frac{\beta}{\gamma}\right)^{-p}|x|^{-p},\quad s\in[0,t] \tag{3.10}$$

is valid on the set $A_{t,\beta}$ for every x with $|x|>\gamma\lambda_t$. Observe that (3.10) is a point-wise estimate for a Brownian trajectory from a "small ball" $A_{t,\beta}$ and for a point x outside a "large ball" $\{y:|y|\le\gamma\lambda_t\}$. On the other hand, (3.9) shows the "small Brownian ball" $A_{t,\beta}$ is "heavy" in the sense that its probability is sufficiently large, in respective logarithmic scale. These observations provide a straightforward tool for proving lower bounds in (2.1) – (3.3).

Since ψ is non-negative and non-decreasing, (3.10) yields

$$\int_{\mathbb{R}^d}\psi\left(\frac{1}{\alpha_t}\xi(t,x)\right)dx\ge\int_{|x|>\gamma\lambda_t}\psi\left(\frac{t}{\alpha_t}(\theta-\varepsilon)\left(1+\frac{\beta}{\gamma}\right)^{-p}|x|^{-p}\right)dx=\left(\frac{t}{\alpha_t}\right)^{\frac{d}{p}}I^{\psi}_{\varepsilon,\beta,\gamma}$$

on $A_{t,\beta}$ with

$$I^{\psi}_{\varepsilon,\beta,\gamma}=\int_{|x|>\gamma}\psi\left((\theta-\varepsilon)\left(1+\frac{\beta}{\gamma}\right)^{-p}|x|^{-p}\right)dx.$$

Together with (1.17) and (3.9), this inequality provides

$$\liminf_{t\to+\infty}\left(\frac{\alpha_t}{t}\right)^{d/p}\log\mathbb{E}\otimes\mathbb{E}_0\exp\left[-\frac{1}{\alpha_t}\int_0^t\overline{V}(B_{\kappa s})\,ds\right]\ge I^{\psi}_{\varepsilon,\beta,\gamma}$$

for every $\varepsilon>0,\beta>0,\gamma>0$. Since

$$\lim_{\varepsilon\to0}\lim_{\gamma\to0}\lim_{\beta\to0}I^{\psi}_{\varepsilon,\beta,\gamma}=\int_{\mathbb{R}^d}\psi\left(\theta|x|^{-p}\right)dx,$$

this completes the proof of the lower bound in (2.1).

Since $(-\varphi)$ is non-increasing and satisfies $-\varphi\ge-1$, (3.10) yields

$$-\int_{\mathbb{R}^d}\varphi\left(\frac{1}{\alpha_t}\xi(t,x)\right)dx\ge-\int_{|x|\le\gamma\lambda_t}dx$$
$$-\int_{|x|>\gamma\lambda_t}\varphi\left(\frac{t}{\alpha_t}(\theta+\varepsilon)\left(1-\frac{\beta}{\gamma}\right)^{-p}|x|^{-p}\right)dx=\left(\frac{t}{\alpha_t}\right)^{\frac{d}{p}}I^{\varphi}_{\varepsilon,\beta,\gamma}$$

on $A_{t,\beta}$ with

$$I^{\varphi}_{\varepsilon,\beta,\gamma}=-\int_{|x|\le\gamma}dx-\int_{|x|>\gamma}\varphi\left((\theta+\varepsilon)\left(1-\frac{\beta}{\gamma}\right)^{-p}|x|^{-p}\right)dx.$$

Since

$$\lim_{\varepsilon\to0}\lim_{\gamma\to0}\lim_{\beta\to0}I^{\varphi}_{\varepsilon,\beta,\gamma}=-\int_{\mathbb{R}^d}\varphi\left(\theta|x|^{-p}\right)dx,$$

this provides the lower bound in (3.4).

Finally, Ξ is non-decreasing in first coordinate and non-increasing in second coordinate. In addition, $\Xi \geq -1$ and hence (3.10) yields in the case $d = p$

$$\int_{\mathbb{R}^d} \Xi\left(\frac{1}{\alpha_t}\xi_{h_t}(t,x), \frac{1}{\alpha_t}\xi^{h_t}(t,x)\right) dx$$
$$\geq \int_{|x|>\gamma\lambda_t} \Xi\left(\frac{t}{\alpha_t}\left[\left\{(\theta-\varepsilon)\left(1+\frac{\beta}{\gamma}\right)^{-d}|x|^{-d}\right\} \wedge \left\{\frac{g\alpha_t}{t}\right\}\right],\right.$$
$$\left.\frac{t}{\alpha_t}\left[\left\{(\theta+\varepsilon)\left(1-\frac{\beta}{\gamma}\right)^{-d}|x|^{-d}\right\} - \frac{tg\alpha_t}{t}\right]_+\right) dx = \left(\frac{t}{\alpha_t}\right) I^{\Xi}_{\varepsilon,\beta,\gamma}$$

on $A_{t,\beta}$ with

$$I^{\Xi}_{\varepsilon,\beta,\gamma} = -\int_{|x|\leq\gamma} dx$$
$$+\int_{|x|>\gamma} \Xi\left(\left[(\theta-\varepsilon)\left(1+\frac{\beta}{\gamma}\right)^{-d}|x|^{-d}\right] \wedge g, \left[(\theta+\varepsilon)\left(1-\frac{\beta}{\gamma}\right)^{-d}|x|^{-d} - g\right]_+\right) dx.$$

Together with (3.1) and (3.9), this inequality provides

$$\liminf_{t\to+\infty}\left(\frac{\alpha_t}{t}\right)\log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\, ds\right] \geq I^{\Xi}_{\varepsilon,\beta,\gamma}$$

for every $\varepsilon > 0, \beta > 0, \gamma > 0$. Since

$$\lim_{\varepsilon\to0}\lim_{\gamma\to0}\lim_{\beta\to0} I^{\Xi}_{\varepsilon,\beta,\gamma} = \int_{\mathbb{R}^d} \Xi\Big((\theta|x|^{-d}) \wedge g, (\theta|x|^{-d} - g)_+\Big)\, dx,$$

this completes the proof of the lower bound in (3.3). $\square$

3.3 Calculation of the integrals.

In the above proof, we have obtained (2.1), (2.9), and (3.3) with the constants represented as certain integrals. Explicit calculation of these integrals can be made in easy and standard way, using sphere substitution and integration-by-parts. For such a calculation of the integral (2.1) we refer to Lemma 7.1 in [13]; calculation of the integral (2.9) is completely analogous and omitted. Here we calculate the integral in (3.3) and prove (2.11).

By sphere substitution, and change of variables,

$$\int_{\mathbb{R}^d} \Xi\Big((\theta|x|^{-d}) \wedge g, (\theta|x|^{-d} - g)_+\Big)\, dx = \omega_d \int_0^\infty \Xi\Big((\theta/r) \wedge g, (\theta/r - g)_+\Big)\, dx$$
$$= \omega_d \int_0^\infty \Big[e^{-\theta/r} - 1 - (\theta/r) \wedge g\Big]\, dr = \theta\omega_d \int_0^\infty \frac{e^{-s} - 1 - s \wedge g}{s^2}\, ds$$
$$= \omega_d\theta\left[\int_0^\infty \frac{e^{-s} - 1 - s \wedge 1}{s^2}\, ds + \log g\right];$$

in the last identity we have used an elementary relation

$$\int_0^\infty \frac{s\wedge g - s\wedge 1}{s^2}\,ds = \log g.$$

Integration by parts and **n.** 538 in [26] give

$$\int_0^\infty \frac{e^{-s}-1-s\wedge 1}{s^2}\,ds = \int_0^t \frac{1-e^{-s}}{s}\,ds - \int_0^\infty \frac{e^{-s}}{s}\,ds = \mathrm{Eu},$$

which completes calculation of the integral in (3.3).

Finally, let K has the form (1.5). Take $h_t = \alpha_t/t$, then $h_t < h$ for t large enough, and

$$\int_{\mathbb{R}^d}\Big(\min(K(y),h) - \alpha_t/t\Big)_+\,dy = \int_{\mathbb{R}^d}(\theta|x|^{-d}\wedge h - h_t)_+\,dx = \omega_d\theta\int_0^\infty \frac{s\wedge h - s\wedge h_t}{s^2}\,ds$$
$$= \omega_d\theta\Big[\log h - \log h_t\Big] = \omega_d\theta\left[\log h + \log\left(\frac{t}{\alpha_t}\right)\right].$$

Combined with (2.10), this calculation provides (2.11). □

4 Quantum regime

4.1 Large deviations.

Our analysis of the asymptotic behavior of the Brownian motion in a renormalized Poisson integral in the quantum regime (i.e. in the critical and heavy scale cases) is based on the following large deviations result. Consider some function $L:\mathbb{R}^d\to\mathbb{R}^+$ and denote

$$\eta(t,x) = \int_0^t L(B_{\kappa s}-x)\,dx. \tag{4.1}$$

Theorem 4.1 *Let, for some sequence $L_n, n\ge 1$ of non-negative continuous compactly supported functions,*

$$L(x) = \sup_{n\ge 1} L_n(x) \tag{4.2}$$

for a.a. $x\in\mathbb{R}^d$. Let $\upsilon:\mathbb{R}^+\to\mathbb{R}$ be an increasing convex function with $\upsilon(0)=0$, and

$$\int_{\mathbb{R}^d}\upsilon(L(x))\,dx < +\infty. \tag{4.3}$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{t\int_{\mathbb{R}^d}\upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\right\}$$
$$= \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\upsilon\left(\int_{\mathbb{R}^d} g^2(y)L(x-y)\,dy\right)\,dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}. \tag{4.4}$$

Proof of Theorem 4.1: the lower bound. By Jensen's inequality,

$$\upsilon\left(\frac{1}{t}\eta(t,x)\right)=\upsilon\left(\frac{1}{t}\int_0^t L(B_{\kappa s}-x)\,ds\right)\le\frac{1}{t}\int_0^t \upsilon\Big(L(B_{\kappa s}-x)\Big)\,ds,$$

and therefore

$$\int_{\mathbb{R}^d}\upsilon\left(\frac{1}{t}\eta(t,x)\right)\,dx\le\frac{1}{t}\int_0^t\int_{\mathbb{R}^d}\upsilon\Big(L(B_{\kappa s}-x)\Big)\,dx\,ds=\int_{\mathbb{R}^d}\upsilon\Big(L(x)\Big)\,dx<+\infty. \tag{4.5}$$

For every $R>0$, we write

$$\int_{\mathbb{R}^d}\upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\ge\int_{[-R,R]^d}\upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx$$

and note that (4.5) provides $\frac{1}{t}\eta(t,\cdot)\in\mathcal{L}_1([-R,R]^d)$ because υ has at least linear growth at $+\infty$.

For a fixed R, denote

$$\mathcal{L}_{1,R}^{+}=\left\{h\in\mathcal{L}_1([-R,R]^d),h\ge 0\right\},$$

and consider a convex function $\Upsilon_R:\mathcal{L}_{1,R}^{+}\to[0,+\infty]$:

$$\Upsilon_R(h)=\int_{[-R,R]^d}\upsilon(h(x))\,dx. \tag{4.6}$$

Denote by B_R the class of bounded measurable functions $f:[-R,R]^d\to\mathbb{R}$, and put

$$C_{\Upsilon,f,R}=\sup\left\{C:C+\int_{[-R,R]^d}f(x)h(x)\,dx\le\Upsilon_R(h),h\in\mathcal{L}_{1,R}^{+}\right\},\quad f\in B_R.$$

Lemma 4.2 *For every $h\in\mathcal{L}_{1,R}^{+}$ with $\Upsilon_R(h)<+\infty$,*

$$\Upsilon_R(h)=\sup_{f\in B_R}\left(C_{\Upsilon,f,R}+\int_{[-R,R]^d}f(x)h(x)\,dx\right). \tag{4.7}$$

Remark 4.3 *This statement is a version of the classic theorem in the finite-dimensional convex analysis about representation of the epigraph of a convex function as an intersection of upper half-spaces, see Theorem 12.1 in [59]. The idea of the proof, in our case, is principally the same, but we have to take care about topological aspects and about the fact that, in general, Υ_R is an improper function.*

Proof: Consider the set

$$\operatorname{epi}\Upsilon_R=\{(h,t):h\in\mathcal{L}_{1,R}^{+},t\ge\Upsilon_R(h)\};$$

clearly, $\operatorname{epi}\Upsilon_R$ is a convex subset of the Banach space $\mathcal{L}_1([-R,R]^d)\times\mathbb{R}$. In addition, this subset is closed by the Fatou lemma. Therefore, the separation theorem (Theorem 9.2 in [61], Chapter II) provides that $\operatorname{epi}\Upsilon_R$ is the intersection of all the closed half-spaces containing $\operatorname{epi}\Upsilon_R$. Note that every continuous linear functional on the space $\mathcal{L}_1([-R,R]^d)\times\mathbb{R}$ has the form (f,a) with $f\in B_R, a\in\mathbb{R}$ and

$$\Big\langle (h,t),(f,a)\Big\rangle = \int_{[-R,R]^d} h(x)f(x)\,dx + at.$$

Take $h_*\in\mathcal{L}_v^+$ with $\Upsilon_R(h_*)<+\infty$, and $t_*<\Upsilon_R(h_*)$. Then $(h_*,t_*)\notin\operatorname{epi}\Upsilon_R$, and therefore there exists (f,a) and $c\in\mathbb{R}$ such that

$$\Big\langle (h_*,t_*),(f,a)\Big\rangle < c, \quad \Big\langle (h,t),(f,a)\Big\rangle \ge c, \quad (h,t)\in\operatorname{epi}\Upsilon_R. \tag{4.8}$$

By the definition of $\operatorname{epi}\Upsilon_R$, if $(h,t)\in\operatorname{epi}\Upsilon_R$ then $(h,t')\in\operatorname{epi}\Upsilon_R$ for every $t'>t$, hence (4.8) is impossible if either $a=0$ or $a<0$. Divide (4.8) by a and denote $f_a=-f/a, c_a=c/a$. Then

$$c_a+\int_{[-R,R]^d} h^*(x)f_a(x)\,dx > t_*, \quad c_a+\int_{[-R,R]^d} h(x)f_a(x)\,dx \le t, \quad (h,t)\in\operatorname{epi}\Upsilon_R. \tag{4.9}$$

Take $t=\Upsilon_R(h)$ in the second inequality in (4.9); this yields $c_a\le C_{\Upsilon,f_a,R}$. Consequently,

$$t_*\le \sup_{f\in B_R}\left(C_{\Upsilon,f,R}+\int_{[-R,R]^d} f(x)h_*(x)\,dx\right),$$

which means that

$$\Upsilon_R(h_*)\le \sup_{f\in B_R}\left(C_{\Upsilon,f,R}+\int_{[-R,R]^d} f(x)h_*(x)\,dx\right)$$

because $t_*<\Upsilon_R(h_*)$ is arbitrary. The inverse inequality is obvious. □

Take $f\in B_R$, then

$$\mathbb{E}_0\exp\left\{t\int_{[-R,R]^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\right\}\ge e^{C_{\Upsilon,f,R}t}\mathbb{E}_0\exp\left\{\int_{[-R,R]^d} f(x)\eta(t,x)dx\right\}.$$

Note that

$$\int_{[-R,R]^d} f(x)\eta(t,x)dx=\int_0^t \hat{f}(B_{\kappa s})ds,$$

where

$$\hat{f}(y)=\int_{[-R,R]^d} f(x)L(y-x)\,dx$$

is a bounded function. Henceforth, by the large deviations result by Kac [36] (see also Theorem 4.1.6 in [12]), we have

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{t\int_{[-R,R]^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\right\}\\
&\qquad\ge C_{\Upsilon,f,R}+\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{\int_0^t \hat{f}(B_{\kappa s})\,ds\right\}\\
&\qquad\ge C_{\Upsilon,f,R}+\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\hat{f}(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^d} \hat{f}(x) g^2(x) dx = \int_{[-R,R]^d} f(x) \Big[\int_{\mathbb{R}^d} g^2(y) L(x-y) dy \Big] dx.$$

Summarizing our proof,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ t \int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big) dx \Big\}$$
$$\geq \sup_{g\in\mathcal{F}_d} \Big\{ C_{\Upsilon,f,R} + \int_{[-R,R]^d} f(x) \Big[\int_{\mathbb{R}^d} g^2(y) L(x-y) dy \Big] dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \Big\}$$

for every $R > 0, f \in B_R$. We take supremum over $f \in B_R$ and get, by Lemma 4.2,

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ t \int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big) dx \Big\} \\ &\qquad \geq \sup_{g\in\mathcal{F}_d} \Big\{ \Upsilon_R \Big(\int_{\mathbb{R}^d} g^2(y) L(\cdot - y) dy \Big) - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \Big\}. \end{aligned} \tag{4.10}$$

Note that by Jensen's inequality for every $g \in \mathcal{F}_d$

$$\begin{aligned} \Upsilon_R \Big(\int_{\mathbb{R}^d} g^2(y) L(\cdot - y) dy \Big) &= \int_{[-R,R]^d} \upsilon \Big(\int_{\mathbb{R}^d} g^2(y) L(x-y) dy \Big) dx \\ &\leq \int_{[-R,R]^d} \int_{\mathbb{R}^d} g^2(y) \upsilon(L(x-y)) dy\, dx \leq \int_{\mathbb{R}^d} \upsilon(L(x))\, dx < +\infty, \end{aligned}$$

which makes it possible to apply Lemma 4.2. Finally, taking supremum over $R > 0$, we obtain the lower bound in (4.4). □

Proof of Theorem 4.1: the upper bound.

Assume first L to be continuous and supported by some cube $[-M, M]^d$. In that case we reduce the proof of the upper bound to application of the large deviation principle for empirical measures of the Brownian motion on a torus. Such a reduction is standard, e.g. [20]; the projection on the torus is required in order to make it possible to use Donsker-Varadhan's large deviation principle for empirical measures of a Markov process with a compact state space, [19].

Note that $\upsilon(u + v) - \upsilon(u) \geq \upsilon(v) - \upsilon(0)$ because of the convexity, and $\upsilon(0) = 0$. Hence the function υ satisfies

$$\upsilon(u + v) \geq \upsilon(u) + \upsilon(v), \quad u, v \geq 0.$$

Thus, for any $N > M$,

$$\begin{aligned} \int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big) dx &= \sum_{z\in\mathbb{Z}^d} \int_{[-N,N]^d} \upsilon\Big(\frac{1}{t}\eta(t, 2Nz + x)\Big) dx \\ &\leq \int_{[-N,N]^d} \upsilon\Big(\frac{1}{t} \sum_{z\in\mathbb{Z}^d} \eta(t, 2Nz + x)\Big) dx = \int_{[-N,N]^d} \upsilon\Big(\frac{1}{t}\tilde{\eta}(t,x)\Big) dx, \end{aligned} \tag{4.11}$$

where

$$\tilde{\eta}(t,x)=\int_0^t \tilde{L}(B_{\kappa s}-x)ds, \quad \tilde{L}(x)=\sum_{z\in\mathbb{Z}^d} L(2Nz+x).$$

Denote by T_d^N the torus of the size $2N$; that is, the cube $[-N,N]^d$ with the sides identified. Let Denote by J_N the projection on this torus: by definition, for $x\in\mathbb{R}^d$ its projection $J_N(x)$ is the unique point $\tilde{x}\in T_d^N$ such that $x-\tilde{x}\in 2N\mathbb{Z}^d$. Denote $B_s^N=J_N(B_\cdot)$, the Brownian motion on the torus T_d^N. With this notation in mind, we rewrite the right hand side term in (4.11):

$$\int_{[-N,N]^d} \upsilon\Big(\frac{1}{t}\tilde{\eta}(t,x)\Big)dx=\int_{T_d^N} \upsilon\Big(\frac{1}{t}\eta^N(t,x)\Big)dx, \quad \eta^N(t,x)=\int_0^t (L\circ J_N)(B_{\kappa s}-x)ds.$$

Consider the empirical measures for the Brownian motion on the torus T_d^N:

$$Q_t^N(A)=\frac{1}{t}\int_0^t \mathbf{1}_{B_{\kappa s}^N\in A}\,ds, \quad A\in\mathcal{B}(T_d^N).$$

Note that

$$\frac{1}{t}\eta^N(t,x)=\int_{T_d^N}(L\circ J_N)(y-x)Q_t^N(dy),$$

and the mapping

$$\mu\mapsto\int_{T_d^N}\upsilon\left(\int_{T_d^N}(L\circ J_N)(y-x)\mu(dy)\right)dx$$

is continuous and bounded on the space of all probability distributions on T_d^N with the metrics of weak convergence. Hence combination of (4.11), the large deviation principle for Q_t^N (Theorem 3 in [19]), and Varadhan's lemma (Proposition 3.8 in [25]) yields

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}_0\exp\Big\{t\int_{\mathbb{R}^d}\upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\Big\}\\ &\qquad\le\sup_{g\in\mathcal{F}_d^N}\left\{\int_{T_d^N}\upsilon\left(\int_{T_d^N}(L\circ J_N)(x-y)g^2(y)\,dy\right)dx-\frac{\kappa}{2}\int_{T_d^N}|\nabla g(y)|^2dy\right\},\end{aligned}$$

where

$$\mathcal{F}_d^N=\Big\{g\in W_2^1(T_d^N):\ \int_{T_d^N}g^2(x)dx=1\Big\}.$$

By smooth truncation, it is easy to verify that

$$\begin{aligned}&\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\upsilon\left(\int_{\mathbb{R}^d}L(x-y)g^2(y)\,dy\right)dx-\frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}\\ &\qquad\le\liminf_{N\to\infty}\sup_{g\in\mathcal{F}_d^N}\left\{\int_{T_d^N}\upsilon\left(\int_{T_d^N}(L\circ J_N)(x-y)g^2(y)\,dy\right)dx-\frac{\kappa}{2}\int_{T_d^N}|\nabla g(y)|^2dy\right\},\end{aligned}$$

which completes the proof.

Finally, we remove the additional regularity assumption on L. Recall the assumption (4.2) and note that one can assume the sequence $L_n, n \geq 1$ to be point-wise increasing, because otherwise one can take $\tilde{L}_n = \max_{k\leq n} L_n$ instead.

Write $\Delta_n = L - L_n$ and

$$\eta(t,x) = \eta_n(t,x) + \zeta_n(t,x), \quad \zeta_n(t,x) = \int_0^t \Delta_n(B_{\kappa s} - x)\, ds.$$

For every $\gamma \in (0,1)$ we have by convexity

$$\upsilon\left(\frac{1}{t}\eta(t,x)\right) \leq \gamma\upsilon\left(\frac{1}{\gamma t}\eta_n(t,x)\right) + (1-\gamma)\upsilon\left(\frac{1}{(1-\gamma)t}\zeta_n(t,x)\right)$$

The Jensen inequality, analogously to (4.5), provides that

$$\int_{\mathbb{R}^d} \upsilon\left(\frac{1}{(1-\gamma)t}\zeta_n(t,x)\right) dx \leq \int_{\mathbb{R}^d} \upsilon\left(\frac{1}{(1-\gamma)}\Delta_n(x)\right) dx.$$

Then from the upper bound with a regular kernel L_n we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}_0 \exp\left\{t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\right\} \\
&\quad \leq \sup_{g\in\mathcal{F}_d}\left\{\gamma\int_{\mathbb{R}^d}\upsilon\left(\frac{1}{\gamma}\int_{\mathbb{R}^d} L_n(x-y)g^2(y)\,dy\right)dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2 dy\right\} \\
&\quad + \int_{\mathbb{R}^d}\upsilon\left(\frac{1}{(1-\gamma)}\Delta_n(x)\right)dx
\end{aligned}$$

for any $n \geq 1$ and $\gamma \in (0,1)$. Passing to the limit first as $n \to \infty$ and then as $\gamma \to 1$ completes the proof. □

4.2 Proof of Theorem 2.1: critical case

The kernel (1.5) has following scaling property: $K(x) = \tau^{-p/2}K(\tau^{-1/2}x)$ for any $\tau > 0$. Then, by the scaling property of the Brownian motion,

$$\xi(t,x) \stackrel{d}{=} \int_0^t K(\tau^{-1/2}B_{s\tau} - x) = \tau^{p/2-1}\int_0^{t\tau} K(B_{\kappa s} - \tau^{1/2}x)\,ds = \tau^{p/2-1}\xi(t\tau, x\tau^{1/2}).$$

Henceforth the integral under the exponent in the right hand side of (1.17), after the variable change $\tau^{1/2}x \mapsto x$, can be written as

$$\tau^{-d/2}\int_{\mathbb{R}^d}\psi\left(\frac{\tau^{p/2-1}}{\alpha_t}\xi(t\tau,x)\right)dx. \tag{4.12}$$

We take $\tau_t = t^{-\frac{2}{d+2}}$. Under such a choice, $\tau_t^{-d/2} = t\tau_t = t^{\frac{d}{d+2}}$. Observe that

$$\frac{\tau^{p/2-1}}{\alpha_t} \sim \frac{1}{\alpha}, \quad t \to \infty$$

because of (1.16). By monotonicity of ψ, we can change the variables $t^{\frac{d}{d+2}} \mapsto t$ and, applying Theorem 4.1 with $L = K, \upsilon(u) = \psi(u/(\alpha \pm \varepsilon))$, obtain

$$\limsup_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\,ds\right]$$
$$\le \sup_{g\in\mathcal{F}_d}\left\{\nu\int_{\mathbb{R}^d}\psi\left(\frac{\theta}{(\alpha-\varepsilon)}\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2 dy\right\},$$

$$\liminf_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\,ds\right]$$
$$\ge \sup_{g\in\mathcal{F}_d}\left\{\nu\int_{\mathbb{R}^d}\psi\left(\frac{\theta}{(\alpha+\varepsilon)}\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2 dy\right\}.$$

Passing to the limit as $\varepsilon \to \infty$ completes the proof of statement II of Theorem 2.1. □

4.3 Proof of Theorem 2.1: heavy scale case

Let us proceed with further transformations of the expression (4.12) for the integral under the exponent in the right hand side of (1.17). Denote $\psi_c(u) = c^{-2}\psi(cu), c > 0$, and $\psi_0(u) = u^2/2$. It can be verified that $\psi_c(u) \uparrow \psi_0(u)$ when $c \downarrow 0$. In particular, $\psi = \psi_1 \le \psi_0$, and hence

$$\tau^{-d/2}\int_{\mathbb{R}^d}\psi\left(\frac{\tau^{p/2-1}}{\alpha_t}\xi(t\tau,x)\right)dx \le \frac{1}{2}\frac{\tau^{p-2-d/2}}{\alpha_t^2}\int_{\mathbb{R}^d}\xi^2(t\tau,x)\,dx$$
$$= \frac{1}{2}\frac{t^2\tau^{p-d/2}}{\alpha_t^2}\int_{\mathbb{R}^d}\left(\frac{1}{t\tau}\xi(t\tau,x)\right)^2 dx.$$

Choose τ_t in such a way that

$$\frac{t^2\tau_t^{p-d/2}}{\alpha_t^2} = t\tau_t,$$

that is,

$$\tau_t = \alpha_t^{-\frac{4}{d+2-2p}}t^{\frac{2}{d+2-2p}}. \tag{4.13}$$

Under such a choice,

$$t\tau_t = \alpha_t^{-\frac{4}{d+2-2p}}t^{\frac{d+4-2p}{d+2-2p}},$$

and the upper bound in (2.3) follows from the upper bound in (4.4) with $\upsilon(u) = \nu u^2/2$. Note that Theorem 4.1 can not be applied to this function υ because (4.3) fails. We refer here to Theorem 1.1 in [15], which together with the Varadhan's lemma provides (2.3).

To get the lower bound, take τ_t from (4.13) and write

$$\tau_t^{-d/2}\int_{\mathbb{R}^d}\psi\left(\frac{\tau_t^{p/2-1}}{\alpha_t}\xi(t\tau_t,x)\right)dx = \frac{\tau_t^{p-2-d/2}}{\alpha_t^2}\tau^{-d/2}\int_{\mathbb{R}^d}\psi_{c_t}\left(\frac{1}{t\tau_t}\xi(t\tau_t,x)\right)dx$$

with

$$c_t = \alpha_t^{-1}t\tau_t^{p/2} = \alpha_t^{\frac{2p}{d+2-2p}-1}t^{1-\frac{p}{d+2-2p}} = t^{\frac{d+2-p}{d+2-2p}}\alpha_t^{-\frac{d+2}{d+2-2p}}.$$

Condition (1.15) and assumption $p < (d+2)/2$ provide $c_t \to 0$. Then, for every fixed $c > 0$, we have $c_t < c$ for t large enough and therefore

$$\liminf_{t\to\infty} \alpha_t^{\frac{4}{d+2-2p}} t^{-\frac{d+4-2p}{d+2-2p}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\int_0^t \overline{V}(B_{\kappa s})\,ds\right]$$
$$\geq \sup_{g\in\mathcal{F}_d}\left\{\nu\int_{\mathbb{R}^d}\psi_c\left(\frac{\theta}{\alpha}\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}.$$

by the lower bound in (4.4) with $\upsilon(u) = \nu\psi_c(u)$. Since $\psi_c(x) \to \psi_0(x) = x^2/2$ monotonously, this proves the lower bound in (2.3). □

4.4 Brownian motion in a Wiener sheet potential

Let us recall briefly the construction of the Brownian motion in a Wiener sheet potential, introduced in ([15]). Let W be a Wiener sheet on $\mathbb{R}^d$, independent on B. Write, for a given shape function K,

$$U(x) = \int_{\mathbb{R}^d} K(y-x)\,W(dy), \quad \mathbf{U}_t = \int_0^t U(B_{\kappa s})\,ds = \int_{\mathbb{R}^d}\xi(t,x)W(dx)$$

for the Wiener sheet potential and respective total net attraction, obtained by a Brownian particle from this potential. If K is of the form (1.5), the potential U is not well defined in mean square sense because $K \notin \mathcal{L}_2(\mathbb{R}^d)$, for any p. On the other hand, for $p \in (d/2, (d+2)/2)$ one has

$$\mathbb{E}_0\int_{\mathbb{R}^d}\xi^2(t,x)dx < +\infty,$$

and consequently $\mathbf{U}_t$ is well defined.

It follows from Corollary 1.5, [15] (with $\beta = 2$) and the variation relation derived in Theorem 1.5, [5] (with p there equal to 1) that under conditions of statement III of Theorem 2.1

$$\lim_{t\to\infty} \alpha_t^{\frac{4}{d+2-2p}} t^{-\frac{d+4-2p}{d+2-2p}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left[-\frac{1}{\alpha_t}\mathbf{U}_t\right]$$
$$= \sup_{g\in\mathcal{F}_d}\left\{\frac{\theta^2}{2}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^p}dy\right)^2 dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2dy\right\}. \tag{4.14}$$

Comparing (4.14) and statement III of Theorem 2.1 below we see that, in the quantum regime, asymptotic behavior of the Brownian motion in a renormalized Poisson potential is principally determined by respective asymptotics of the Brownian motion in a Wiener sheet potential. Such a relation between these random potentials is quite natural, according to the following version of the central limit theorem.

Proposition 4.4 *Let K has the form (1.5) with $p \in (d/2, (d+2)/2)$. Then for every $t > 0$ the distribution of*

$$\nu^{-1/2}\int_0^t \overline{V}(B_{\kappa s})\,ds \tag{4.15}$$

w.r.t. $\mathbb{P}\otimes\mathbb{P}_0$ weakly converges as $\nu\to\infty$ to the distribution of $\mathbf{U}_t$ w.r.t. $\mathbb{P}^W\otimes\mathbb{P}_0$, where $\mathbb{P}^W$ denotes the distribution of the Wiener sheet W.

Proof: Characteristic function of (4.15) is equal

$$\mathbb{E}_0 \exp\left[\nu \int_{\mathbb{R}^d} \left(e^{iz\nu^{-1/2}\xi(t,x)} - 1 - iz\nu^{-1/2}\xi(t,x)\right) dx\right], \tag{4.16}$$

see Proposition 2.2 in [14]. For every $z \in \mathbb{C}, t > 0, x \in \mathbb{R}^d$,

$$\nu\left(e^{iz\nu^{-1/2}\xi(t,x)} - 1 - iz\nu^{-1/2}\xi(t,x)\right) \to -\frac{z^2}{2}\xi^2(t,x), \quad \nu \to \infty.$$

In addition,

$$\nu\left|e^{iz\nu^{-1/2}\xi(t,x)} - 1 - iz\nu^{-1/2}\xi(t,x)\right| \leq C_0 z^2 \xi^2(t,x)$$

with some constant C_0. Since for $p \in (d/2, (d+2)/2)$

$$\mathbb{E}_0 e^{C\int_{\mathbb{R}^d} \xi^2(t,x)\,dx} < \infty, \quad t > 0, C > 0$$

([15]), we have by the dominated convergence theorem that the characteristic functions (4.16) converge point-wise to

$$\mathbb{E}_0 \exp\left[-\frac{z^2}{2}\int_{\mathbb{R}^d} \xi^2(t,x)\,dx\right],$$

which is just the characteristic function of $\mathbf{U}_t$. □

5 The integrated density of states

5.1 Proof of Proposition 2.6

Statement (a). We proceed in two steps. First, we show that, under conditions of the proposition, almost all realizations of Q are bounded from below on a given cube. We consider the case $Q = \overline{V}$; the case of a partially renormalized potential is quite analogous. To simplify notation, we assume in the sequel $\nu = 1$.

Write

$$\overline{V}(x) = \int_{\mathbb{R}^d} K^g(x-y)[\omega(dy) - dy] + \int_{\mathbb{R}^d} K_g(x-y)[\omega(dy) - dy]$$

The function K^g is supported by some ball $\{x : |x| \leq R\}$, which brings the lower bound $-\omega_d R^d$ for the first summand. Henceforth, without loss of generality we can remove this term. In what follows, we consider a renormalized Poisson potential with K_g instead of K. To simplify the notation, we just consider $\overline{V}$ assuming that K is bounded, Lipschitz, and belong to $W_2^1(\mathbb{R}^d)$.

It is a simple observation that for a smooth compactly supported function $L : \mathbb{R}^d \to \mathbb{R}$, every realization of respective renormalized Poisson potential belongs to $W_2^1(\mathbb{R}^d)$, and

$$\nabla \int_{\mathbb{R}^d} L(x-y)[\omega(dx) - dy] = \int_{\mathbb{R}^d} \nabla L(x-y)[\omega(dy) - dy]. \tag{5.1}$$

This, by usual approximation argument, provides that almost all realizations of the renormalized Poisson potential with kernel K belong to $W_2^1(\mathbb{R}^d)$, and (5.1) holds true in Sobolev sense for $K = L$.

Since K is Lipschitz, ∇K is bounded; the function K is bounded, as well. Then Proposition 2.7 in [14] provides

$$\mathbb{E}\exp\Big(\big|\overline{V}(0)\big| + \big|\nabla\overline{V}(0)\big|\Big) < +\infty.$$

By shift invariance, this gives

$$\mathbb{E}\int_U \exp\Big(\big|\overline{V}(x)\big| + \big|\nabla\overline{V}(x)\big|\Big)\, dx < +\infty.$$

Therefore, almost all realizations of $\overline{V}$ belong to $W_\infty^1(U) = \bigcap_{p>1} W_p^1(U)$, and henceforth are continuous by Sobolev's inclusion theorem (e.g. [3]). In particular, these realizations are bounded.

The second part of the proof is represented by the following deterministic lemma.

Lemma 5.1 *Let $Q : \mathbb{R}^d \to \mathbb{R}$ be a function bounded from below, and U be a given cube. Denote*

$$Q_N = Q \wedge N,$$

and consider the Schrödinger operators $H_U^{Q_N}, N \geq 1$ with the Dirichlet boundary conditions, defined by (2.12) with Q replaced by Q_N.

Then

I. $R_{t,U}^{Q_N} = e^{-tH_U^{Q_N}}, t \geq 0$ converge strongly as $N \to \infty$ to a continuous semigroup $R_{t,U}^{Q}, t \geq 0$ of self-adjoint operators in $L_2(U, dx)$. $R_{t,U}^{Q}, t \geq 0$ admits the Feynman-Kac representation (2.13).

II. Every operator $R_{t,U}^{Q}, t \geq 0$ is of trace class.

III. For the generator H_U^Q of the semigroup $R_{t,U}^{Q}, t \geq 0$, the function (2.14) is well defined, and its Laplace transform admits the representation

$$\int_{\mathbb{R}} e^{-\lambda t} dN_U^Q(\lambda) = (2\pi\kappa t)^{-\frac{d}{2}} \frac{1}{|U|} \int_U \mathbb{E}_{0,0}^{\kappa t}\left(\exp\left[-\int_0^t Q(B_{\kappa s} + x)\, ds\right] \chi_{U,t}(B_\cdot + x)\right) dx.$$

Proof: For every N, the semigroup $R_{t,U}^{Q_N} = e^{-tH_U^{Q_N}}, t \geq 0$ admits the Feynman-Kac representation (2.13). An alternative form of this representation ([66], p.13) is that $R_{t,U}^{Q_N}$ is an integral operator with the kernel

$$r_{t,U}^{Q_N}(x, y) = p_t(x, y)\mathbb{E}_{x,y}^t\left(\exp\left[-\int_0^t Q_N(B_{\kappa s})\, ds\right] \chi_{U,t}(B_\cdot)\right), \tag{5.2}$$

where

$$p_t(x, y) = (2\pi\kappa t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2\kappa t}}$$

is the transition probability density for the process $B_{\kappa s}$, $s \geq 0$, and $\mathbb{E}^t_{x,y}$ denotes the expectation w.r.t. law of the Brownian bridge which takes values x and y at $s = 0$ and $s = t$, respectively. By Proposition 3.2 and Proposition 3.5 in [66], for a given N the function $r^{Q_N}_{t,U}$ is continuous and symmetric. Then, by the monotone convergence theorem, there exists a monotonous limit

$$r^Q_{t,U}(x,y) = \lim_{N\to\infty} r^{Q_N}_{t,U}(x,y) = (2\pi\kappa t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2\kappa t}} \mathbb{E}^t_{x,y}\left(\exp\left[-\int_0^t Q(B_{\kappa s})\,ds\right]\chi_{U,t}(B.)\right),$$

which is bounded and symmetric. Integral operators $R^{Q_N}_{t,U}$ converge to the operator $R^Q_{t,U}$ with kernel $r^Q_{t,U}$ in the Hilbert-Schmidt norm because respective kernels converge in $L_2(U\times U, dx)$. This proves statement I immediately. Since

$$R^Q_{t,U} = R^Q_{t/2,U} R^Q_{t/2,U},$$

and every $R^Q_{t/2,U}$ is a Hilbert-Schmidt operator, the operator $R^Q_{t,U}$ is of trace class. This proves statement II.

We have just proved that $R^Q_{t,U}$ is of trace class, which means that it has a purely point spectrum and

$$\text{Trace}\, R^Q_{t,U} = \sum_k \lambda_{k,Q,t} < +\infty,$$

where $\{\lambda_{k,Q,t}\}$ denote eigenvalues of $R^Q_{t,U}$, counted with their multiplicities. By the spectral decomposition theorem, this yields that the generator H^Q_U of the semigroup $R^Q_{t,U}$, $t \geq 0$ has a purely point spectrum, locally finite on every interval $(-\infty, \lambda]$. In addition, $\lambda_{k,Q,t} = e^{-\lambda_{k,Q} t}$, where $\{\lambda_{k,Q}\}$ are respective eigenvalues of H^Q_U, counted with their multiplicities. Therefore the function (2.14) is well defined, and its Laplace transform has the form

$$\int_{\mathbb{R}} e^{-\lambda t} dN^Q_U(\lambda) = \frac{1}{|U|}\sum_k e^{-t\lambda_{k,U}} = \frac{1}{|U|}\text{Trace}\, R^Q_{t,U} = \frac{1}{|U|}\int_{U\times U} r^Q_{t/2,U}(x,y) r^Q_{t/2,U}(y,x)\,dxdy,$$

in the last equality we have used the standard relation (e.g. [34], Chapter III, §9)

$$\text{Trace}\, A^*A = \|A\|^2_{HS}$$

with the Hilbert-Shmidt norm of the operator A in the right hand side. By the Feynman-Kac representation (2.13) and the Markov property of the Brownian bridge, the last integral can be written as

$$(2\pi\kappa t)^{-\frac{d}{2}} \frac{1}{|U|}\int_U \mathbb{E}^{\kappa t}_{x,x}\left(\exp\left[-\int_0^t Q(B_{\kappa s})\,ds\right]\chi_{U,t}(B.)\right)\,dx.$$

which completes the proof of the lemma. □

Statement (b): sketch of the proof. In the second part of the proposition, the classic argument which goes back to [54] is applicable. In order to keep the exposition self-sufficient, we give a brief sketch of this argument here.

The random fields $Q = \overline{V}$ and $Q = V^h$ are *ergodic* (or *metrically transitive*) in the sense that the σ-algebra generated by functionals, invariant w.r.t. the transformations

$$S_h : Q(\cdot) \mapsto Q(\cdot + h), \quad h \in \mathbb{R}^d,$$

is degenerate. The argument here is a straightforward modification of the classic one for one-dimensional moving average integrals, see Theorem 1.1 and Example 3 in Chapter XI, [21]. Then the Birkhoff's ergodic theorem (its modification for random fields, e.g. Chapter 6.5 in [2]) yields that, for any integrable function f on the space of realizations of the field Q,

$$\lim_{U \uparrow \mathbb{R}^d} \frac{1}{|U|} \int_U f(Q(\cdot + x))\, dx = \mathbb{E} f(Q) \tag{5.3}$$

both almost surely and in mean sense.

By Proposition 2.7 in [14], $\mathbb{E}e^{-cQ(0)} < +\infty$ for every $c > 0$. This, by the Jensen inequality, provides

$$\mathbb{E} \otimes \mathbb{E}^{\kappa t}_{0,0} \exp\left[-\int_0^t Q(B_{\kappa s})\, ds\right] < +\infty, \quad t \geq 0; \tag{5.4}$$

the argument here is the same as at the beginning of Section 3.1. Then (5.3) applied to the function

$$f : q \mapsto \mathbb{E}^{\kappa t}_{0,0} \exp\left[-\int_0^t q(B_{\kappa s})\, ds\right]$$

yields

$$\lim_{U \uparrow \mathbb{R}^d} \frac{1}{|U|} \int_U \mathbb{E}^{\kappa t}_{0,0} \left(\exp\left[-\int_0^t Q(B_{\kappa s} + x)\, ds\right]\right) dx = \mathbb{E} \otimes \mathbb{E}^{\kappa t}_{0,0} \exp\left[-\int_0^t Q(B_{\kappa s})\, ds\right] \tag{5.5}$$

both in mean and in a.s. sense. Straightforward calculation shows that, with probability 1,

$$\lim_{U \uparrow \mathbb{R}^d} \frac{1}{|U|} \int_U \chi_{U,t}(B_\cdot + x)\, dx = 1.$$

Together with mean convergence (5.5), this provides

$$\lim_{U \uparrow \mathbb{R}^d} \frac{1}{|U|} \int_U \mathbb{E}^{\kappa t}_{0,0} \left(\exp\left[-\int_0^t Q(B_{\kappa s} + x)\, ds\right] (1 - \chi_{U,t}(B_\cdot + x))\right) dx = 0,$$

which completes the proof. □

5.2 Proof of Theorem 2.5.

Classic regime. Arguments in Section 3.1 are process insensitive. Henceforth, the upper bounds in (2.1), (2.9), and (3.3) hold true with $\mathbb{E}_0$ replaced by $\mathbb{E}^{\kappa t}_{0,0}$.

On the other and, Brownian bridge measures enjoy the scaling property similar to the Brownian one ([66], p. 140): its law $\mathbb{P}^t_{0,0}$ is the image of measure of $\mathbb{P}^1_{0,0}$ under the map

$$w(\cdot) \mapsto \sqrt{t} w\left(\frac{\cdot}{t}\right).$$

In addition, the Brownian bridge measure $\mathbb{P}^1_{0,0}$ has the small balls asymptotics similar to the Brownian one [62]:

$$\varepsilon^2 \log \mathbb{P}^1_{0,0}\left(\sup_{s\in[0,1]} |B(s)| \le \varepsilon\right) \to -\frac{1}{2} j^2_{\frac{d-2}{2}},$$

where $j_{\frac{d-2}{2}}$ is the smallest positive root of the Bessel function $J_{\frac{d-2}{2}}$. Henceforth the heavy small ball argument from Section 3.2 can be applied to get the lower bounds in (2.1), (2.9), and (3.3) with $\mathbb{E}^{\kappa t}_{0,0}$ instead of $\mathbb{E}_0$.

The argument which deduce (2.10) from (3.3) is process insensitive.□

Quantum regime. The upper bound in (4.4) with $\mathbb{E}^{\kappa t}_{0,0}$ instead of $\mathbb{E}_0$ can be deduced from the same upper bound in its original form. In the proof, we combine the standard trick based on the Markov property of the Brownian bridge (e.g. Lemma 3 in [28]) with the "universal" upper bound provided by the convexity, see the end of Section 4.1.

Write

$$\eta(t,x) = \eta(t-1,x) + \zeta(t,x), \quad \zeta(t,x) = \int_{t-1}^t L(B_{\kappa s} - x)\, ds.$$

For every $\gamma \in (0,1)$ we have by convexity

$$\upsilon\left(\frac{1}{t}\eta(t,x)\right) \le \gamma\upsilon\left(\frac{1}{\gamma t}\eta(t-1,x)\right) + (1-\gamma)\upsilon\left(\frac{1}{(1-\gamma)t}\zeta(t,x)\right).$$

Analogously to (4.5), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \upsilon\left(\frac{1}{(1-\gamma)t}\zeta(t,x)\right) dx &\le \int_{t-1}^t \int_{\mathbb{R}^d} \upsilon\left(\frac{1}{(1-\gamma)t} L(B_{\kappa s} - x)\right) dx\, ds \\ &= \int_{\mathbb{R}^d} \upsilon\left(\frac{1}{(1-\gamma)t} L(x)\right) dx.\end{aligned}$$

The last term vanishes when $t\to\infty$. Therefore, for every $\gamma \in (0,1)$,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}^{\kappa t}_{0,0} \exp\left\{t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\right\} \\ &\qquad \le \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}^{\kappa t}_{0,0} \exp\left\{\gamma t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{\gamma t}\eta(t-1,x)\Big)dx\right\}.\end{aligned}$$

On the other hand, applying the Markov property at time $t-1$, we arrive at

$$\begin{aligned}\mathbb{E}^{\kappa t}_{0,0} \exp\left\{\gamma t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{\gamma t}\eta(t-1,x)\Big)dx\right\} &= \mathbb{E}_0 \exp\left\{\gamma t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{\gamma t}\eta(t-1,x)\Big)dx\right\} p_1(B_{\kappa(t-1)},0) \\ &\le \mathbb{E}_0 \exp\left\{\gamma t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{\gamma t}\eta(t,x)\Big)dx\right\} p_1(B_{\kappa(t-1)},0).\end{aligned}$$

Because

$$p_1(x,y) \le (2\pi\kappa)^{-\frac{d}{2}}, \quad x,y\in\mathbb{R}^d,$$

we get from the upper bound in (4.4) that, for every $\gamma \in (0,1)$,

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}^{\kappa t}_{0,0} \exp\Big\{t\int_{\mathbb{R}^d} \upsilon\Big(\frac{1}{t}\eta(t,x)\Big)dx\Big\} \le \sup_{g\in\mathcal{F}_d}\left\{\gamma \int_{\mathbb{R}^d} \upsilon\left(\frac{1}{\gamma}\int_{\mathbb{R}^d} L(x-y)g^2(y)\,dy\right)\,dx - \frac{\kappa}{2}\int_{\mathbb{R}^d}|\nabla g(y)|^2 dy\right\}.$$

Passing to the limit $\gamma \to 1$ completes the proof of the upper bound.

The lower bound in (4.4) with $\mathbb{E}^{\kappa t}_{0,0}$ instead of $\mathbb{E}_0$ can be obtained by almost the same argument as it was used to prove lower bound in (4.4) in its initial form. Only the minor changes of the argument is required; let us discuss these changes.

Consider the large deviation result by Kac, which was the basic point in the proof of the lower bound:

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}_0 \exp\left\{\int_0^t \hat{f}(B_{\kappa s})\,ds\right\} \ge C_{\Upsilon,f,R} + \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\hat{f}(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2 dx\right\}. \tag{5.6}$$

Note that, under $\mathbb{P}_0$

$$B_{\kappa s} - \frac{s}{\kappa t}B_{\kappa t}$$

is a Gaussian process with the covariance $\kappa(s\wedge s') - \kappa\frac{ss'}{t}$, and therefore has the distribution $\mathbb{P}^t_{0,0}$ ([66], p.140). Then, for any Lipschiz continuous bounded function $\hat{f}$, (5.6) holds true with $\mathbb{E}^{\kappa t}_{0,0}$ instead of $\mathbb{E}_0$.

Note that the statement of the Lemma 4.2 still holds true when B_R is replaced with any class of functions $K \subset B_R$ separating points in $\mathcal{L}_1([-R,R]^d)$; in particular one can take $K = BL_R$, the class of Lipschitz continuous bounded functions. Clearly, for $f \in BL_R$ the function

$$\hat{f}(y) = \int_{[-R,R]^d} f(x)L(y-x)\,dx$$

is Lipschitz continuous and bounded. Applying the modified (5.6) and proceeding literally as in the proof of the lower bound in Theorem 4.1, we get the required lower bound.

Once we have proved the modified large deviation asymptotics (4.4), we can repeat the arguments from Sections 4.2 and 4.3 (with the the scaling property of the Brownian bridge used instead of the same property of the Brownian motion), and deduce (2.2), (2.3) with $\mathbb{E}_0$ replaced by $\mathbb{E}^{\kappa t}_{0,0}$.

□

5.3 Proof of Theorem 2.7.

Theorem 9.7 in [56], Chapter IV gives (2.16) as a straightforward corollary of (2.1). Because $t \mapsto t\log t$ is a regularly varying function of the order 1, this theorem is not applicable when (2.17) is considered.

From (2.11) (with $\mathbb{E}_{0,0}^{\kappa t}$ instead of $\mathbb{E}_0$) we have

$$\log \int_{\mathbb{R}} e^{-(\lambda+\nu\omega_d\theta(\log h+\mathrm{Eu}))t} dN^{V^h}(\lambda) = \nu\omega_d\theta t\log t + o(t), \quad t\to+\infty.$$

Henceforth the proof of (2.17) by elementary transformations can be reduced to the proof of the following lemma.

Lemma 5.2 *Let, for a non-negative and non-decreasing function* $N(\lambda), \lambda\in\mathbb{R}$,

$$\log \int_{\mathbb{R}} e^{-\lambda t} dN(\lambda) = at\log t + o(t), \quad t\to+\infty \tag{5.7}$$

with some $a > 0$.

Then

$$\log N(\lambda) = -a\exp\left[-\frac{\lambda}{a}-1\right]\Big(1+o(1)\Big), \quad \lambda\to-\infty. \tag{5.8}$$

Proof: The upper bound, in a standard way, is provided by the Chebyshev inequality: for every $t > 0$, $\lambda < 0$

$$\log N(\lambda) \le \lambda t + \log\int_{\mathbb{R}} e^{-\chi t} dN(\chi). \tag{5.9}$$

Take $t_\lambda = \exp[-\lambda/a-1]$, the solution of the minimization problem

$$t_\lambda = \arg\ \min\Big(\lambda t + at\log t\Big).$$

Clearly, $t_\lambda\to+\infty, \lambda\to-\infty$. By (5.7) and (5.9),

$$\log N(\lambda) \le -at_\lambda + o(t_\lambda), \quad \lambda\to-\infty,$$

which gives the upper bound in (5.8).

Assume that the lower bound in (5.8) fails; that is, there exist $b > a$ and a sequence $\lambda_n\to-\infty$ such that

$$\log N(\lambda_n) \le -b\exp\left[-\frac{\lambda_n}{a}-1\right], \quad n\ge 1. \tag{5.10}$$

Fix $c < a$ and $\delta > 0$, which will be specified below. Since the upper bound in (5.8) is already proved, there exists $\Lambda_c < 0$ such that

$$\log N(\lambda) \le -c\exp\left[-\frac{\lambda}{a}-1\right], \quad \lambda < \Lambda_c. \tag{5.11}$$

Let n be large enough for $\lambda_n < \Lambda_c$. Denote $t_n = e^{-(\lambda_n-\delta)/a-1}$, and write

$$\begin{aligned}\int_{\mathbb{R}} e^{-\lambda t_n} dN(\lambda) &= \int_{-\infty}^{\Lambda_c} e^{-\lambda t_n} N(\lambda)\,d\lambda + \left[e^{-\Lambda_c t_n} + \int_{\Lambda_c}^{\infty} e^{-\lambda t_n} dN(\lambda)\right] = \int_{-\infty}^{\lambda_n-2\delta} e^{-\lambda t_n} N(\lambda)\,d\lambda \\ &+ \int_{\lambda_n-2\delta}^{\lambda_n} e^{-\lambda t_n} N(\lambda)\,d\lambda + \int_{\lambda_n}^{\Lambda_c} e^{-\lambda t_n} N(\lambda)\,d\lambda + \left[e^{-\Lambda_c t_n} + \int_{\Lambda_c}^{\infty} e^{-\lambda t_n} dN(\lambda)\right] \\ &= I_n^1 + I_n^2 + I_n^3 + I_n^4.\end{aligned}$$

Let us estimate $I_n^1 - I_n^4$. Clearly,

$$I_n^4 \le Ce^{-\Lambda_c t_n} \tag{5.12}$$

with appropriate constant C. Assumption (5.10) yields, via monotonicity,

$$\begin{aligned} I_n^2 &\le 2\delta \exp\left[-(\lambda_n - 2\delta)t_n - be^{-\lambda_n/a-1}\right] = 2\delta \exp\left[(\delta + a\log t_n + a)t_n - be^{-\delta/a}t_n\right] \\ &= 2\delta \exp\left[at_n \log t_n - (be^{-\delta/a} - a - \delta)t_n\right]; \end{aligned} \tag{5.13}$$

here we have used the relation

$$\lambda_n = -a\log t_n - a + \delta,$$

which comes from the definition of t_n.

Consider the function $\Theta : \lambda \mapsto -\lambda t_n - ce^{-\lambda/a-1}$. Straightforward computation shows that, assuming

$$c > ae^{-\delta/a}, \tag{5.14}$$

its derivative is increasing on $(-\infty, \lambda_n - 2\delta]$, and

$$\Theta_n'(\lambda_n - 2\delta) = \left(\frac{c}{a}e^{\delta/a} - 1\right) t_n > 1$$

for n large enough. Then, with (5.11) in mind, we get

$$I_n^1 \le \int_{-\infty}^{\lambda_n - 2\delta} e^{\lambda - \lambda_n + 2\delta} e^{\Theta_n(\lambda_n - 2\delta)}\, d\lambda = e^{\Theta_n(\lambda_n - 2\delta)} = \exp[at_n \log t_n - (ce^{\delta/a} - a - \delta)t_n]. \tag{5.15}$$

Similar argument leads to inequality

$$I_n^3 \le \int_{\lambda_n}^{+\infty} e^{-\lambda + \lambda_n} e^{\Theta_n(\lambda_n)}\, d\lambda = e^{\Theta_n(\lambda_n)} = \exp[at_n \log t_n - (\delta + ce^{\delta/a} - a)t_n]. \tag{5.16}$$

Now, we can finalize the proof. Take $\delta > 0$ such that $be^{-\delta/a} - a - \delta > 0$. Note that

$$ae^{\delta/a} - a - \delta > 0, \quad \delta + ae^{\delta/a} - a > 0.$$

Therefore $c < a$ can be chosen in such a way that (5.14) holds true and

$$ce^{\delta/a} - a - \delta > 0, \quad \delta + ce^{\delta/a} - a > 0.$$

Under such a choice of the constants δ and c, inequalities (5.12), (5.13) (5.15), and (5.16) provide that, for n large enough,

$$\log \int_{\mathbb{R}} e^{-\lambda t_n} dN(\lambda) \le at_n \log t_n - \varepsilon t_n$$

with some positive ε. This contradicts to (5.7) and proves that assumption (5.10) is impossible. □

Remark 5.3 *In the proof of the lower bound in Lemma 5.2, we have combined the upper bound from the same lemma with the estimates, typical for the Laplace method (e.g. [16]). Note that such a structure of the proof is similar to the one for the Gärtner-Ellis theorem (see Section 1.1 in [12]), although we can not deduce the statement of the lemma from the Gärtner-Ellis theorem directly.*

5.4 Proof of Theorem 2.8.

Our argument is based on the following version of the Gärtner-Ellis theorem.

Lemma 5.4 *Consider a sequence $N_n, n \geq 1$ of non-negative monotonous functions on $\mathbb{R}$, which vanish at $-\infty$, and assume that there exist $a > 1, c > 0$, and a sequence $\Upsilon_n \to +\infty$ such that*

$$\frac{1}{\Upsilon_n} \log \int_{\mathbb{R}} e^{-\mu \Upsilon_n x} dN_n(x) \to I(\mu) = c\mu^a, \quad n \to \infty, \quad \mu > 0. \tag{5.17}$$

Then

$$\frac{1}{\Upsilon_n} \log N_n(-x) = -I^*(x)\Big(1 + o(1)\Big), \quad n \to \infty$$

uniformly by $x \in [A, B]$ for every $[A, B] \subset (0, +\infty)$. Here

$$I^*(x) = \sup_{\mu > 0} \Big[\mu x - I(\mu)\Big] = \Big(c(a-1)\Big)^{-\frac{1}{a-1}} \left(\frac{a-1}{a}\right)^{\frac{a}{a-1}} x^{\frac{a}{a-1}}.$$

The only difference between conditions of Lemma 5.4 and standard assumptions of the Gärtner-Ellis theorem is that functions N_n are not assumed to be distribution functions, and are allowed to define non-probability measures. One can see easily that this difference is inessential, and Lemma 5.4 can be proved in the same way with the Gärtner-Ellis theorem (or with Lemma 5.2 above).

Corollary 5.5 *Let the field $\{N(\lambda, \gamma), \lambda \in \mathbb{R}, \gamma > 0\}$ be such that*

(a) every function $N(\cdot, \gamma), \gamma > 0$ is non-negative and non-decreasing;

(b) for every $t > 0$,

$$\tilde{N}(t, \gamma) := \int_{\mathbb{R}} e^{-\lambda t} N(d\lambda, \gamma) < \infty;$$

(c) for given $a > 1, b \in \mathbb{R}$, $c > 0$, and given sequences $\lambda_n > 0, \gamma_n > 0, n \geq 1$ with $\lambda_n^a \gamma_n^{-b} \to \infty$,

$$\log \tilde{N}\left(\mu \lambda_n^{\frac{1}{a-1}} \gamma_n^{-\frac{b}{a-1}}, \gamma_n\right) = c\mu^a \lambda_n^{\frac{a}{a-1}} \gamma_n^{-\frac{b}{a-1}} \Big(1 + o(1)\Big), \quad n \to \infty, \quad \mu > 0. \tag{5.18}$$

Then

$$\log N(-\lambda_n x, \gamma_n) = \Big(c(a-1)\Big)^{-\frac{1}{a-1}} \left(\frac{a-1}{a}\right)^{\frac{a}{a-1}} \lambda_n^{\frac{a}{a-1}} \gamma_n^{-\frac{b}{a-1}} x^{\frac{a}{a-1}} \Big(1 + o(1)\Big), \quad n \to \infty \tag{5.19}$$

uniformly by $x \in [A, B]$ for every $[A, B] \subset (0, +\infty)$.

Proof: Put $\Upsilon_n = \lambda_n^{\frac{a}{a-1}} \gamma_n^{-\frac{b}{a-1}}$,

$$N_n(x) = N(\lambda_n x, \gamma_n), \quad x \in \mathbb{R}.$$

Because $a > 1$ and $\lambda_n^a \gamma_n^{-b} \to \infty$, we have $\Upsilon_n \to \infty$. Condition (5.18) provides (5.17). Henceforth (5.19) follows by Lemma 5.4. □

Now, we can finalize the proof of Theorem 2.8.

Statement I. Take

$$a = b = d/p, \quad c = \nu \omega_d \theta^{d/p} \left(\frac{p}{d-p} \right) \Gamma \left(\frac{2p-d}{p} \right).$$

For given sequences $\lambda_n < 0$, $\gamma_n > 0, n \geq 1$ and arbitrary $\mu > 0$ denote

$$t_n = \mu(-\lambda_n)^{\frac{1}{a-1}} \gamma_n^{-\frac{b}{a-1}}, \quad \alpha_{t_n} = 1/\gamma_n. \tag{5.20}$$

Condition $(-\lambda_n)^{\frac{p}{d}}/\gamma_n \to \infty$ yields $t_n \to \infty$, and condition $(-\lambda_n)^{\frac{d+2-p}{2}}/\gamma_n \to \infty$ yields $\alpha_{t_n} = o(t_n)$. Therefore (5.18) is provided by (2.1). In addition, we have $(-\lambda_n)/\gamma_n \to \infty$ because $p/d < 1$, $(d+2-p)/2 > 1$, and consequently $(-\lambda_n)^a \gamma_n^{-b} \to \infty$. Applying Corollary 5.5 with $x = -1$ and λ_n replaced by $-\lambda_n$, we obtain the required statement.

Statement II. Take

$$a = \frac{d+4-2p}{d+2-2p}, \quad b = \frac{4}{d+2-2p}, \quad c = \mathsf{C}_2$$

and keep the notation (5.19). Condition $(-\lambda_n)^{\frac{d+4-2p}{4}}/\gamma_n \to \infty$ yields $(-\lambda_n)^a \gamma_n^{-b} \to \infty$. Condition $(-\lambda_n)^{\frac{d+2-p}{2}}/\gamma_n \to 0$ yields $t_n^{\frac{d+2-p}{d+2}} = o(\alpha_{t_n})$. Finally, these two conditions yield $\lambda_n \to 0-$, and consequently $(-\lambda_n)^{\frac{d+2-2p}{4}}/\gamma_n \to \infty$ which is equivalent to $t_n \to \infty$.

Therefore (5.18) is provided by (2.3). Applying Corollary 5.5 with $x = -1$ and λ_n replaced by $-\lambda_n$, we obtain the required statement.

Statement III. In the critical case, one can transform easily (2.2) to

$$\lim_{t\to\infty} \left(\frac{\alpha_t}{t} \right)^{d/p} \log \mathbb{E} \otimes \mathbb{E}_0 \exp \left[-\frac{1}{\alpha_t} \int_0^t \overline{V}(B_{\kappa s})\, ds \right] = \mathsf{C}_\psi.$$

Using this relation and following the proof of statement I with appropriate modifications, we obtain the required statement. □

Acknowledgement. The authors are grateful for the referee for useful comments.

References

[1] de Acosta, A. (1983). Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm. *Ann. Probab.* **11**, 78 – 101.

[2] Adler, R.J. (1981) The Geometry of Random Fields, New York, Wiley.

[3] Adams, R.A. (1975). Sobolev spaces, Pure and Applied Mathematics, 65, New York-London: Academic Press.

[4] Anderson, P.W. (1958). Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492 – 1505.

[5] Bass, R., Chen, X. and Rosen, J. (2009). Large deviations for Riesz potentials of additive processes. *Annales de l'Institut Henry Poincare* **45**, 626 – 666.

[6] van den Berg, M., Bolthausen, E. and den Hollander, F. (2005). Brownian survival among Poissonian traps with random shapes at critical intensity. *Probab. Theory Related Fields* **132** 163 – 202.

[7] Bezerra, S., Tindel, S. and Viens, F. (2008). Superdiffusivity for a Brownian polymer in a continuous Gaussian environment. *Ann. Probab.* **36**, 1642 – 1675.

[8] Biskup, M. and König, W. (2001). Screening effect due to heavy lower tails in one-dimensional parabolic Anderson model. *J. Statist. Phys.* **102**, 1253 – 1270.

[9] Borkar, V. S. *Probability Theory: An advanced Course.* Springer, New York, NY 1995.

[10] Carmona, R.A. and Molchanov, S.A. (1994). Parabolic Anderson problem and intermittency. Amer. Math. Soc., Providence, RI.

[11] Carmona, R.A. and Viens, F.G. (1998). Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter, *Stochastics,* **62(3&4)**, 251 – 273.

[12] Chen, X. *Random Walk Intersections: Large Deviations and Related Topics.* Math. Surv. Mono. **157**, Providence 2009.

[13] Chen, X. (2010). Quenched asymptotics for Brownian motion of renormalized Poisson potential and for the related parabolic Anderson models. *Ann. Probab.* (to appear).

[14] Chen, X. and Kulik, A. M. (2011). Brownian motion and parabolic Anderson model in a renormalized Poisson potential. *Annales de l'Institut Henry Poincare*, DOI: 10.1214/11-AIHP419.

[15] Chen, X. and Rosen, J. (2010). Large deviations and renormalization for Riesz potentials of stable intersection measures. *Stoch. Proc. Appl.* (to appear)

[16] Copson, E.T. (1965). Asymptotic expansions, Cambridge Uni. Press, Cambridge.

[17] Cranston, M., Gauthier, D. and Mountford, T. S. (2010). On large deviations for the parabolic Anderson model. *Probab. Theory Related Fields* **147**, 349 – 378.

[18] Dalang, R. C. and Mueller, C. (2009). Intermittency properties in a hyperbolic Anderson problem. *Annales de l'Institut Henry Poincare* **45**, 1150 – 1164.

[19] Donsker, M. D. and Varadhan, S. R. S. (1975). Asymptotic evaluation of certain Markov process expectations for large time. I. *Comm. Pure Appl. Math.,* **28(1)**, 1 -– 47.

[20] Donsker, M. D. and Varadhan, S. R. S. (1975). Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math., 28(4)*, 525 – 565.

[21] Doob, J. L. (1953). Stochastic processes, New York, Wiley.

[22] Erdös, L. (1998). Lifschitz tail in a magnetic field: The nonclassical regime. *Probab. Theory Relat. Fields* **112**, 321 -– 371.

[23] Erdös, L. (2001). Lifschitz tail in a magnetic field: Coexistence of classical and quantum behavior in the borderline case. *Probab. Theory Relat. Fields* **121**, 219 -– 236.

[24] Feller, W. (1971) An Introduction to Probability Theory and Its Applications, vol. 2, Wiley.

[25] Feng, J. and Kurtz, T.G. (2006). Large Deviations for Stochastic Processes, Mathematical Surveys and Monographs, vol. 131, American Mathematical Society, Providence, RI.

[26] Fikhtengoltz, G. (1948). A Course in Differential and Integral Calculus [in Russian], Moscow: Gostekhizdat.

[27] Florescu, I. and Viens, F. (2006). Sharp estimation of the almost-sure Lyapunov exponent for the Anderson model in continuous space. *Probab. Theory Related Fields* **135**, 603 – 644.

[28] Fukushima, R. (2010). Second order asymptotics for Brownian motion among a heavy tailed Poissonian potential. (manuscript)

[29] Gärtner, J., den Hollander, F. and Maillard, G.(2007). Intermittency on catalysts: symmetric exclusion. *Electron. J. Probab.* **12**, 516 – 573.

[30] Gärtner, J. and König, W. (2000). Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10**, 192 – 217.

[31] Gärtner, J., König, W. and Molchanov, S. (2000). Almost sure asymptotics for the continuous parabolic Anderson model. *Probab. Theory Related Fields* **118**, 547 – 573.

[32] Gärtner, J. and Molchanov, S. A. (1990). Parabolic problem for the Anderson model. *Comm. Math. Phys.* **132**, 613 – 655.

[33] Germinet, F., Hislop, P. and Klein, A. (2007). Localization for Schrödinger operators with Poisson random potential. *J. Europ. Math. Soc.* **9**, 577 – 607.

[34] Gohberg, I. C. and Krein, M. G. (1969). Introduction to the theory of linear nonselfadjoint operators, Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, R.I.

[35] Hundertmark, D., Kirsch, W. and Warzel, S. (2003). Classical magnetic Lifshits tails in three space dimensions: Impurity potentials with slow anisotropic decay. *Markov Process. Relat. Fields* **9**, 651 – 660.

[36] Kac, M. (1950). On Some Connections between Probability Theory and Differential and Integral Equations, in Proc. 2nd Berkeley Symp. Math. Statist. Probab., Univ. of Calif. Press.

[37] Kirsch, W., Khorunzhiy, O. and Müller, P. (2006). Lifshits tails for spectra of Erdös–Rényi random graphs. *Ann. Appl. Probab.* **16**, 295 – 309.

[38] Kirsch, W. and Martinelli, F. (1982). On the density of states of Schrödinger operators with a random potential. *J. Phys. A,* **15(7)**, 2139 – 2156.

[39] Kirsch, W. and Martinelli, F. (1983). Large deviations and Lifshitz singularity of the integrated density of states of random Hamiltonians. *Commun. Math. Phys.* **89**, 27 – 40.

[40] Kirsch, W. and Martinelli, F. (1982). On the spectrum of Schrodinger operators with a random potential. *Comm. Math. Phys.* **85(3)**, 329–350.

[41] Kirsch, W. and Metzger, B. (2007) The integrated density of states for random Schrodinger operators. In: Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, Part 2, Proc. Symp. Pure Math., vol. 76, 649 – 696, Amer. Math. Soc., Providence, RI.

[42] Kirsch, W. and Simon, B. (1986). Lifshitz tails for periodic plus random potentials. *J. Stat. Phys.* **42**, 799 – 808.

[43] Kirsch, W. and Warzel, S. (2005). Lifshits tails caused by anisotropic decay: the emergence of a quantum-classical regime. *Math. Phys. Anal. Geom.* **8**, 257 – 285.

[44] Kirsch, W. and Warzel, S. (2006). Anderson localization and Lifshits tails for random surface potentials. *J. Funct. Anal.* **230**, 222 – 250.

[45] Klopp, F. (2002). Weak disorder localization and Lifshitz tails: Continuous Hamiltonians. *Ann. Henri Poincaré* **3**, 711 – 737.

[46] Klopp, F. and Pastur, L. (1999). Lifshitz tails for random Schrödinger operators with negative singular Poisson potential. *Commun. Math. Phys.* **206**, 57 – 103.

[47] Komorowski, T. (2000). Brownian motion in a Poisson obstacle field. *Séminaire Bourbaki,* **1998/99**, 91 – 111.

[48] Leschke, H. and Warzel, S. (2004). Quantum-classical transitions in Lifshits tails with magnetic fields. *Phys. Rev. Lett.* **92**, 086402.

[49] Lifshitz, I. (1963). Structure of the energy spectrum of impurity bands in disordered solid solutions. *Soviet Physics JETP* **17**, 1159 – 1170.

[50] Lifshitz, I. (1965). Energy spectrum structure and quantum states of disordered condensed systems. *Soviet Physics Usp.* **7**, 549 – 573.

[51] Marcus, M. B. and Rosinski, J. (2005). Continuity and boundedness of infinitely divisible process: A Poisson point process approach. *J. Theor. Probab.* **18**, 109 – 160.

[52] Nakamura, S. (2000). Lifshitz tail for Schrödinger operator with random magnetic field. *Commun. Math. Phys.* **214**, 565 – 572.

[53] Ôkura, H. (1981). An asymptotic property of a certain Brownian motion expectation for large time. *Proc. Japan Acad. Ser. A Math. Sci.,* **57(3)**, 155 – 159.

[54] Pastur, L. A. (1973). Spectra of random self-adjoint operators. *Russian Math. Surveys,* **28(1)**, 1 – 67.

[55] Pastur, L. A. (1977). The behavior of certain Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential. *Teoret. Mat. Fiz.,* **32(1)**, 88 – 95.

[56] Pastur, L. and Figotin, A. (1991). Spectra of Random and Almost-periodic Operators, Springer, Berlin.

[57] Povel, T. (1999). Confinement of Brownian motion among Poissonian obstacles in $\mathbb{R}^d, d \geq 3$. *Probab. Theory Related Fields* **114**, 177 – 205.

[58] Rajput, B S.and Rosinski, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82**, 451 – 487.

[59] Rockafellar, R.T. (1970). Convex analysis, Princeton University Press, Princeton, NJ.

[60] Rosinski, J. (1989). On path properties of certain infinitely divisible process. *Stoc. Proc. Appl* **33**, 73 – 87.

[61] Schaefer, H.H. (1971). Topological vector spaces, Springer, Berlin.

[62] Shi, Z. (1996) Small ball probabilities for a Wiener process under weighted sup-norms, with applications to the supremum of the Bessel local times. *J. Theor. Probab.,* **9(4)**, 915 – 929.

[63] Simon, B. (1985). Lifshitz tails for the Anderson model. *J. Phys.* **38**, 65 – 76.

[64] Stollmann, P. (1999). Lifshitz asymptotics via linear coupling of disorder. *Math. Phys. Anal. Geom.* **2**, 279 – 289.

[65] Stolz, G. (2000). Non-monotonic random Schrödinger operators: the Anderson model. *Journal of Math. Analysis and Appl.* **248 (1)**, 173 – 183.

[66] Sznitman, A-L. (1998). Brownian motion, obstacles and random media. Springer-Verlag, Berlin.

Xia Chen
Department of Mathematics
University of Tennessee
Knoxville TN 37996, USA
xchen@math.utk.edu

Alexy M. Kulik
Institute of Mathematics
Ukrainian National Academy of Sciences
Kyiv 01601, Ukraine
kulik@imath.kiev.ua