Spatial Brownian motion in renormalized Poisson potential: A critical case

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Abstract

Let B_s be a three dimensional Brownian motion and $\omega(dx)$ be an independent Poisson field on $\mathbb{R}^3$. It is proved that for any $t > 0$, conditionally on $\omega(\cdot)$,

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \begin{cases} < \infty \ a.s. & \text{if } \theta < 1/16, \\ = \infty \ a.s. & \text{if } \theta > 1/16, \end{cases} \tag{$*$}$$

where $\overline{V}(x)$ is the renormalized Poisson potential

$$\overline{V}(x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \big[\omega(dy) - dy\big].$$

Then the long term behavior of the quenched exponential moment $(*)$ is determined for $\theta \in (0, 1/16)$ in the form of integral tests.

This paper exhibits and builds upon the interrelation between the exponential moment $(*)$ and the celebrated Hardy's inequality

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \le 4\|\nabla f\|_2^2, \qquad f \in W^{1,2}(\mathbb{R}^3).$$

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1 Introduction

Consider a particle moving randomly according to a standard d-dimensional Brownian motion B_s in $\mathbb{R}^d$. Independently, there is a family of obstacles randomly placed in the space $\mathbb{R}^d$ according to a Poisson field $\omega(dx)$ (i.e., a Poisson random measure). Assume that each obstacle has mass 1 and the Poisson field $\omega(dx)$ has the Lebesgue measure dx as its intensity measure. Throughout this paper, "$\mathbb{P}_z$" and "$\mathbb{E}_z$" will respectively stand for the probability law and expectation relative to Brownian motion B_s with $B_0 = z$. Notation "$\mathbb{P}$" and "$\mathbb{E}$" will be used for the probability law and expectation, respectively, relative to Poisson field $\omega(dx)$.

Given a *shape function* $K(x)$ (as known by mathematicians, and a *point-mass potential* by physicists) on $\mathbb{R}^d$, the potential associated with the random mass distribution $\omega(\cdot)$ is given by

$$V(x) = \int_{\mathbb{R}^d} K(x-y)\,\omega(dy),$$

and is called a *Poisson potential.* The quantity

$$t^{-1}\int_0^t V(B_s)\,ds$$

represents the average Poisson potential along a Brownian trajectory. More important quantities of interest are the respective *annealed* and *quenched* exponential moments

$$\mathbb{E}\otimes\mathbb{E}_0\exp\left\{\pm\int_0^t V(B_s)\,ds\right\} \text{ and } \mathbb{E}_0\exp\left\{\pm\int_0^t V(B_s)\,ds\right\}. \tag{1.1}$$

Knowledge of their asymptotic behavior at $t\to\infty$ is fundamental to our understanding of parabolic Anderson models (see Corollary 2.4). The reader is referred to [1], [2], [3], [8], [9], [10], [11], [14], [15], [16], [23], [24], [29], and [30] for the existing literature on this topic.

In the classical literature on this subject, the function $K(x)$ of the Poisson potential was assumed to be bounded and/or compactly supported. However, in physics many point-mass potential functions are unbounded. For example, in scattering theory, power potentials $K(x) = \mp|x|^{-p}$ ($d = 3$) play a significant role, see [20], [21]. The parameter $p > 0$ is called the index of attraction or repulsion, respectively. When $p = 1$, we have Coulomb interaction. $K(x) = |x|^{-4}$ is referred to as Maxwellian potential, $K(x) = -|x|^{-4}$ is important in the study of ionized gases, $K(x) = |x|^{-2}$ is known as a centrifugal potential, see [20, Chapters 1-7, 3-6].

Donsker and Varadhan [10], Pastur [23], and Fukushima [13] studied the asymptotics in (1.1) with the *negative* signs for the case $K(x) = |x|^{-p}$. Specifically, in [10] and [23] the asymptotics of the annealed moment were obtained when $p > d+2$ and $d < p < d+2$, respectively. In [13] both annealed and quenched moments are determined for

$d < p < d+2$. In these papers the singularity of $K(x)$ was circumvented by applying truncations near the origin.

When $p \le d$, the Poisson potential becomes infinite a.s. To deal with this problem, in the recent paper [6], the renormalized Poisson potential

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^p} \big[\omega(dy) - dy\big] \qquad x \in \mathbb{R}^d$$

was introduced; it exists as a random integral if and only if $d/2 < p < d$, see Corollary 1.3 and physical arguments behind the renormalization in [6]. In the same work, integrabilities associated with the construction of the annealed and quenched exponential moments

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\bigg\{ \pm\theta \int_0^t \overline{V}(B_s)\, ds\bigg\} \ \text{ and } \ \mathbb{E}_0 \exp\bigg\{ \pm\theta \int_0^t \overline{V}(B_s)\, ds\bigg\}$$

were investigated. In the range $d/2 < p < d$, the time-integral

$$\int_0^t \overline{V}(B_s)\, ds$$

is well defined and satisfies the annealed integrability (and therefore quenched integrability as well)

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\bigg\{ -\theta \int_0^t \overline{V}(B_s)\, ds\bigg\} < \infty$$

for every $\theta > 0$ and $t > 0$. We refer the recent papers [5] and [7] for the study on the asymptotics of quenched and annealed negative exponential moments, respectively.

However, the case of exponential moments with positive coefficient is far more delicate. By [6, Theorem 1.4], for every $\theta > 0$ and $t > 0$

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\bigg\{ \theta \int_0^t \overline{V}(B_s)\, ds\bigg\} = \infty.$$

On the other hand, by [6, Theorem 1.5] the quenched exponential moment exists for any $\theta > 0$, $t > 0$, and $p < 2$, as we have with probability 1,

$$\mathbb{E}_0 \exp\bigg\{ \theta \int_0^t \overline{V}(B_s) ds\bigg\} \begin{cases} < \infty \ a.s. & \text{if } p < 2, \\ = \infty \ a.s. & \text{if } p > 2. \end{cases} \tag{1.2}$$

Furthermore, the first author recently observed in [5] that

$$\lim_{t\to\infty} \frac{1}{t}\Big(\frac{\log\log t}{\log t}\Big)^{\frac{2}{2-p}} \log \mathbb{E}_0 \exp\bigg\{ \theta \int_0^t \overline{V}(B_s) ds\bigg\} \tag{1.3}$$
$$= \frac{1}{2} p^{\frac{p}{2-p}} (2-p)^{\frac{4-p}{2-p}} \Big(\frac{d\theta\sigma(d,p)}{2+d-p}\Big)^{\frac{2}{2-p}} \qquad a.s. - \mathbb{P},$$

where $\sigma(d,p) > 0$ is the best constant of the inequality

$$\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^p} dx \le C \|f\|_2^{2-p} \|\nabla f\|_2^p, \quad f \in W^{1,2}(\mathbb{R}^d). \tag{1.4}$$

The only unanswered case is $p = 2$ and necessarily $d = 3$ (recall the constraint $d/2 < p < d$); we will call it the *critical case*. Our results will justify this name.

The present paper is devoted to the study of the quenched exponential moment

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)\, ds\right\}, \quad \theta > 0, \;\; t > 0 \tag{1.5}$$

in the critical case, i.e., when

$$\overline{V}(x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \left[\omega(dy) - dy\right], \quad x \in \mathbb{R}^3. \tag{1.6}$$

Remark that in physics $K(x) = \pm|x|^{-2}$ ($x \in \mathbb{R}^3$) is a transition potential. It lies on the boundary between the classes of regular ($p < 2$) and singular ($p > 2$) potentials separating fundamentally different physical systems, see [12, Section II]. For example, in nonrelativistic quantum mechanics a particle in an attractive singular potential has infinite negative energy. The particle in this case "falls" to the center with infinite velocity. However, if $p < 2$, the energy is finite, solutions to physical problems are uniquely given, and there is no problem with their physical interpretation, see [12, Sections I–II.A].

It was already noticed in [6, Theorem 1.5] that the quenched exponential moment in (1.5) is infinite a.s. for θ sufficiently large and all $t > 0$. A natural question is whether this is true for all $\theta > 0$. Fortunately, the answer is negative. If so, what is the critical value θ_0 where the phase transition occurs? (It is even not clear that θ_0 must be deterministic.) We prove that $\theta_0 = 1/16$. Then we establish the asymptotic behavior of the quenched exponential moment in (1.5), showing that it is fundamentally different from (1.3) since the strong law of large numbers does not hold in the critical case. These results are the consequences of the interrelation with Hardy's inequality (2.9) via a chain of asymptotic equivalences sketched in (2.10). In conclusion, the critical case of $p = 2$ is substantially different from the other cases. The only continuity appears in Hardy's inequality, where a formal substitution of $p = 2$ in (1.4) gives (2.9).

The paper is organized as follows. In section 2 we present main results and their application to the parabolic Anderson model. In section 3, we develop key tools for the estimations needed in later sections. Some of these tools are interesting for their own novelty. Slepian-type correlation inequalities for infinite divisible fields (cf. [28]) are provided (Lemma 3.1) for the proof of (2.12), where the random variables $\omega(z + Q_{b\delta})$ ($z \in 2\delta\mathbb{Z}^3$) are correlated as $b > 1$. An estimation by a chaining maximal inequality (Lemma 3.2) allows the truncation of the Poisson potential at the proper level.

Feynman-Kac formula plays a crucial role in the proof of the main results in this paper. A clean and simple minorization bound (Lemma 3.4) for Brownian density killed upon exit leads to a Feynman-Kac lower bound (Lemma 3.5) adoptable to our setting. For the Feynman-Kac upper bound (Lemma 3.6) with the random potential $\overline{V}(\cdot)$, we use the independence between the Brownian exit time and Brownian exit location from a ball centered at 0. The lower and upper bounds for the main theorems are proved in the sections 4 and 5, respectively. The main ingredients in these two sections are the estimation of the principal eigenvalues of the correspondent initial-boundary value problems that leads to the relation suggested by (2.10) and, the strong laws for extreme values of the Poisson field indicated by (2.11)–(2.12). Section 6 is devoted to Hardy's inequality and related facts.

2 Main results

From now on we will assume $d = 3$, $p = 2$ and that the renormalized Poisson potential $\overline{V}(x)$ is given by (1.6), if not otherwise stated.

Theorem 2.1 *For every $t > 0$,*

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \begin{cases} < \infty \ a.s. & \text{if } \theta < 1/16, \\ = \infty \ a.s. & \text{if } \theta > 1/16. \end{cases} \tag{2.1}$$

In view of the limit law (1.3) obtained in the non-critical case, a natural problem is the asymptotic behaviors in the critical case. Recall that a positive function $\gamma(t)$ on $\mathbb{R}^+$ is said to be regularly varying at infinity if the limit

$$\lim_{t\to\infty} \frac{\gamma(\lambda t)}{\gamma(t)} = c(\lambda)$$

exists for each $\lambda > 0$. A regularly varying function $\gamma(t)$ is said to be slowly varying at infinity, if $c(\lambda) \equiv 1$. From Karamata theory, every regularly varying function $\gamma(t)$ has a representation $\gamma(t) = t^\beta l(t)$, where β is a constant and $l(t)$ is a slowly varying function.

Throughout, $l(t)$ will stand for a slowly varying function at infinity.

Theorem 2.2 *For every $\theta \in (0, 1/16)$*

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \tag{2.2}$$

$$= \begin{cases} 0 \ a.s. & \text{if } \int_1^\infty \frac{dt}{t\cdot l(t)} < \infty, \\ \infty \ a.s. & \text{if } \int_1^\infty \frac{dt}{t\cdot l(t)} < \infty, \end{cases}$$

where $k = \lfloor (8\theta)^{-1} \rfloor$ *is the integer part of* $(8\theta)^{-1}$.

Theorem 2.3 *For every* $\theta \in (0, 1/16)$

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \tag{2.3}$$
$$= \begin{cases} 0 \quad a.s. \quad if \displaystyle\int_1^\infty \frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt = \infty \text{ for some } c > 0, \\ \infty \quad a.s. \quad if \displaystyle\int_1^\infty \frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt < \infty \text{ for every } c > 0, \end{cases}$$

where $k = \lfloor (8\theta)^{-1} \rfloor$ *is as in Theorem 2.2.*

Theorems 2.2–2.3 show rather unexpected behavior of the quenched exponential moments with regard to θ. Indeed, putting θ into different sub-intervals of the partition

$$\left(0, \frac{1}{16}\right) = \left(\frac{1}{24}, \frac{1}{16}\right) \cup \bigcup_{k=3}^\infty \left(\frac{1}{8(k+1)}, \frac{1}{8k}\right]$$

leads to different asymptotic rates. On the other hand, moving θ around within the same sub-interval does not bring any change to the asymptotic behavior of the system.

Our main results indicates that as far as the strong limit is concerned, there is not "right" deterministic normalization to the logarithm of the quenched exponential moment in the critical setting $p = 2$ and $d = 3$. Indeed, by Theorems 2.2–2.3, for any $0 < \theta < 1/16$, and for any positive deterministic function $\gamma(t)$ regularly varying at infinity, with probability 1

$$\limsup_{t\to\infty} \gamma(t)^{-1} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} = 0 \text{ or } \infty$$

and

$$\liminf_{t\to\infty} \gamma(t)^{-1} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} = 0 \text{ or } \infty.$$

This pattern sharply contrasts (1.3) observed in the non-critical setting.

Letting $l(t)$ be some specific functions, we get the following results:

$$\limsup_{n\to\infty} t^{-\frac{k+1}{k-1}} (\log t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} = \infty \quad a.s.$$

On the other hand, for any $\delta > 0$

$$\limsup_{n\to\infty} t^{-\frac{k+1}{k-1}} \big((\log t)(\log\log)^{1+\delta}\big)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} = 0 \quad a.s.$$

As for the liminf behavior,

$$\liminf_{n\to\infty} t^{-\frac{k+1}{k-1}}(\log\log t)^{\frac{2}{3(k-1)}}\log \mathbb{E}_0\exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\} = 0 \quad a.s.$$

On the other hand, for any $l(t) \gg \log\log t$ as $t\to\infty$,

$$\liminf_{n\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{\frac{2}{3(k-1)}}\log \mathbb{E}_0\exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\} = \infty \quad a.s.$$

Theorem 2.1 provides solution to the parabolic Anderson equation

$$\begin{cases} \partial_t u(t,x) = \kappa\Delta u(t,x) + \theta\overline{V}(x)u(t,x) \\ \\ u(0,x) = 1 \end{cases} \tag{2.4}$$

where $\kappa > 0$ is a constant called diffusion coefficient. Indeed, consider the time-space field

$$u_\theta(t,x) = \mathbb{E}_x\exp\left\{\theta\int_0^t \overline{V}(B_{2\kappa s})ds\right\} = \mathbb{E}_x\exp\left\{\frac{\theta}{2\kappa}\int_0^{2\kappa t} \overline{V}(B_s)ds\right\}. \tag{2.5}$$

By translation invariance of the Poisson field, for any $x\in\mathbb{R}^d$

$$\Big\{u_\theta(t,x);\ \ t\ge 0\Big\} \overset{d}{=} \Big\{u_\theta(t,0);\ \ t\ge 0\Big\}. \tag{2.6}$$

By Theorem 2.1, $u_\theta(t,x) < \infty$ a.s. for every $x\in\mathbb{R}^d$ and $t>0$ when $\theta < \kappa/8$. The argument same as the one for Proposition 1.6, [6] concludes that when $\theta<\kappa/8$, $u_\theta(t,x)$ is a mild solution to the equation (2.4) in the sense that

$$\int_0^t p_{2\kappa(t-s)}(x-y)|\overline{V}(y)|u_\theta(s,y)dyds < \infty \qquad x\in\mathbb{R}^3,\ \ t>0$$

and

$$u_\theta(t,x) = 1 + \theta\int_0^t p_{2\kappa(t-s)}(x-y)\overline{V}(y)u_\theta(s,y)dyds \qquad x\in\mathbb{R}^3,\ \ t>0$$

where $p_t(x)$ is the Brownian density.

Further, Theorem 2.2 and Theorem 2.3 lead to the long term property of the stochastic partial different equation (2.4).

Corollary 2.4 *Under $d=3$ and $p=2$, the random field $u_\theta(t,x)<\infty$ for all $\theta<\kappa/8$ and $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$, and $u_\theta(t,x)=\infty$ for all $\theta>\kappa/8$ and $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$. When $\theta<\kappa/8$, $u_\theta(t,x)$ is a mild solution to the equation (2.4) and further, for any $x\in\mathbb{R}^3$,*

$$\limsup_{t\to\infty} t^{-\frac{i+1}{i-1}} l(t)^{-\frac{2}{3(i-1)}}\log u_\theta(t,x) = \begin{cases} 0 & a.s.\ if\ \displaystyle\int_1^\infty \frac{dt}{t\cdot l(t)} < \infty \\ \\ \infty & a.s.\ if\ \displaystyle\int_1^\infty \frac{dt}{t\cdot l(t)} = \infty \end{cases} \tag{2.7}$$

$$\liminf_{t\to\infty} t^{-\frac{i+1}{i-1}} l(t)^{\frac{2}{3(i-1)}} \log u_\theta(t,x) \tag{2.8}$$
$$= \begin{cases} 0 \quad a.s. \text{ if } \int_1^\infty \frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt = \infty \text{ for some } c>0 \\ \\ \infty \quad a.s. \text{ if } \int_1^\infty \frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt < \infty \text{ for every } c>0 \end{cases}$$

where $i = \lfloor (4\theta)^{-1}\kappa \rfloor$ is the integer part of $(4\theta)^{-1}\kappa$.

Given the non-deterministic asymptotic behaviors observed from Theorem 2.2 and Theorem 2.3, the weak law (if any) becomes an interesting problem. In view of Theorem 2.2 and Theorem 2.3, one might expect that the process

$$t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\}$$

converges to a non-degenerated distribution. We leave this problem to future study.

The critical ($p=2$) and non-critical ($p<2$) settings depend on the environment in different ways and therefore are treated differently. In the non-critical case, the quantity

$$\mathbb{E}_0 \exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\}$$

is made by letting Brownian particle stay in a slowly shrinking neighborhood that provides maximal energy from Poisson field among all same-size neighborhoods in a large ball of the radius (roughly) t. Consequently, the limit in (1.3) depends on the extreme values of the Poisson potential $\overline{V}(\cdot)$ over a group of shrinking neighborhoods.

In contrary, the limsup in Theorem 2.2 and the liminf in Theorem 2.3 correspond with the value ∞ to the existence (with a proper asymptotic intensity) of the neighborhoods in which the number of Poisson obstacles exceeds the fixed level $k = [(8\theta)^{-1}]$ within proper distance, and with the value 0 to the absence of such neighborhoods. The central piece behind this strategy is the celebrated Hardy's inequality (Lemma 6.1) which states that

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2}dx \le 4\|\nabla f\|_2^2 \qquad f\in W^{1,2}(\mathbb{R}^3). \tag{2.9}$$

As a consequence (Lemma 6.2) of Hardy's inequality,

$$H(\theta) \equiv \sup_{g\in\mathcal{F}_3}\left\{\theta\int_{\mathbb{R}^3}\frac{g^2(x)}{|x|^2}dx - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\} = \begin{cases} 0 & \text{if } \theta\le 1/8 \\ \\ \infty & \text{if } \theta > 1/8. \end{cases}$$

The connection of Theorem 2.2 and Theorem 2.3 to Hardy's inequality is described roughly by the following almost sure asymptotic relation:

$$\begin{aligned}
&\log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \\
&\approx \log \mathbb{E}_0\left[\exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\};\ \max_{0\le s\le t}|B_s|\le R\right] \\
&\approx t^{\frac{k+1}{k-1}} l(t)^{\pm\frac{2}{3(k-1)}}\left\{o(1)+A(t)H\Big(\theta \max_{\substack{|z|\le R\\ z\in 2\delta\mathbb{Z}^3}}\omega(z+Q_{b\delta})\Big)\right\}
\end{aligned} \tag{2.10}$$

where R increases to ∞ and $\delta>0$ decreases to zero with suitable polynomial rates as $t\to\infty$, where $z+Q_{b\delta}$ represents the cubic $z+[-b\delta, b\delta]^3$ with b being a fixed constant, and where $A(t)$ ranges from a constant (in the argument for the lower bound) to a function increasing to ∞ at a considerable speed (in the argument for the upper bound).

Our strategy is to let the Brownian particle spend significant portion of the duration $[0,t]$ in one of the δ-neighborhoods within the distance R. A principle of choosing R and δ is to make alternation between the behaviors

$$\limsup_{t\to\infty} \max_{\substack{|z|\le R\\ z\in 2\delta\mathbb{Z}^3}}\omega(z+Q_{b\delta})\le (8\theta)^{-1} \quad a.s. \tag{2.11}$$

$$\liminf_{t\to\infty} \max_{\substack{|z|\le R\\ z\in 2\delta\mathbb{Z}^3}}\omega(z+Q_{b\delta})\le (8\theta)^{-1} \quad a.s. \tag{2.12}$$

and their opposites.

Comparing to the extreme value problem in the non-critical setting, the strong laws in (2.11) and (2.12) are much more sensitive to truncation radius R (more precisely, to the number of the δ-neighborhoods that covers the ball $\{|x|\le R\}$), as they corresponds to the polynomial (rather than exponential) decay of the Poissonian tail. To validate the first step in (2.10) in the argument for the upper bounds, on the other hand, one has to take R significantly larger than it is in the proof for the lower bounds. The impact of larger R in (2.11) and (2.12) can be counter-balanced by taking smaller δ, even though this action leads to a further increase of the number of the δ-neighborhoods. The cost of this strategy turns out to be a possibly very large function $A(t)$ appearing on the right hand side of (2.10). An observation unique to the critical setting is the irrelevance of $A(t)$ to the asymptotic behaviors of the system, as the quantity

$$H\Big(\theta \max_{\substack{|z|\le R\\ z\in 2\delta\mathbb{Z}^3}}\omega(z+Q_{b\delta})\Big)$$

is equal to zero eventually (under (2.11)) or infinitely often (under (2.12)).

3 Basic estimates

In this section we give some auxiliary results that will be used in our proofs. We state them separately for a convenient reference. For future reference, all results in this section are established in the space $\mathbb{R}^d$ for $d \geq 1$, except Lemma 3.6 where $d = 3$.

3.1 Association of infinitely divisible fields

Recall that random variables $X_1, \ldots, X_n$ are said to be *associated* if for any bounded measurable functions $f, g : \mathbb{R}^n \mapsto \mathbb{R}$ non-decreasing (equivalently, non-increasing) in each coordinate

$$\mathrm{Cov}(f(X), g(X)) \geq 0. \tag{3.1}$$

Association is a fairly strong property exhibiting positive dependence.

Consider now a non-negative random measure M on $\mathbb{R}^d$, taking independent values on disjoint sets such that $M(A)$ is infinitely divisible with the characteristic function

$$\mathbb{E}\exp\{-uM(A)\} = \exp\Big\{ - m(A) \int_0^\infty (1 - e^{-us})\, \rho(ds) \Big\}, \quad u > 0, \tag{3.2}$$

for every Borel set $A \subset \mathbb{R}^d$ with $m(A) < \infty$, where m is a σ-finite measure on $\mathbb{R}^d$ and ρ is a measure on $(0, \infty)$ such that $\int_0^\infty \min\{s, 1\}\, \rho(ds) < \infty$. M can be viewed as a distribution of obstacles in $\mathbb{R}^d$ having random locations and random masses, so we call it an *infinitely divisible random field.* M is a Poisson field if $m(dx) = dx$ and $\rho(ds) = \delta_1(ds)$. See [25] for more information on infinitely divisible random measures.

Lemma 3.1 *Let M be infinitely divisible random field. Then $M(A_1), \ldots, M(A_n)$ are associated for any Borel sets A_j with $m(A_j) < \infty$, $j = 1, \ldots, n$. In particular, for all $c_1, \ldots, c_n \in \mathbb{R}$*

$$\mathbb{P}(M(A_1) \leq c_1, \ldots, M(A_n) \leq c_n) \geq \prod_{j=1}^n \mathbb{P}(M(A_j) \leq c_j) \tag{3.3}$$

and

$$\mathbb{P}(M(A_1) \geq c_1, \ldots, M(A_n) \geq c_n) \geq \prod_{j=1}^n \mathbb{P}(M(A_j) \geq c_j). \tag{3.4}$$

Proof: It follows from (3.2) that $X = (M(A_1), \ldots, M(A_n))$ has infinitely divisible distribution without Gaussian part and its Lévy measure is concentrated on $\mathbb{R}^d_+$. Thus the components of X are associated [26] (see also [27] for more information on association of infinitely divisible random vectors).

Applying (3.1) recursively for $f = \prod_{j=1}^{n-1} \mathbf{1}_{(-\infty, c_j]}$ and $g = \mathbf{1}_{(-\infty, c_n]}$ ($f = \prod_{j=1}^{n-1} \mathbf{1}_{[c_j, \infty)}$ and $g = \mathbf{1}_{[c_n, \infty)}$, respectively) we obtain (3.3) ((3.4), respectively). $\square$

3.2 Truncating Poisson potentials

In this subsection we study a family of Poisson potentials generated by a smooth truncation of the singular potential kernel $K(x) = |x|^{-p}$, where $p \in (d/2, d)$. The following notation will be used throughout this paper:

α: $\mathbb{R}^+ \longrightarrow [0,1]$ denotes a fixed smooth function with the following properties: $\alpha(\lambda) = 1$ on $[0,1]$, $\alpha(\lambda) = 0$ for $\lambda \geq 3$ and $-1 \leq \alpha'(\lambda) \leq 0$. Let $a > 0$ be fixed but arbitrary. Put

$$L_a(x) = \frac{1 - \alpha(a^{-1}|x|)}{|x|^p} \quad \text{and} \quad \overline{V}_{a,\epsilon}(x) = \int_{\mathbb{R}^d} L_a(x-y)\big[\omega(\epsilon dy) - \epsilon dy\big]. \tag{3.5}$$

Let $D \subset \mathbb{R}^d$ be a fixed bounded set.

Lemma 3.2 *For any $\theta > 0$ and fixed $a > 0$*

$$\mathbb{E}\exp\Big\{\theta \sup_{x\in D} |\overline{V}_{a,1}(x)|\Big\} < \infty. \tag{3.6}$$

Further, for given $\theta > 0$ one can take $a > 0$ large enough so

$$\sup_{0<\epsilon<1} \mathbb{E}\exp\Big\{\theta(\log\epsilon^{-1}) \sup_{x\in D} |\overline{V}_{a,\epsilon}(x)|\Big\} < \infty. \tag{3.7}$$

Proof: Due to similarity, we only prove (3.7). Write

$$\Psi(\lambda) = e^{\lambda} - 1 - \lambda \qquad \lambda \in \mathbb{R}.$$

We have for $\theta > 0$

$$\begin{aligned}\mathbb{E}\exp\Big\{\pm\theta(\log\epsilon^{-1})\overline{V}_{a,\epsilon}(0)\Big\} &= \exp\Big\{\epsilon\int_{\mathbb{R}^d}\Psi\Big(\pm\theta(\log\epsilon^{-1})L_a(y)\Big)\,dy\Big\}\\ &\leq \exp\Big\{\epsilon\int_{\mathbb{R}^d}\Psi\Big(\theta(\log\epsilon^{-1})L_a(y)\Big)\,dy\Big\},\end{aligned}$$

where the inequality follows from the fact that $\Psi(-\lambda) \leq \Psi(\lambda)$ for any $\lambda \geq 0$. Therefore,

$$\mathbb{E}\exp\Big\{\theta(\log\epsilon^{-1})|\overline{V}_{a,\epsilon}(0)|\Big\} \leq 2\exp\Big\{\epsilon\int_{\mathbb{R}^d}\Psi\Big(\theta(\log\epsilon^{-1})L_a(y)\Big)\,dy\Big\}.$$

By a change of variable,

$$\begin{aligned}\int_{\mathbb{R}^d}\Psi\Big(\theta(\log\epsilon^{-1})L_a(y)\Big)\,dy &= (\log\epsilon^{-1})^{d/p}\int_{\mathbb{R}^d}\Psi\Big(\theta L_{a(\log\epsilon^{-1})^{-1/p}}(x)\Big)\,dx\\ &\leq (\log\epsilon^{-1})^{d/p}\int_{\{|x|\geq a(\log\epsilon^{-1})^{-1/p}\}}\Psi\Big(\theta|x|^{-p}\Big)\,dx\\ &\leq (\log\epsilon^{-1})^{d/p}\Big\{\int_{\{1\wedge a(\log\epsilon^{-1})^{-1/p}\leq|x|\leq 1\}} + \int_{\{|x|\geq 1\}}\Big\}\Psi\Big(\theta|x|^{-p}\Big)\,dx\\ &\leq (\log\epsilon^{-1})^{d/p}\Big\{C\exp\Big\{\theta a^{-p}\log\epsilon^{-1}\Big\} + \int_{\{|x|\geq 1\}}\Psi\Big(\theta|x|^{-p}\Big)\,dx\Big\},\end{aligned} \tag{3.8}$$

where the last step follows from the bound $\Psi(\lambda) \le e^{\lambda}$ ($\lambda > 0$). Since

$$\int_{\{|x|\ge 1\}} \Psi\Big(\theta|x|^{-p}\Big)\,dx < \infty,$$

we get for $a > \theta^{1/p}$

$$\sup_{0<\epsilon<1} \mathbb{E}\exp\Big\{\theta(\log\epsilon^{-1})|\overline{V}_{a,\epsilon}(0)|\Big\} < \infty. \tag{3.9}$$

Similarly as at the beginning of the proof, for any $x, y \in D$ with $x \ne y$,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta(\log\epsilon^{-1})\frac{|\overline{V}_{a,\epsilon}(x) - \overline{V}_{a,\epsilon}(y)|}{|x-y|}\Big\}\\
&\quad\le 2\exp\Big\{\epsilon\int_{\mathbb{R}^d}\Psi\Big(\frac{\theta\log\epsilon^{-1}}{|x-y|}|L_a(x-z) - L_a(y-z)|\Big)\,dz\Big\}.
\end{aligned}$$

By the mean value theorem we obtain for all $x, y \in \mathbb{R}^d$

$$\begin{aligned}
|L_a(z-x) - L_a(z-y)| &\le Ca^{-1}|x-z|^{-p}\mathbf{1}(|x-z| > a)|x-y|\\
&\quad + Ca^{-1}|y-z|^{-p}\mathbf{1}(|y-z| > a)|x-y|,
\end{aligned}$$

where $C = p+1$. Using this estimate and the convexity of Ψ we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Psi\Big(\frac{\theta\log\epsilon^{-1}}{|x-y|}|L_a(x-z) - L_a(y-z)|\Big)\,dz\\
&\quad\le \frac{1}{2}\int_{\{|x-z|>a\}}\Psi\Big(2\theta Ca^{-1}\log\epsilon^{-1}|x-z|^{-p})\Big)\,dz\\
&\qquad + \frac{1}{2}\int_{\{|y-z|>a\}}\Psi\Big(2\theta Ca^{-1}\log\epsilon^{-1}|y-z|^{-p})\Big)\,dz\\
&\quad = (\log\epsilon^{-1})^{d/p}\int_{\{|z|\ge a(\log\epsilon^{-1})^{-1/p}\}}\Psi\Big(2\theta Ca^{-1}|z|^{-p}\Big)\,dz.
\end{aligned}$$

By the same estimate as in (3.8) we get

$$\sup_{0<\epsilon<1}\sup_{x\ne y}\mathbb{E}\exp\Big\{\theta(\log\epsilon^{-1})\frac{|\overline{V}_{a,\epsilon}(x) - \overline{V}_{a,\epsilon}(y)|}{|x-y|}\Big\} < \infty.$$

By Theorem D.6, p.313, [4] we

$$\sup_{0<\epsilon<1} \mathbb{E}\exp\Big\{\theta(\log\epsilon^{-1})\sup_{x,y\in D}|\overline{V}_{a,\epsilon}(x) - \overline{V}_{a,\epsilon}(y)|\Big\} < \infty. \tag{3.10}$$

So the desired conclusion follows from (3.9) and (3.10). □

Using above lemma, we derive the following almost sure bounds

Lemma 3.3 *For any $a > 0$*

$$\lim_{R\to\infty} (\log R)^{-1} \sup_{|x|\le R} |\overline{V}_{a,1}(x)| = 0 \quad a.s. \tag{3.11}$$

Further, for any positive sequence ϵ_n such that

$$\limsup_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n} < 1$$

and any constants $\beta > 0$, the strong law

$$\lim_{n\to\infty} \sup_{|x|\le \epsilon_n^{-\beta}} |\overline{V}_{a,\epsilon_n}(x)| = 0 \quad a.s. \tag{3.12}$$

holds for sufficiently large n.

Proof: The ball $\{x \in \mathbb{R}^d;\ |x| \le R\}$ is covered by roughly CR^d unit balls. By homogeneity of the field $\overline{V}(\cdot)$, for each $\delta > 0$

$$\mathbb{P}\Big\{\sup_{|x|\le R} |\overline{V}_{a,1}(x)| \ge \delta \log R\Big\} \le CR^d \mathbb{P}\Big\{\sup_{|x|\le 1} |\overline{V}_{a,1}(x)| \ge \delta \log R\Big\}$$
$$\le CR^{-(\theta\delta - d)} \mathbb{E} \exp\Big\{\theta \sup_{|x|\le 1} |\overline{V}_{a,1}(x)|\Big\}.$$

Take θ sufficiently large so $\theta\delta - d \ge 1$. By (3.6) the exponential moment on the right hand side is finite. Thus,

$$\sum_n \mathbb{P}\Big\{\sup_{|x|\le 2^n} |\overline{V}_{a,1}(x)| \ge \delta \log 2^n\Big\} < \infty.$$

Notice that $\delta > 0$ can be arbitrarily small. By Borel-Cantelli lemma

$$\lim_{n\to\infty} (\log 2^n)^{-1} \sup_{|x|\le 2^n} |\overline{V}_{a,1}(x)| = 0 \quad a.s.$$

Hence, (3.11) follows from the fact that $\sup_{|x|\le R} |\overline{V}_{a,1}(x)|$ is non-decreasing in R.

We now come to the proof of (3.12). First notice that the ball $B(0, \epsilon_n^{-\beta})$ can be covered by $C\epsilon_n^{-d\beta}$ balls of radius 1. Thus, for each $\delta > 0$

$$\mathbb{P}\Big\{\sup_{|x|\le \epsilon_n^{-\beta}} |\overline{V}_{a,\epsilon_n}(x)| \ge \delta\Big\} \le C\epsilon_n^{-d\beta} \mathbb{P}\{\sup_{|x|\le 1} |\overline{V}_{a,\epsilon_n}(x)| \ge \delta\Big\}$$
$$\le C\epsilon_n^{\theta\delta - d\beta} \mathbb{E} \exp\Big\{\theta\big(\log \epsilon_n^{-1}\big) \sup_{|x|\le 1} |\overline{V}_{a,\epsilon_n}(x)|\Big\}.$$

Take $\theta > 0$ sufficiently large so $\theta\delta - d\beta \ge 1$. By (3.7), the exponential moment on the right hand side is bounded uniformly over n when $a > 0$ is sufficiently large. Hence,

$$\sum_n \mathbb{P}\Big\{\sup_{|x|\le \epsilon_n^{-\beta}} |\overline{V}_{a,\epsilon_n}(x)| \ge \delta\Big\} < \infty.$$

Therefore, (3.12) follows from Borel-Cantelli lemma. □

3.3 Lower bound on Brownian motion before it exits a ball

$B(x,r)$ will denote a ball in $\mathbb{R}^d$ with center at x and radius r.

Lemma 3.4 *For every $R, t > 0$ and a Borel set $A \subset \mathbb{R}^d$*

$$\mathbb{P}_0\big(B_t \in A,\ \max_{0\le s\le t}|B_s| \le 2R\big) \geq \mathbb{P}_0\big(B_t \in A\cap B(0,R)\big)\ \mathbb{P}_0\big(\max_{0\le s\le 1}|B_s^0| \le Rt^{-1/2}\big), \tag{3.13}$$

where B_s^0 is the Brownian bridge in $\mathbb{R}^d$.

Proof: Let $B_s^0 = B_s - sB_1$ be a Brownian bridge on $[0,1]$ taking values in $\mathbb{R}^d$ and let Z be independent of B_s standard Gaussian random vector in $\mathbb{R}^d$. Then $X_s = B_s^0 + sZ$ is a standard Brownian motion on $0 \le s \le 1$ in $\mathbb{R}^d$. We have by scaling

$$\begin{aligned}
\mathbb{P}(B_t \in A,\ \max_{0\le s\le t}|B_s| \le 2R) &= \mathbb{P}(t^{1/2}B_1 \in A,\ \max_{0\le s\le 1}|B_s| \le 2Rt^{-1/2}) \\
&= \mathbb{P}(t^{1/2}Z \in A,\ \max_{0\le s\le 1}|B_s^0 + sZ| \le 2Rt^{-1/2}) \\
&\geq \mathbb{P}(t^{1/2}Z \in A,\ |Z| \le Rt^{-1/2},\ \max_{0\le s\le 1}|B_s^0| \le Rt^{-1/2}) \\
&= \mathbb{P}(B_t \in A\cap B(0,R))\,\mathbb{P}(\max_{0\le s\le 1}|B_s^0| \le Rt^{-1/2}),
\end{aligned}$$

which proves (3.13). □

3.4 Bounds by Feynman-Kac functionals

Given a bounded open domain $D \subset \mathbb{R}^d$, let $W^{1,2}(D)$ be the Sobolev space over D, defined to be the closure of the inner product space consists of the infinitely differentiable functions compactly supported in D under the Sobolev norm

$$\|g\|_H = \Big\{\|g\|^2_{\mathcal{L}^2(D)} + \|\nabla g\|^2_{\mathcal{L}^2(D)}\Big\}^{1/2}.$$

Define

$$\mathcal{F}_d(D) = \Big\{g \in W^{1,2}(D);\ \ \int_D g^2(x)dx = 1\Big\}. \tag{3.14}$$

For any measurable function ζ on D, put

$$\lambda_\zeta(D) = \sup_{g\in\mathcal{F}_d(D)}\Big\{\int_D \zeta(x)g^2(x)dx - \frac{1}{2}\int_D |\nabla g(x)|^2dx\Big\}. \tag{3.15}$$

Clearly, $\lambda_\xi(D) \le \lambda_\eta(D)$ and $\lambda_\zeta(D) \le \lambda_\zeta(D')$ whenever $\xi(x) \le \eta(x)$ $(x \in D)$ and $D \subset D'$.

Let

$$\tau_D = \inf\{s \ge 0; \quad B_s \notin D\}.$$

For any $r > 0$, define $T_r = \tau_{B(0,r)}$ with $B(0,r) = \{x \in \mathbb{R}^d; \quad |x| < r\}$.

Lemma 3.5 *Let $R > 0$ and let $\zeta(x)$ be a function on $\mathbb{R}^d$ such that $K = \sup_{x \in B(0,2R)} \zeta(x) < \infty$. We have that for any $t, t_0 > 0$ satisfying $t_0 < t$,*

$$\int_{B(0,R)} \mathbb{E}_x\bigg[\exp\bigg\{\int_0^t \zeta(B_s)ds\bigg\}; T_R \ge t\bigg]dx \tag{3.16}$$
$$\ge (2\pi t_0)^{d/2} e^{-t_0 K} \exp\Big\{(t+t_0)\lambda_\zeta\big(B(0,R)\big)\Big\},$$

$$\mathbb{E}_0\bigg[\exp\bigg\{\int_0^t \zeta(B_s)ds\bigg\}; T_{2R} \ge t\bigg] \tag{3.17}$$
$$\ge \mathbb{P}_0\Big\{\max_{0 \le s \le 1} |B_s^0| \le R t_0^{-1/2}\Big\} \exp\Big\{-2t_0 K - \frac{R^2}{2t_0}\Big\} \exp\Big\{-t\lambda_\zeta\big(B(0,R)\big)\Big\}$$

where B_s^0 $(0 \le s \le 1)$ is a d-dimensional Brownian bridge.

Proof: By a standard procedure of approximation we may assume that $\zeta(\cdot)$ is Hölder continuous. By Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x\bigg[\exp\bigg\{\int_0^t \zeta(B_s)ds\bigg\}dx; \ T_R \ge t\bigg]$$

solves the initial-boundary value problem

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\Delta u(t,x) + \zeta(x)u(t,x) & (t,x) \in (0,\infty) \times B(0,R), \\ u(0,x) = 1 & x \in B(0,R), \\ u(t,x) = 0 & t > 0 \text{ and } |x| = R. \end{cases}$$

Let $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \cdots$ be the eigenvalues of the operator $(1/2)\Delta + \zeta$ in $\mathcal{L}^2\big(B(0,R)\big)$ with zero boundary condition and initial value 1 in $B(0,R)$ and let $e_k \in \mathcal{L}^2\big(B(0,R)\big)$ be an orthogonal basis corresponding to $\{\lambda_k\}$. By (2.31) in [15],

$$\mathbb{E}_x \exp\bigg\{\int_0^t \zeta(B_s)ds\bigg\}\delta_x(B_t); \ T_R \ge t\bigg] = \sum_{k=1}^\infty e^{t\lambda_k} e_k^2(x) \ge e^{t\lambda_1} e_1^2(x)$$

where $\delta_x(\cdot)$ is the Dirac function on $\mathbb{R}^d$ with concentration at x.

Noticing the fact that $\lambda_1 = \lambda_\zeta\big(B(0,R)\big)$ and integrating both sides we have

$$\int_{B(0,R)} \mathbb{E}_x \exp\left\{\int_0^t \zeta(B_s)ds\right\}\delta_x(B_t);\ T_R \geq t\Bigg] dx \geq \exp\Big\{t\lambda_\zeta\big(B(0,R)\big)\Big\}.$$

In addition, by Markov property

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left\{\int_0^t \zeta(B_s)ds\right\}\delta_x(B_{t+1});\ T_R \geq t\right] \\
&\leq e^{t_0K}\mathbb{E}_x\left[\exp\left\{\int_0^{t-t_0} \zeta(B_s)ds\right\}\delta_x(B_t);\ T_R \geq t-t_0\right] \\
&= e^{t_0K}\mathbb{E}_x\left[\exp\left\{\int_0^{t-t_0} \zeta(B_s)ds\right\}p_{t_0}(B_{t-t_0}-x);\ T_R \geq t-t_0\right]
\end{aligned}$$

where

$$p_{t_0}(y) = \frac{1}{(2\pi t_0)^{d/2}}\exp\left\{-\frac{|y|^2}{2t_0}\right\} \leq \frac{1}{(2\pi t_0)^{d/2}} \qquad y\in\mathbb{R}^d.$$

Hence, we have proved that

$$\begin{aligned}
&\int_{B(0,R)} \mathbb{E}_x \exp\left\{\int_0^{t-t_0} \zeta(B_s)ds\right\};\ T_R \geq t-t_0\Bigg] dx \\
&\qquad \geq (2\pi t_0)^{d/2}e^{-t_0K}\exp\Big\{t\lambda_\zeta\big(B(0,R)\big)\Big\}.
\end{aligned} \tag{3.18}$$

Replacing t by $t+t_0$ leads to (3.16).

On the other hand, using Markov property again

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\int_0^t \zeta(B_s)ds\right\};T_{2R}\geq t\right] \geq e^{-t_0K}\mathbb{E}_0\left[\exp\left\{\int_{t_0}^t \zeta(B_s)ds\right\};T_{2R}\geq t\right] \\
&= e^{-t_0K}\int_{B(0,2R)}\tilde{p}_{t_0}(x)\mathbb{E}_x\left[\exp\left\{\int_0^{t-t_0}\zeta(B_s)ds\right\};T_{2R}\geq t-t_0\right]dx \\
&\geq e^{-t_0K}\int_{B(0,R)}\tilde{p}_{t_0}(x)\mathbb{E}_x\left[\exp\left\{\int_0^{t-t_0}\zeta(B_s)ds\right\};T_{R}\geq t-t_0\right]dx
\end{aligned}$$

where $\tilde{p}_{t_0}(x)$ is the density function of the measure

$$\mu_{t_0}(A) = \mathbb{P}_0\Big\{B_{t_0}\in A,\ T_{2R}\geq t_0\Big\} \qquad A\subset\mathbb{R}^d.$$

By (3.13),

$$\tilde{p}_{t_0}(x) \geq \mathbb{P}_0\Big\{ \max_{0\leq s\leq 1} |B^0_s| \leq R t_0^{-1/2}\Big\} \frac{1}{(2\pi t_0)^{d/2}} \exp\Big\{ -\frac{|x|^2}{2t_0}\Big\}$$
$$\geq \mathbb{P}_0\Big\{ \max_{0\leq s\leq 1} |B^0_s| \leq R t_0^{-1/2}\Big\} \frac{1}{(2\pi t_0)^{d/2}} \exp\Big\{ -\frac{R^2}{2t_0}\Big\} \quad x \in B(0,R).$$

Thus, we conclude that

$$\mathbb{E}_0\bigg[\exp\Big\{\int_0^t \zeta(B_s)ds\Big\}; T_{2R} \geq t\bigg]$$
$$\geq e^{-t_0 K}\mathbb{P}_0\Big\{ \max_{0\leq s\leq 1} |B^0_s| \leq R t_0^{-1/2}\Big\} \frac{1}{(2\pi t_0)^{d/2}} \exp\Big\{ -\frac{R^2}{2t_0}\Big\}$$
$$\times \int_{B(0,R)} \mathbb{E}_x\bigg[\exp\Big\{\int_0^{t-t_0} \zeta(B_s)ds\Big\}; T_R \geq t-t_0\bigg] dx.$$

Finally, (3.17) follows from (3.18). □

Lemma 3.6 *Let $d=3$. For any $\delta>0$ with $\{|x|\leq\delta\}\subset D$,*

$$\mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\};\ \tau_D \geq 2t\bigg]$$
$$\leq \exp\Big\{\theta t \sup_{|x|\leq\delta/2} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\} + \frac{6|D|}{\pi\delta^3}\mathbb{E}_0 \exp\Big\{\sqrt{2\delta}T_1\theta \sup_{x\in D}|\overline{V}_{\frac{\delta}{6},1}(x)|\Big\}\exp\Big\{t\lambda_{\theta\overline{V}}(D)\Big\}$$

conditioning on the event $\Big\{\omega\Big\{|x|\leq\delta\Big\}=0\Big\}$, where $|D|$ is the volume of D and $\overline{V}_{a,\epsilon}(\cdot)$ is defined in (3.5).

Proof: Notice that $\alpha(\lambda)=0$ for $\lambda\geq 3$. Thus, on the event $\{\omega(|x|\leq\delta)=0\}$,

$$\int_{\mathbb{R}^3}\frac{\alpha(6\delta^{-1}|x|)}{|x|^2}\omega(dy) \leq \int_{\{|y|>\delta\}}\frac{1_{\{|y-x|\leq\delta/2\}}}{|x|^2}\omega(dy) = 0$$

whenever $|x|\leq\delta/2$. Consequently,

$$\overline{V}(x) = -C_\delta + \overline{V}_{\frac{\delta}{6},1}(x) \leq \overline{V}_{\frac{\delta}{6},1}(x)$$

where

$$C_\delta = \int_{\mathbb{R}^3}\frac{\alpha(6\delta^{-1}|x|)}{|x|^2}dx.$$

For any $r < \delta/2$, therefore

$$\int_0^{T_r\wedge t}\overline{V}(B_s)ds = \int_0^{\tau_r\wedge t}\overline{V}_{\frac{\delta}{6},1}(B_s)ds \le t\sup_{|x|\le\delta}|\overline{V}_{\frac{\delta}{6},1}(x)|.$$

Thus,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\};\ \tau_D\ge 2t\right] \qquad (3.19)\\
&\quad\le \mathbb{E}_0\left[\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\};\ T_r\le t,\ \tau_D\ge 2t\right] + \exp\left\{t\sup_{|x|\le\delta}|\overline{V}_{\frac{\delta}{6},1}(x)|\right\}.
\end{aligned}$$

Write $\tau_D' = \inf\{t\ge T_r;\ \ B_s\notin D\}$. By Markov property,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\};\ T_r\le t,\ \tau_D\ge 2t\right]\\
&\le \mathbb{E}_0\left[\exp\left\{\theta T_r\Big(\sup_{|x|\le\delta}|\overline{V}_{\frac{\delta}{6},1}(x)| - C_\delta\Big)\right\}\exp\left\{\theta\int_{T_r}^t\overline{V}(B_s)ds\right\};\ T_r\le t,\ \tau_D'\ge 2t\right]\\
&= \mathbb{E}_0\left[\exp\left\{\theta T_r\Big(\sup_{|x|\le\delta}|\overline{V}_{\frac{\delta}{6},1}(x)| - C_\delta\Big)\right\}u_0(t-T_r, B_{T_r});\ T_r\le t\right],
\end{aligned}$$

where

$$\begin{aligned}
u_0(s,x) &= \mathbb{E}_x\left[\exp\left\{\theta\int_0^s\overline{V}(B_u)du\right\};\ \tau_D\ge t+s\right]\\
&\le \mathbb{E}_x\left[\exp\left\{\theta\int_0^s\overline{V}(B_u)du\right\};\ \tau_D\ge t\right] \equiv u_1(s,x) \quad (0\le s\le t).
\end{aligned}$$

Notice that on $\{T_r\le t\}$

$$\begin{aligned}
u_1(t-T_r, B_{T_r}) &\le \exp\left\{-\theta T_r\inf_{x\in D}\overline{V}(x)\right\}\mathbb{E}_{B_{T_r}}\left[\exp\left\{\theta\int_0^t\overline{V}(B_u)du\right\};\ \tau_D\ge t\right]\\
&= \exp\left\{-\theta T_r\inf_{x\in D}\overline{V}(x)\right\}u_2(t, B_{T_r}) \quad \text{(say)}.
\end{aligned}$$

Summarizing our estimate,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\};\ T_r\le t,\ \tau_D\ge 2t\right]\\
&\quad\le \mathbb{E}_0\left[\exp\left\{\theta T_r\Big(\sup_{|x|\le\delta}|\overline{V}_{\frac{\delta}{6},1}(x)| - C_\delta - \inf_{x\in D}\overline{V}(x)\Big)\right\}u_2(t, B_{T_r})\right].
\end{aligned}$$

Recall the classic facts that T_r and B_{T_r} are independent and that B_{T_r} is uniformly distributed on the sphere $\{|x| = r\}$. So the right hand side is equal to

$$\mathbb{E}_0 \exp\left\{\theta T_r\Big(\sup_{|x|\le\delta} |\overline{V}_{\frac{\delta}{6},1}(x)| - C_\delta - \inf_{x\in D}\overline{V}(x)\Big)\right\}\frac{1}{4\pi r^2}\int_{\{|x|=r\}} u_2(t,x)dx.$$

Using fact that $\{|x| \le \delta\} \subset D$ and the bound

$$-\overline{V}(x) \le C_\delta - \overline{V}_{\frac{\delta}{6},1}(x) \le C_\delta + |\overline{V}_{\frac{\delta}{6},1}(x)|$$

we have that

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta T_r\Big(\sup_{|x|\le\delta} |\overline{V}_{\frac{\delta}{6},1}(x)| - C_\delta - \inf_{x\in D}\overline{V}(x)\Big)\right\} \le \mathbb{E}_0 \exp\left\{2\theta T_r \sup_{x\in D} |\overline{V}_{\frac{\delta}{6},1}(x)|\right\}\\
&\le \mathbb{E}_0 \exp\left\{2\theta T_\delta \sup_{x\in D} |\overline{V}_{\frac{\delta}{6},1}(x)|\right\} = \mathbb{E}_0 \exp\left\{\sqrt{2\delta}\theta T_1 \sup_{x\in D} |\overline{V}_{\frac{\delta}{6},1}(x)|\right\}.
\end{aligned}$$

Here we have used the fact that $T_r \le T_{\delta/2} \overset{d}{=} \sqrt{\delta/2}T_1$.

By (3.19), we conclude that

$$\begin{aligned}
&(4\pi r^2)\mathbb{E}_0\left[\exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\};\ \tau_D \ge 2t\right]\\
&\le \mathbb{E}_0 \exp\left\{\sqrt{2\delta}\theta T_1 \sup_{x\in D} |\overline{V}_{\frac{\delta}{6},1}(x)|\right\}\int_{\{|x|=r\}} u_2(t,x)dx\\
&+ (4\pi r^2)\exp\left\{t\sup_{|x|\le\delta} |\overline{V}_{\delta/6,1}(x)|\right\}.
\end{aligned}$$

Integrating the variable r over $[0, \delta/2]$ on the both sides,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\};\ \tau_D \ge 2t\right]\\
&\le \frac{6}{\pi\delta^3}\mathbb{E}_0 \exp\left\{\sqrt{2\delta}\theta T_1 \sup_{x\in D} |\overline{V}_{\frac{\delta}{6},1}(x)|\right\}\int_{\{|x|\le r\}} u_2(t,x)dx\\
&+ \exp\left\{t\sup_{|x|\le\delta} |\overline{V}_{\frac{\delta}{6},1}(x)|\right\}.
\end{aligned}$$

Finally, the desired conclusion follows from the bound

$$\int_{\{|x|\le r\}} u_2(t,x)dx \le \int_D u_2(t,x)dx \le |D|\exp\left\{t\lambda_{\theta\overline{V}}(D)\right\}$$

where the second step follows from Lemma 4.1 in [5]. □

4 Lower bounds

We establish the lower bounds requested by Theorem 2.1, Theorem 2.2 and Theorem 2.3. Let t be either fixed (as in Theorem 2.1 or increase to infinity (as in Theorem 2.2 and Theorem 2.3). Let $\epsilon \to 0$ and $R \to \infty$ either as sequences (when t is fixed) or as functions of t (when $t \to \infty$). The constraint assumed here is that $R^2\epsilon^{2/3}t^{-1} \ge c$ eventually for some constant $c > 0$. Other relations among the parameters introduced above will be specified later according to the context.

By Brownian scaling,

$$\begin{aligned}
\mathbb{E}_0 \exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\} &= \mathbb{E}\exp\Big\{\theta\int_0^{t\epsilon^{-2/3}} \overline{V}_\epsilon(B_s)ds\Big\}\\
&\ge \mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^{t\epsilon^{-2/3}} \overline{V}_\epsilon(B_s)ds\Big\};\ T_{2R} \ge t\epsilon^{-2/3}\bigg],
\end{aligned}$$

where

$$\overline{V}_\epsilon(x) = \int_{\mathbb{R}^3} \frac{1}{|y-x|^2}\big[\omega(\epsilon dy) - \epsilon dy\big].$$

Let $r > 0$ and $a > 0$ be two large but fixed numbers with $r < a$. Consider the decomposition

$$\overline{V}_\epsilon(x) = \overline{V}_{a,\epsilon}(x) + V_{a,\epsilon}(x) - \epsilon C_a$$

where $\overline{V}_{a,\epsilon}(x)$ is defined as in (3.5),

$$V_{a,\epsilon}(x) = \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|y-x|)}{|y-x|^2}\,\omega(\epsilon dy) \ \text{ and } \ C_a = \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x|)}{|x|^2}\,dx.$$

We have

$$\begin{aligned}
&\mathbb{E}_0 \exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\} \qquad (4.1)\\
&\ge \exp\Big\{-\theta t\epsilon^{-2/3}\Big(C_a\epsilon + \sup_{x\in B(0,2R)}|\overline{V}_{a,\epsilon}(x)|\Big)\Big\}\\
&\quad\times \mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^{t\epsilon^{-2/3}} V_{a,\epsilon}(B_s)ds\Big\};\ T_{2R} \ge t\epsilon^{-2/3}\bigg].
\end{aligned}$$

Let $\delta > 0$ be a small but fixed number satisfying $r+\delta < a$. For any $z \in 2r\mathbb{Z}^3\cap B(0, R-r)$ and $x \in \mathbb{R}^d$

$$\begin{aligned}
\theta V_{a,\epsilon}(x) &\ge \theta\int_{\{|y-z|\le\delta\}} \frac{\alpha(a^{-1}|y-x|)}{|y-x|^2}\omega(\epsilon dy)\\
&\ge \theta\omega\big(B(\epsilon^{1/3}z, \epsilon^{1/3}\delta)\big)\frac{\alpha\big(a^{-1}(|z-x|+\delta)\big)}{(|z-x|+\delta)^2} \equiv \zeta_\epsilon^z(x).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^{t\epsilon^{-2/3}} V_{a,\epsilon}(B_s)ds\Big\};\ T_{2R}\ge t\epsilon^{-2/3}\bigg]\\
&\quad\ge \mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^{t\epsilon^{-2/3}} \zeta_\epsilon^z(B_s)ds\Big\};\ T_{2R}\ge t\epsilon^{-2/3}\bigg]\\
&\quad\ge \mathbb{P}_0\Big\{\max_{0\le s\le 1}|B_s^0|\le \frac{R\epsilon^{1/3}}{\delta\sqrt{t}}\Big\}\exp\Big\{t\epsilon^{-2/3}\lambda_{\zeta_\epsilon^z}\big(B(0,R)\big)\Big\}\\
&\qquad\times\exp\Big\{-2\theta\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)t\epsilon^{-2/3}-\frac{R^2\epsilon^{2/3}}{2\delta^2 t}\Big\},
\end{aligned}$$

where the last step follows from (3.17) in Lemma 3.5 with t being replaced by $t\epsilon^{-2/3}$ and $t_0=\delta^2 t\epsilon^{-2/3}$, and from the observation

$$0\le \zeta_\epsilon^z(x)\le \delta^{-2}\theta\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\qquad x\in\mathbb{R}^d.$$

Notice that $B(z,r)\subset B(0,R)$ and that $r+\delta<a$ leads to

$$\zeta_\epsilon^z(x)=\theta\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\frac{1}{(|z-x|+\delta)^2}\qquad x\in B(z,r).$$

By substitution $g(x)\mapsto g(x-z)$, therefore,

$$\begin{aligned}
&\lambda_{\zeta_\epsilon^z}\big(B(0,R)\big)\ge \lambda_{\zeta_\epsilon^z}\big(B(z,r)\big)\\
&\quad=\sup_{g\in\mathcal{F}_3(B(z,r))}\Big\{\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\theta\int_{B(z,r)}\frac{g^2(x)}{(|x-z|+\delta)^2}dx-\frac12\int_{B(z,r)}|\nabla g(x)|^2dx\Big\}\\
&\quad=\sup_{g\in\mathcal{F}_3(B(0,r))}\Big\{\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\theta\int_{B(0,r)}\frac{g^2(x)}{(|x|+\delta)^2}dx-\frac12\int_{B(0,r)}|\nabla g(x)|^2dx\Big\}\\
&\quad=H_{r,\delta}\Big(\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\theta\Big)
\end{aligned}$$

where the function $H_{r,\delta}(\cdot)$ is defined as

$$H_{r,\delta}(\theta)=\sup_{g\in\mathcal{F}_3(B(0,r))}\Big\{\theta\int_{B(0,r)}\frac{g^2(x)}{(|x|+\delta)^2}dx-\frac12\int_{B(0,r)}|\nabla g(x)|^2dx\Big\}.$$

Summarizing our estimates since (4.1),

$$\begin{aligned}
&\mathbb{E}_0\exp\Big\{\theta\int_0^t\overline{V}(B_s)ds\Big\}\\
&\quad\ge \mathbb{P}_0\Big\{\max_{0\le s\le 1}|B_s^0|\le\frac{R\epsilon^{1/3}}{\delta\sqrt{t}}\Big\}\exp\Big\{-\frac{R^2\epsilon^{2/3}}{2\delta^2 t}\Big\}\exp\Big\{t\epsilon^{-2/3}H_{r,\delta}\Big(\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\theta\Big)\Big\}\\
&\qquad\times\exp\Big\{-\theta t\epsilon^{-2/3}\Big(C_a\epsilon+\sup_{x\in B(0,R)}|\overline{V}_{a,\epsilon}(x)|+2\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\big)\Big)\Big\}.
\end{aligned}$$

By the assumption that $R^2\epsilon^{2/3}/t$ is eventually bounded from below, there is a constant $\gamma > 0$ such that

$$\mathbb{P}_0\Big\{\max_{0\le s\le 1}|B_s^0|\le \frac{R\epsilon^{1/3}}{\delta\sqrt{t}}\Big\}\ge\gamma$$

eventually holds. Taking maximum over $z \in 2r\mathbb{Z}^3\cap B(0, R-r)$,

$$\begin{aligned}
&\mathbb{E}_0\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\} &(4.2)\\
&\quad\ge\gamma\exp\left\{-\frac{R^2\epsilon^{2/3}}{2\delta^2 t}\right\}\exp\left\{t\epsilon^{-2/3}H_{r,\delta}\Big(\max_{z\in 2r\mathbb{Z}^3\cap B(0,R-r)}\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\theta\Big)\right\}\\
&\qquad\times\exp\left\{-\theta t\epsilon^{-2/3}\Big(C_a\epsilon+\sup_{x\in B(0,R)}|\overline{V}_{a,\epsilon}(x)|+2\max_{z\in 2r\mathbb{Z}^3\cap B(0,R-r)}\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\Big)\right\}.
\end{aligned}$$

A version of (4.2) is also needed and is derived as follows: By the Brownian scaling,

$$\int_{B(0,\epsilon^{1/3}R)}\mathbb{E}_x\exp\left\{\int_0^t\overline{V}(B_s)ds\right\}dx=\epsilon\int_{B(0,R)}\mathbb{E}_x\exp\left\{\int_0^{t\epsilon^{-2/3}}\overline{V}_\epsilon(B_s)ds\right\}dx.$$

Following the decomposition of $\overline{V}_\epsilon(\cdot)$ the same way as above and then applying (3.16) (instead of (3.17)) with $t_0 = 1$, we have

$$\begin{aligned}
&\int_{B(0,\epsilon^{1/3}R)}\mathbb{E}_x\exp\left\{\int_0^t\overline{V}(B_s)ds\right\}dx &(4.3)\\
&\quad\ge(2\pi)^{d/2}\epsilon\exp\left\{-\delta^{-2}\max_{z\in 2r\mathbb{Z}^3\cap B(0,R-r)}\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3})\big)\right\}\\
&\qquad\times\exp\left\{-t\epsilon^{-2/3}\theta\Big(\epsilon C_a+\sup_{x\in B(0,R)}|\overline{V}_{a,\epsilon}(x)|\Big)\right\}\\
&\qquad\times\exp\left\{(1+t\epsilon^{-2/3})H_{r,\delta}\Big(\max_{z\in 2r\mathbb{Z}^3\cap B(0,R-r)}\omega\big(B(\epsilon^{1/3}z,\epsilon^{1/3}\delta)\big)\theta\Big)\right\}.
\end{aligned}$$

4.1 Lower bound for Theorem 2.1

We show that when $\theta > 1/16$,

$$\mathbb{E}_0\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\}=\infty\quad a.s.\quad\forall t>0. \tag{4.4}$$

Let t be fixed. Taking $\epsilon = 2^{-3n}$ and $R = \delta 2^{2n}$ in (4.2) gives

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \qquad (4.5)\\
&\geq \gamma \exp\left\{-\frac{2^{2n}}{2t}\right\}\exp\left\{t2^{2n}H_{r,\delta}\Big(\theta \max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big)\Big)\right\}\\
&\times \exp\left\{-\theta t 2^{2n}\left\{C_a 2^{-3n} + \sup_{x\in B(0,\delta 2^{2n})}|\overline{V}_{a,2^{-3n}}(x)|\right.\right.\\
&\qquad\left.\left. + 2\max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big)\right)\right\}.
\end{aligned}$$

By (3.12),

$$\lim_{n\to\infty} \sup_{x\in B(0,\delta 2^{2n})} |\overline{V}_{a,2^{-3n}}(x)| = 0 \quad a.s. \tag{4.6}$$

when $a > 0$ is sufficiently large.

We now prove that

$$\limsup_{n\to\infty} \max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) = 2 \quad a.s. \tag{4.7}$$

By homogeneity and increment independence of the Poisson field, The random variables

$$\omega\big(B(2^{-n}z, 2^{-n}\delta)\big);\quad z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)$$

are i.i.d's. Hence,

$$\begin{aligned}
&\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \geq 3\Big\}\\
&\quad\leq \#\big\{2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)\big\}\mathbb{P}\Big\{\omega\big(B(0,2^{-n}\delta)\big)\geq 3\Big\}\\
&\quad\leq C2^{6n}\Big((2^{-n}\delta)^3\Big)^3 = O\Big(2^{-3n}\Big\}.
\end{aligned}$$

Thus,

$$\sum_n \mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \geq 3\Big\} < \infty.$$

By Borel-Cantelli lemma and by the fact that the random variable

$$\max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(z, 2^{-n}\delta)\big)$$

takes integer-values,

$$\limsup_{n\to\infty} \max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \leq 2 \quad a.s.$$

On the other hand, write $A_n = B(0, \delta 2^{2n} - r) \setminus B(0, \delta 2^{2(n-1)})$.

$$\begin{aligned}
&\mathbb{P}\Big\{ \max_{z\in 2r\mathbb{Z}^3\cap A_n} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \le 1\Big\} \\
&= \Big(1 - \mathbb{P}\Big\{\omega\big(B(0, 2^{-n}\delta)\big) \ge 2\Big\}\Big)^{\#\{2r\mathbb{Z}^3\cap A_n\}} \\
&\le \Big(1 - c\delta^3 2^{-6n}\Big)^{\#\{2r\mathbb{Z}^3\cap A_n\}} \le \exp\{-c_0\delta^3\}.
\end{aligned}$$

So we have that

$$\sum_n \mathbb{P}\Big\{ \max_{z\in 2r\mathbb{Z}^3\cap A_n} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \ge 2\Big\} = \infty.$$

Notice that the sequence

$$\max_{z\in 2r\mathbb{Z}^3\cap A_n} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \quad n = 1, 2, \cdots$$

is an independent sequence. By Borel-Cantelli lemma

$$\begin{aligned}
&\limsup_{n\to\infty} \max_{z\in 2r\mathbb{Z}^3\cap B(0,\delta 2^{2n}-r)} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \\
&\ge \limsup_{n\to\infty} \max_{z\in 2r\mathbb{Z}^3\cap A_n} \omega\big(B(2^{-n}z, 2^{-n}\delta)\big) \ge 2 \quad a.s.
\end{aligned}$$

By the fact that $\theta > 16^{-1}$ and by Lemma 6.2,

$$\lim_{\substack{\delta\to 0^+ \\ r\to\infty}} H_{r,\delta}(2\theta) = \sup_{g\in\mathcal{F}_3} \left\{ 2\theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx \right\} = \infty.$$

Therefore, one can take δ sufficiently small, and r sufficiently large, so we have $H_{r,\delta}(2\theta) > 2\theta + 2^{-1}t^{-2}$. Finally, the requested (4.4) follows from (4.5), (4.6), (4.7). $\Box$

4.2 Lower bound for Theorem 2.2

Recall that $0 < \theta < 1/16$ and $k = [(8\theta)^{-1}]$. We prove

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{ \theta \int_0^t \overline{V}(B_s) ds \right\} = \infty \quad a.s. \tag{4.8}$$

under the assumption

$$\int_1^\infty \frac{dt}{t\cdot l(t)} = \infty. \tag{4.9}$$

Taking $t_n = 2^n$, $\epsilon = \epsilon_n = t_n^{-\frac{3}{k-1}} l(t_n)^{-\frac{1}{k-1}}$ and $R = R_n = \delta t_n^{\frac{k+1}{k-1}} l(t_n)^{\frac{2}{3(k-1)}}$ in (4.2) gives

$$\begin{aligned}
&\mathbb{E}_0 \exp\Big\{\theta \int_0^{t_n} \overline{V}(B_s)ds\Big\} \qquad (4.10)\\
&\geq \gamma \exp\Big\{ -\frac{1}{2} t_n^{\frac{k+1}{k-1}} l(t_n)^{\frac{2}{3(k-1)}}\Big\}\\
&\quad\times \exp\Big\{ t_n^{\frac{k+1}{k-1}} l(t_n)^{\frac{2}{3(k-1)}} H_{r,\delta}\Big(\theta \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z, \epsilon_n^{1/3}\delta)\big)\Big)\Big\}\\
&\quad\times \exp\Big\{ -\theta t_n^{\frac{k+1}{k-1}} l(t_n)^{\frac{2}{3(k-1)}}\Big(C_a\epsilon_n + \sup_{x\in B(0,R_n)} |\overline{V}_{a,\epsilon_n}(x)|\\
&\qquad\qquad + 2 \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z, \epsilon_n^{1/3}\delta)\big)\Big)\Big\}.
\end{aligned}$$

By (3.12),

$$\lim_{n\to\infty} \sup_{x\in B(0,R_n)} |\overline{V}_{a,\epsilon_n}(x)| = 0 \quad a.s. \tag{4.11}$$

as $a > 0$ is sufficiently large.

In addition,

$$\begin{aligned}
&\mathbb{P}\Big\{ \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z, \epsilon_n^{1/3}\delta)\big) \geq k+2\Big\}\\
&\leq C t_n^{\frac{3(k+1)}{k-1}} l(t_n)^{\frac{2}{(k-1)}} \mathbb{P}\Big\{\omega\big(B(0, \epsilon_n^{1/3}\delta)\big) \geq k+2\Big\}\\
&\leq C t_n^{-\frac{3}{k-1}} l(t_n)^{-\frac{k}{k-1}}.
\end{aligned}$$

Consequently,

$$\sum_n \mathbb{P}\Big\{ \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z, \epsilon_n^{1/3}\delta)\big) \geq k+2\Big\} < \infty.$$

By Borel-Cantelli lemma,

$$\limsup_{n\to\infty} \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z, \epsilon_n^{1/3}\delta)\big) \leq k+1 \quad a.s. \tag{4.12}$$

On the other hand, let $A_n = B(0, R_n - r) \setminus B(0, R_{n-1} - r)$.

$$\begin{aligned}
&\mathbb{P}\Big\{ \max_{z\in 2r\mathbb{Z}^3\cap A_n} \omega\big(B(\epsilon_n^{1/3}z, \epsilon_n^{1/3}\delta)\big) \leq k\Big\}\\
&= \Big(1 - \mathbb{P}\Big\{\omega\big(B(0, \epsilon_n^{1/3}\delta)\big) \geq k+1\Big\}\Big)^{\#\{2r\mathbb{Z}^3\cap A_n\}}.
\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^3\cap A_n}\omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big)\ge k+1\Big\}\\&\quad\sim\#\{2r\mathbb{Z}^3\cap A_n\}\mathbb{P}\Big\{\omega\big(B(0,\epsilon_n^{1/3}\delta)\big)\ge k+1\Big\}\ge c_0 l(t_n)^{-1},\end{aligned}$$

where $c_0>0$ is a constant independent of n. By (4.9),

$$\sum_n l(t_n)^{-1}=\infty.$$

By Borel-Cantelli lemma,

$$\begin{aligned}&\limsup_{n\to\infty}\max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)}\omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big)\\&\qquad\ge\limsup_{n\to\infty}\max_{z\in 2r\mathbb{Z}^3\cap A_n}\omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big)\ge k+1\quad a.s.\end{aligned}\tag{4.13}$$

By (4.10), (4.11), (4.12), (4.13),

$$\begin{aligned}&\limsup_{n\to\infty} t_n^{-\frac{k+1}{k-1}}l(t_n)^{-\frac{2}{3(k-1)}}\log\mathbb{E}_0\exp\left\{\theta\int_0^{t_n}\overline{V}(B_s)ds\right\}\\&\qquad\ge H_{r,\delta}\big((k+1)\theta\big)-2\theta(k+1)-2^{-1}\quad a.s.\end{aligned}$$

Notice that $(k+1)\theta>8^{-1}$. By Lemma 6.2, letting $r\to\infty$ and $\delta\to 0^+$ on the right hand side leads to (4.8). $\Box$

4.3 Lower bound for Theorem 2.3

We prove that

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}}l(t)^{\frac{2}{3(k-1)}}\log\mathbb{E}_0\exp\left\{\theta\int_0^t\overline{V}(B_s)ds\right\}=\infty\quad a.s.\tag{4.14}$$

under the assumption that

$$\int_1^\infty\frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt<\infty\quad\forall c>0.\tag{4.15}$$

This time we use (4.3) instead of (4.2). Taking $t_n=2^n$, $\epsilon=\epsilon_n=t_n^{-\frac{3}{k-1}}l(t_n)\big)^{\frac{1}{k-1}}$,

$R = R_n = t_n^{\frac{k+1}{k-1}} l(t_n)^{-\frac{2}{3(k-1)}}$ in (4.3) gives

$$\int_{B(0,\epsilon_n^{1/3}R_n)} \mathbb{E}_x \exp\Big\{\int_0^{t_n} \overline{V}(B_s)ds\Big\}dx \tag{4.16}$$
$$\geq (2\pi)^{3/2}\epsilon_n \exp\Big\{-\delta^{-2} \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3})\big)\Big\}$$
$$\times \exp\Big\{-t_n^{\frac{k+1}{k-1}} l(t_n)^{-\frac{2}{3(k-1)}}\theta\Big(\epsilon C_a + \sup_{x\in B(0,R_n)}|\overline{V}_{a,\epsilon_n}(x)|\Big)\Big\}$$
$$\times \exp\Big\{\Big(1+t_n^{\frac{k+1}{k-1}} l(t_n)^{-\frac{2}{3(k-1)}}\Big)H_{r,\delta}\Big(\max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big)\theta\Big)\Big\}.$$

We now show that for any $\delta > 0$ and $r > 0$,

$$\liminf_{n\to\infty} \max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big) \geq k+1 \quad a.s. \tag{4.17}$$

Indeed, by independence

$$\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big) \leq k\Big\}$$
$$= \Big(1-\mathbb{P}\Big\{\omega\big(B(0,\epsilon_n^{1/3}\delta)\big)\geq k+1\Big\}\Big)^{\#\{2r\mathbb{Z}^3\cap B(0,R_n-r)\}}.$$

By the fact that

$$\mathbb{P}\Big\{\omega\big(B(0,\epsilon_n^{1/3}\delta)\big)\geq k+1\Big\} \sim \frac{1}{(k+1)!}\Big(\frac{4}{3}\pi\delta^3\epsilon_n\Big)^{k+1}$$
$$= \frac{1}{(k+1)!}\Big(\frac{4}{3}\pi\delta^3\Big)^{k+1} t_n^{-\frac{3(k+1)}{k-1}} l(t_n)^{\frac{k+1}{k-1}},$$

there is a constant $c = c(k,\delta,r) > 0$ such that

$$\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^3\cap B(0,R_n-r)} \omega\big(B(\epsilon_n^{1/3}z,\epsilon_n^{1/3}\delta)\big) \leq k\Big\} \leq \exp\Big\{-c\cdot l(t_n)\Big\}$$

for large n. By (4.15),

$$\sum_n \exp\Big\{-c\cdot l(t_n)\Big\} < \infty.$$

Hence, (4.17) follows from Borel-Cantelli lemma.

Notice that (4.11) and (4.12) remain true in this setting. By (4.16) and (4.17), therefore,

$$\liminf_{n\to\infty} t_n^{-\frac{k+1}{k-1}} l(t_n)^{\frac{2}{3(k-1)}} \int_{B(0,R_n\epsilon_n^{1/3})} \mathbb{E}_x \exp\Big\{\int_0^{t_n} \overline{V}(B_s)ds\Big\}dx = \infty. \tag{4.18}$$

For any large $t > 0$, let n be such that $t_n \le t \le t_{n+1}$. Then

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \ge \mathbb{E}_0\left[\exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\};\ T_{R_n\epsilon_n^{1/3}} \ge t - t_{n-1}\right].$$

Notice that for any $x \in \mathbb{R}^d$

$$\begin{aligned}\overline{V}(x) &= \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}\epsilon_n^{-1/3}|y-x|)}{|y-x|^2}\omega(dy) - C_a\epsilon_n^{1/3} \\ &\quad + \int_{\mathbb{R}^3} \frac{1-\alpha(a^{-1}\epsilon_n^{-1/3}|y-x|)}{|y-x|^2}\big[\omega(dy) - dy\big] \\ &\ge -C_a\epsilon_n^{1/3} + \epsilon_n^{-2/3}\overline{V}_{a,\epsilon_n}(\epsilon_n^{-1/3}x).\end{aligned}$$

So we have

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \\ &\ge \exp\left\{-\theta t_{n+1}\Big(C_a\epsilon_n^{1/3} + \epsilon_n^{-2/3} \sup_{x\in B(0,R_n)} |\overline{V}_{a,\epsilon_n}(x)|\Big)\right\} \\ &\quad \times \mathbb{E}_0\left[\exp\left\{\theta \int_{t-t_{n-1}}^t \overline{V}(B_s)ds\right\};\ T_{R_n\epsilon_n^{1/3}} \ge t - t_{n-1}\right].\end{aligned}$$

By Markov property,

$$\begin{aligned}&\mathbb{E}_0\left[\exp\left\{\theta \int_{t-t_{n-1}}^t \overline{V}(B_s)ds\right\};\ T_{R_n\epsilon_n^{1/3}} \ge t - t_{n-1}\right] \\ &= \int_{B(0,R_n\epsilon_n^{-1/3})} \tilde{p}_{t-t_{n-1}}(x)\mathbb{E}_x \exp\left\{\theta \int_0^{t_{n-1}} \overline{V}(B_s)ds\right\}dx \\ &\ge \int_{B(0,\,R_{n-1}\epsilon_{n-1}^{-1/3})} \tilde{p}_{t-t_{n-1}}(x)\mathbb{E}_x \exp\left\{\theta \int_0^{t_{n-1}} \overline{V}(B_s)ds\right\}dx\end{aligned}$$

where $\tilde{p}_{t-t_{n-1}}(x)$ is the density function of the measure

$$\mu_{t-t_{n-1}}(A) = \mathbb{P}_0\big\{B_{t-t_{n-1}} \in A,\ T_{R_n\epsilon_n^{1/3}} \ge t - t_{n-1}\big\} \qquad A \subset \mathbb{R}^d.$$

Notice that $R_{n-1}\epsilon_{n-1}^{-1/3} \le 2^{-1}R_n\epsilon_n^{-1/3}$ for large n. By Lemma 3.4,

$$\begin{aligned}&\tilde{p}_{t-t_{n-1}}(x) \\ &\ge \mathbb{P}_0\Big\{\max_{0\le s\le 1}|B_s^0| \le R_n\epsilon_n^{1/3}(t-t_{n-1})^{-1/2}\Big\}\big(2\pi(t-t_{n-1})\big)^{-3/2}\exp\left\{-\frac{|x|^2}{2(t-t_{n-1})}\right\} \\ &\ge \mathbb{P}_0\Big\{\max_{0\le s\le 1}|B_s^0| \le R_n\epsilon_n^{1/3}t_{n+1}^{-1/2}\Big\}\big(2\pi t_{n+1}\big)^{-3/2}\exp\left\{-\frac{R_{n-1}^2\epsilon_{n-1}^{2/3}}{2t_{n-1}}\right\} \\ &\ge \gamma 2^{-3n/2}\exp\left\{-\frac{1}{2}t_{n-1}^{\frac{k+1}{k-1}}l(t_{n-1})^{\frac{2}{3(k-1)}}\right\}\end{aligned}$$

for every $x \in B(0,\, R_{n-1}\epsilon_{n-1}^{-1/3})$, where $\gamma > 0$ is a constant independent of t.

Summarizing our computation,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\}\\
&\geq \gamma 2^{-3n/2} \exp\left\{ -\theta t_{n+1}\Big(C_a\epsilon_n^{1/3} + \epsilon_n^{-2/3} \sup_{x\in B(0,R_n)} |\overline{V}_{a,\epsilon_n}(x)|\Big)\right\}\\
&\ \times \exp\left\{ -\frac{1}{2} t_{n-1}^{\frac{k+1}{k-1}} l(t_{n-1})^{\frac{2}{3(k-1)}}\right\} \int_{B(0,\, R_{n-1}\epsilon_{n-1}^{-1/3})} \mathbb{E}_x \exp\left\{\theta \int_0^{t_{n-1}} \overline{V}(B_s)ds\right\}dx
\end{aligned}$$

when $t_n \leq t \leq t_{n+1}$ for large n. In view of (4.11) and (4.18), this leads to (4.14). □

5 Upper bounds

In this section we install the upper bounds requested by Theorem 2.1, Theorem 2.2, and Theorem 2.3. Through this section $0 < \theta < 1/16$. Recall that $k = [(8\theta)^{-1}]$ and that for any open set $D \subset \mathbb{R}^3$ and the function $\zeta(\cdot)$ on D, $\lambda_\zeta(D)$ is defined by the variation given in (3.15). For each $R > 0$, write $Q_R = (-R, R)^d$.

5.1 Asymptotics for the principal eigenvalues

By $0 < \theta < 1/16$ we have that $k = [(8\theta)^{-1}] \geq 2$. Write

$$R_k(t) = \begin{cases} t^{\frac{k}{k-2}} l(t)^{\frac{2}{3(k-2)}} & \text{when } k \geq 3 \\ t^3 l(t)^{2/3} & \text{when } k = 2. \end{cases} \tag{5.1}$$

Lemma 5.1

$$\lim_{t\to\infty} t^{-\frac{2}{k-1}} l(t)^{-\frac{2}{3(k-1)}} \lambda_{\theta\overline{V}}\big(Q_{R_k(t)}\big) = 0 \quad a.s.$$

under the assumption

$$\int_1^\infty \frac{dt}{t\cdot l(t)} < \infty. \tag{5.2}$$

Proof: We first consider the case $k \geq 3$. Let $M > 0$ be fixed but arbitrary. Write

$$r(t) = M\big(tl(t)^{1/3}\big)^{\frac{1}{(k-1)(k-2)}}, \quad \epsilon(t) = \big(t^3 l(t)\big)^{-\frac{k}{(k-1)(k-2)}}$$

$$\delta(t) = \epsilon(t)^{1/3} r(t) = M\big(tl(t)^{1/3}\big)^{-\frac{1}{k-2}}.$$

Decompose $\overline{V}$ as follows:

$$\overline{V}(x) = \int_{\mathbb{R}^3} \frac{\alpha\big(\delta(t)^{-1}|y-x|\big)}{|y-x|^2}\big[\omega(dy) - dy\big] + \int_{\mathbb{R}^3} \frac{1-\alpha\big(\delta(t)^{-1}|y-x|\big)}{|y-x|^2}\big[\omega(dy) - dy\big].$$

For the first term

$$\int_{\mathbb{R}^3} \frac{\alpha\big(\delta(t)^{-1}|y-x|\big)}{|y-x|^2}\big[\omega(dy) - dy\big] \le \int_{\mathbb{R}^3} \frac{\alpha\big(\delta(t)^{-1}|y-x|\big)}{|y-x|^2}\omega(dy)$$
$$= \epsilon(t)^{-2/3} \int_{\mathbb{R}^3} \frac{\alpha\big(r(t)^{-1}|y-\epsilon(t)^{-1/3}x|\big)}{|y-\epsilon(t)^{-1/3}x|^2}\omega(\epsilon(t)dy) = \epsilon(t)^{-2/3}\xi_{r,\epsilon}\big(\epsilon(t)^{-1/3}x\big),$$

where

$$\xi_{r,\epsilon}(x) = \xi_{r(t),\epsilon(t)}(x) = \int_{\mathbb{R}^3} \frac{\alpha\big(r(t)^{-1}|y-x|\big)}{|y-x|^2}\omega(\epsilon(t)dy).$$

As for the second term

$$\int_{\mathbb{R}^3} \frac{1-\alpha\big(\delta(t)^{-1}|y-x|\big)}{|y-x|^2}\big[\omega(dy) - dy\big]$$
$$= a^2\delta(t)^{-2}\int_{\mathbb{R}^3} \frac{1-\alpha\big(a^{-1}|y-a\delta(t)^{-1}x|\big)}{|y-a\delta(t)^{-1}x|^2}\big[\omega(a^{-3}\delta(t)^3dy) - a^{-3}\delta(t)^3dy\big]$$
$$= a^2\delta(t)^{-2}\overline{V}_{a,\tilde{\delta}(t)}\big(a\delta(t)^{-1}x\big)$$

where $\tilde{\delta}(t) = a^{-3}\delta(t)^3$, the random field $\overline{V}_{a,\epsilon}(\cdot)$ is defined in (3.5) and the constant $a > 0$ will be specified later.

By triangle inequality and by the substitution $g(x) \mapsto \epsilon(t)^{-1/2}g\Big(x\epsilon(t)^{-1/3}\Big)$,

$$\lambda_{\theta\overline{V}}(Q_{R_k(t)}) \le \epsilon(t)^{-2/3}\lambda_{\theta\xi_{r,\epsilon}}(Q_{\epsilon^{-1/3}(t)R_k(t)}) + \theta a^2\delta^{-2}(t) \sup_{x\in a\delta(t)^{-1}Q_{R_k(t)}} |\overline{V}_{a,\tilde{\delta}(t)}(x)|. \quad (5.3)$$

By Proposition 1 in [14], there is a non-negative and continuous function $\Phi(x)$ on $\mathbb{R}^3$ whose support is contained in the 1-neighborhood of the grid $2r(t)\mathbb{Z}^3$, such that

$$\lambda_{\xi_{r,\epsilon}-\Phi^y}(Q_{\epsilon^{-1/3}(t)R_k(t)}) \le \max_{z\in 2r(t)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t)R_k(t)+2r(t)}} \lambda_{\xi_{r,\epsilon}}(z + Q_{r(t)+1}) \quad y \in Q_{r(t)}$$

where $\Phi^y(x) = \Phi(x+y)$. In addition, $\Phi(x)$ is periodic with period $2r(t)$:

$$\Phi(x + 2r(t)z) = \Phi(x); \quad x \in \mathbb{R}^3, \ \ z \in \mathbb{Z}^3$$

and there is a constant $K > 0$ independent of $r(t)$ and t such that

$$\int_{Q_r} \Phi(x)dx \le \frac{K}{r(t)}.$$

By periodicity, therefore,

$$\eta(x) \equiv \frac{1}{\big(2r(t)\big)^3}\int_{Q_r}\Phi^y(x)dy = \frac{1}{\big(2r(t)\big)^3}\int_{Q_r}\Phi(y)dy \le \frac{K}{8r(t)^4}.$$

So we have

$$\begin{aligned}\lambda_{\xi_{r,\epsilon}}(Q_{\epsilon^{-1/3}(t)R_k(t)}) &\le \frac{K}{8r(t)^4} + \lambda_{\xi_{r,\epsilon}-\eta}(Q_{\epsilon^{-1/3}(t)R_k(t)})\\ &\le \frac{K}{8r(t)^4} + \frac{1}{\big(2r(t)\big)^3}\int_{Q_{r(t)}}\lambda_{\xi_{r,\epsilon}-\Phi^y}(Q_{\epsilon^{-1/3}(t)R_k(t)})dy\\ &\le \frac{K}{8r(t)^4} + \max_{z\in 2r(t)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t)R_k(t)+2r(t)}}\lambda_{\xi_{r,\epsilon}}(z+Q_{r(t)+1}),\end{aligned}$$

where the second step follows from Jensen inequality.

Summarizing the estimate since (5.3),

$$\begin{aligned}&t^{-\frac{2}{k-1}}l(t)^{-\frac{2}{3(k-1)}}\lambda_{\theta\overline{V}}\big(Q_{R_k(t)}\big)\\ &\quad\le \theta a^2M^{-2}\sup_{x\in a\delta(t)^{-1}Q_{R_k(t)}}|\overline{V}_{a,\tilde{\delta}(t)}(x)| + \frac{K}{8M^4}\\ &\quad\quad+\big(tl(t)^{1/3}\big)^{\frac{2}{(k-1)(k-2)}}\max_{z\in 2r(t)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t)R_k(t)+2r(t)}}\lambda_{\theta\xi_{r,\epsilon}}(z+Q_{r(t)+1}).\end{aligned}\tag{5.4}$$

Take $t_n = 2^n$. By (3.12),

$$\lim_{n\to\infty}\sup_{x\in a\delta(t_n)^{-1}Q_{R_k(t_n)}}|\overline{V}_{a,\tilde{\delta}(t_n)}(x)| = 0 \quad a.s.\tag{5.5}$$

when a is sufficiently large.

We now prove that

$$\mathbb{P}\Big\{\max_{z\in 2r(t_n)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t_n)R_k(t_n)+2r(t_n)}}\lambda_{\xi_{r(t_n),\epsilon(t_n)}}(z+Q_{r(t_n)+1}) = 0 \text{ eventually in } n\Big\} = 1.\tag{5.6}$$

Notice that

$$\begin{aligned}&\mathbb{P}\Big\{\max_{z\in 2r(t_n)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t_n)R_k(t_n)+2r(t_n)}}\lambda_{\theta\xi_{r(t_n),\epsilon(t_n)}}(z+Q_{r(t_n)+1})\ne 0\Big\}\\ &\quad\le \#\Big\{2r(t_n)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t_n)R_k(t_n)+2r(t_n)}\Big\}\mathbb{P}\Big\{\lambda_{\theta\xi_{r(t_n),\epsilon(t_n)}}(Q_{r(t_n)+1})\ne 0\Big\}.\end{aligned}$$

Recall that truncation function $\alpha(\cdot)$ is supported on $[0,3]$. For any $g \in \mathcal{F}_3(Q_{r(t_n)+1})$,

$$\begin{aligned}
\int_{Q_{r(t_n)+1}} \xi_{r(t_n),\epsilon(t_n)}(x) g^2(x)\, dx &= \int_{\mathbb{R}^3} \left[\int_{Q_{r_n(t)+1}} \frac{\alpha(r(t_n)^{-1}|y-x|)}{|y-x|^2} g^2(y) dy \right] \omega\big(\epsilon(t_n) dx\big) \\
&= \int_{Q_{5r(t_n)}} \left[\int_{Q_{r(t_n)+1}} \frac{\alpha(r(t_n)^{-1}|y-x|)}{|y-x|^2} g^2(y) dy \right] \omega\big(\epsilon(t_n) dx\big) \\
&\le \omega(Q_{5\delta(t_n)}) \sup_{x \in \mathbb{R}^3} \int_{Q_{r(t_n)+1}} \frac{g^2(y)}{|y-x|^2} dy
\end{aligned}$$

when $r(t_n) \ge 1$. Therefore,

$$\begin{aligned}
&\lambda_{\theta \xi_{r(t_n),\epsilon(t_n)}}(Q_{r(t_n)+1}) \qquad (5.7) \\
&\le \sup_{g \in \mathcal{F}_3(Q_{r(t_n)+1})} \left\{ \omega(Q_{5\delta(t_n)}) \theta \sup_{x \in \mathbb{R}^3} \int_{Q_{r(t_n)+1}} \frac{g^2(y)}{|y-x|^2} dy - \frac{1}{2} \int_{Q_{r_n(t)+1}} |\nabla g(y)|^2 dy \right\} \\
&\le \sup_{g \in \mathcal{F}_3} \left\{ \omega(Q_{5\delta(t_n)}) \theta \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{g^2(y)}{|y-x|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(y)|^2 dy \right\} \\
&= \sup_{x \in \mathbb{R}^3} \sup_{g \in \mathcal{F}_3} \left\{ \omega(Q_{5\delta(t_n)}) \theta \int_{\mathbb{R}^3} \frac{g^2(y)}{|y-x|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(y)|^2 dy \right\} \\
&= \sup_{g \in \mathcal{F}_3} \left\{ \omega(Q_{5\delta(t_n)}) \theta \int_{\mathbb{R}^3} \frac{g^2(y)}{|y|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(y)|^2 dy \right\},
\end{aligned}$$

where the last step follows from shifting invariance.

Notice that $k\theta \le 8^{-1}$. By Lemma 6.2 we obtain the bound

$$\begin{aligned}
&\mathbb{P}\Big\{ \max_{z \in 2r(t_n)\mathbb{Z}^3 \cap Q_{2\epsilon^{-1/3}(t_n) R_k(t_n) + 2r(t_n)}} \lambda_{\theta \xi_{a,\epsilon(t_n)}}(z + Q_{r(t_n)+1}) \ne 0 \Big\} \\
&\le C t_n^{\frac{3(k+1)}{k-2}} l(t_n)^{\frac{3}{k-2}} \mathbb{P}\Big\{ \omega(Q_{5\delta(t_n)}) \ge k+1 \Big\} \le C l(t_n)^{-1}.
\end{aligned}$$

By (5.2),

$$\sum_n l(t_n)^{-1} < \infty.$$

Hence, (5.6) follows from Borel-Cantelli lemma.

Since the second term in (5.4) can be arbitrarily small by making M sufficiently large, by (5.5) and (5.6),

$$\limsup_{n \to \infty} t_n^{-\frac{2}{k-1}} l(t_n)^{-\frac{2}{3(k-1)}} \lambda_{\theta \overline{V}}\big(Q_{R_k(t_n)}\big) \le 0 \quad a.s.$$

Notice that $\lambda_{\theta \overline{V}}\big(Q_{R_k(t)}\big)$ is non-decreasing in t. We have completed the proof in the case $k \ge 3$.

The case $k=2$ follows from the same argument with

$$r(t) = M\big(tl(t)^{1/3}\big)^{1/2}, \quad \epsilon(t) = \big(t^3 l(t)\big)^{-2},$$
$$\delta(t) = \epsilon(t)^{1/3} r(t) = M\big(tl(t)^{1/3}\big)^{-3/2}.$$

□

Write

$$S_k(t) = \begin{cases} t^{\frac{k}{k-2}} l(t)^{-\frac{2}{3(k-2)}} & \text{when } \ k \geq 3 \\ \\ t^3 l(t)^{-2/3} & \text{when } \ k = 2. \end{cases} \tag{5.8}$$

Lemma 5.2

$$\liminf_{t\to\infty} t^{-\frac{2}{k-1}} l(t)^{\frac{2}{3(k-1)}} \lambda_{\theta\overline{V}}\big(Q_{S_k(t)}\big) = 0 \quad a.s.$$

under the assumption that there is $c_0 > 0$ such that

$$\int_1^\infty \frac{1}{t} \exp\{-cl(t)\}\, dt \begin{cases} = \infty & \textit{when } \ c < c_0 \\ \\ < \infty & \textit{when } \ c > c_0. \end{cases} \tag{5.9}$$

Proof: We first consider the case $k \geq 3$. Let $u > 0$ and $M > 0$ be fixed but arbitrary. Write

$$r(t) = M\big(tl(t)^{-1/3}\big)^{\frac{1}{(k-1)(k-2)}}, \quad \epsilon(t) = u^{-3}\big(t^3 l(t)^{-1}\big)^{-\frac{k}{(k-1)(k-2)}}$$
$$\delta(t) = \epsilon(t)^{1/3} r(t) = \Big(\frac{M}{u}\Big)\big(tl(t)^{-1/3}\big)^{-\frac{1}{k-2}}.$$

Similar to (5.4),

$$\begin{aligned} &t^{-\frac{2}{k-1}} l(t)^{\frac{2}{3(k-1)}} \lambda_{\theta\overline{V}}\big(Q_{S_k(t)}\big) \\ &\qquad \leq \theta a^2 \Big(\frac{u}{M}\Big)^2 \sup_{x\in a\delta(t)^{-1}Q_{R_k(t)}} |\overline{V}_{a,\tilde{\delta}(t)}(x)| + \frac{Ku^2}{8M^4} \\ &\qquad + u^2\big(tl(t)^{-1/3}\big)^{\frac{2}{(k-1)(k-2)}} \max_{z\in 2r(t)\mathbb{Z}^3 \cap Q_{2\epsilon^{-1/3}(t)S_k(t)+2r(t)}} \lambda_{\theta\xi_{r,\epsilon}}(z + Q_{r(t)+1}), \end{aligned} \tag{5.10}$$

where $\tilde{\delta} = \tilde{\delta}(t) = a^{-3}\delta(t)^3$, the random field $\overline{V}_{a,\epsilon}(x)$ is defined in (3.5), and

$$\xi_{r,\epsilon}(x) = \xi_{r(t),\epsilon(t)}(x) = \int_{\mathbb{R}^3} \frac{\alpha\big(r(t)^{-1}|y-x|\big)}{|y-x|^2} \omega(\epsilon(t)dy).$$

Same as (5.7),

$$\lambda_{\theta\xi_{r,\epsilon}}(z + Q_{r(t)+1}) \leq \sup_{g\in\mathcal{F}_3} \Big\{\omega\big(\epsilon(t)^{1/3}(z + Q_{5r(t)})\big)\theta \int_{\mathbb{R}^3} \frac{g^2(y)}{|y|^2} dy - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(y)|^2 dy\Big\}$$

for each $z \in 2r(t)\mathbb{Z}^d \cap Q_{2\epsilon^{-1/3}(t)S_k(t)+2r(t)}$. Thus,

$$\begin{aligned}
&\max_{z\in 2r(t)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t)S_k(t)+2r(t)}} \lambda_{\theta\xi_{r,\epsilon}}(z+Q_{r(t)+1})\\
&\le \sup_{g\in\mathcal{F}_3}\Big\{\theta\Big(\max_{z\in 2r(t)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t)S_k(t)+2r(t)}} \omega\big(\epsilon(t)^{1/3}(z+Q_{5r(t)})\big)\Big)\int_{\mathbb{R}^3}\frac{g^2(y)}{|y|^2}dy\\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(y)|^2dy\Big\}\\
&=\sup_{g\in\mathcal{F}_3}\Big\{\theta\Big(\max_{z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}} \omega(z+Q_{5\delta(t)})\Big)\int_{\mathbb{R}^3}\frac{g^2(y)}{|y|^2}dy-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(y)|^2dy\Big\}.
\end{aligned}$$

By Lemma 6.2, therefore,

$$\begin{aligned}
&\Big\{\max_{z\in 2r(t)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t)S_k(t)+2r(t)}} \lambda_{\xi_{r,\epsilon}}(z+Q_{r(t)+1})=0\Big\}\\
&\quad\supset\Big\{\max_{z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}} \omega(z+Q_{5\delta(t)})\le k\Big\}.
\end{aligned}\tag{5.11}$$

Unfortunately, the random variables

$$\omega(z+Q_{5\delta(t)})\qquad z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}$$

are not independent. So we apply Slepian-type domination (Lemma 3.1):

$$\begin{aligned}
&\mathbb{P}\Big\{\max_{z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}} \omega(z+Q_{5\delta(t)})\le k\Big\}\\
&\qquad\qquad\ge\Big(\mathbb{P}\Big\{\omega(Q_{5\delta(t)})\le k\Big\}\Big)^{\#\{2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}\}}.
\end{aligned}$$

It is straightforward to check that

$$\mathbb{P}\Big\{\omega(Q_{5\delta(t)})\ge k+1\Big\}\sim\frac{(10u^{-1}M)^{3(k+1)}}{(k+1)!}\Big(t^3 l(t)^{-1}\Big)^{-\frac{k+1}{k-2}}\qquad (t\to\infty)$$

and that

$$\#\{2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}\}\sim\Big(\frac{u}{M}\Big)^3 t^{\frac{3(k+1)}{k-2}} l(t)^{-\frac{3}{k-2}}\qquad (t\to\infty).$$

Hence, there is a constant C_k independent of u and M such that

$$\mathbb{P}\Big\{\max_{z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}} \omega(z+Q_{5\delta(t)})\le k\Big\}\ge\exp\Big\{-C_k\Big(\frac{M}{u}\Big)^{3k} l(t)\Big\}\tag{5.12}$$

for large t. In connection to (5.10), our strategy is to make u^2/M^4, M/u sufficiently small, and to make u and M sufficiently large.

Fix a constant $\tilde{c}$ satisfying

$$\frac{k-1}{3k}c_0 < \tilde{c} < c_0.$$

Define $\{t_n\}$ as following:

$$t_1 = 1, \quad t_{n+1} = t_n \exp\Big\{\tilde{c}l(t_n)\Big\} \qquad n = 1, 2, \cdots .$$

By (3.12),

$$\lim_{n\to\infty} \sup_{x\in Q_{\epsilon^{-1/3}(t_n)S_k(t_n)}} |\overline{V}_{a,\tilde{\delta}(t_n)}(x)| = 0 \qquad a.s. \tag{5.13}$$

for sufficiently large a.

We now prove that

$$\mathbb{P}\bigg\{\max_{z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}} \omega(z + Q_{5\delta(t)}) \le k \text{ i.o.}\bigg\} = 1 \tag{5.14}$$

Write

$$\begin{aligned}
H_n &= \max_{z\in 2\delta(t)\mathbb{Z}^3\cap Q_{2S_k(t)+2\delta(t)}} \omega(z + Q_{5\delta(t)}),\\
A_n &= Q_{2S_k(t_{n+1})+2\delta(t_{n+1})} \setminus Q_{2S_k(t_n)+b\delta(t_n)},\\
Z_n &= \max_{z\in 2\delta(t_n)\mathbb{Z}^3\cap A_n} \omega(z + Q_{5\delta(t_{n+1})}),\\
\widetilde{Z}_n &= \max_{z\in 2\delta(t_n)\mathbb{Z}^3\cap Q_{2S_k(t_n)+b\delta(t_n)}} \omega(z + Q_{5\delta(t_{n+1})}),
\end{aligned}$$

where $b > 0$ is a constant which is large enough to make sure that the random variables $Z_1, Z_2, \cdots$ are independent.

We have that $H_{n+1} = \max\{Z_n, \widetilde{Z}_n\}$. Notice that

$$\begin{aligned}
\mathbb{P}\{\widetilde{Z}_n \ge k+1\} &\le \#\{2\delta(t_n)\mathbb{Z}^3 \cap Q_{2S_k(t_n)+b\delta(t_n)}\}\mathbb{P}\Big\{\omega\big(Q_{5\delta(t_{n+1})}\big) \ge k+1\Big\}\\
&\le C t_n^{\frac{3(k+1)}{k-1}} l(t_n)^{-\frac{3}{k-2}} t_{n+1}^{-\frac{3(k+1)}{k-1}} l(t_{n+1})^{\frac{k+1}{k-2}}\\
&= C l(t_n)^{-\frac{3}{k-2}} l(t_{n+1})^{\frac{k+1}{k-2}} \exp\Big\{-\frac{3\tilde{c}(k+1)}{k-2} l(t_n)\Big\}.
\end{aligned}$$

Since $l(t)$ is slow-varying,

$$l(t_{n+1}) = l\Big(t_n \exp\{\tilde{c}l(t_n)\}\Big) \le l(t_n)\exp\Big\{o\big(l(t_n)\Big\} = \exp\Big\{o\big(l(t_n)\big)\Big\}$$

for large n. Therefore, we obtain the bound

$$\mathbb{P}\{\widetilde{Z}_n \geq k+1\} \leq C \exp\Big\{-\frac{3k\tilde{c}}{k-2} l(t_n)\Big\} \quad (n \to \infty).$$

For any $c > c_0$, on the other hand,

$$\begin{aligned}\infty > \int_1^\infty \frac{1}{t} \exp\{-cl(t)\} dt = \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{1}{t} \exp\{-cl(t)\} dt \\ \geq \sum_{n=1}^\infty \frac{t_{n+1}-t_n}{t_{n+1}} \exp\{-cl(t_{n+1})\} \geq \delta \sum_{n=1}^\infty \exp\{-cl(t_{n+1})\}\end{aligned}$$

So we have that

$$\sum_n \mathbb{P}\{\widetilde{Z}_n \geq k+1\} < \infty.$$

By Borel-Cantelli lemma

$$\mathbb{P}\{\widetilde{Z}_n \leq k \text{ eventually in } n\} = 1. \tag{5.15}$$

By (5.12),

$$\mathbb{P}\{Z_n \leq k\} \geq \mathbb{P}\{H_{n+1} \leq k\} \geq \exp\Big\{-C_k\Big(\frac{M}{u}\Big)^{3k} l(t_{n+1})\Big\}.$$

Pick c_1 satisfying $\tilde{c} < c_1 < c_0$ and make M/u so small that

$$C_k\Big(\frac{r}{u}\Big)^{3k} < c_1 - \tilde{c}$$

We have

$$\begin{aligned}\infty = \int_1^\infty \frac{1}{t} \exp\{-c_1 l(t)\} dt = \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{1}{t} \exp\{-c_1 l(t)\} dt \\ \leq \sum_{n=1}^\infty \frac{t_{n+1}-t_n}{t_n} \exp\{-c_1 l(t_n)\} \leq \sum_{n=1}^\infty \exp\{-(c_1-\tilde{c}) l(t_n)\}.\end{aligned}$$

Consequently,

$$\sum_n \mathbb{P}\{Z_n \leq k\} = \infty.$$

Applying Borel-Cantelli lemma to the independent sequence $\{Z_n\}$ we have

$$\mathbb{P}\{Z_n \leq k \text{ i.o.}\} = 1.$$

This, together with (5.15), leads to (5.14). By (5.11) and (5.14),

$$\mathbb{P}\Big\{\max_{z\in 2r(t_n)\mathbb{Z}^3\cap Q_{2\epsilon^{-1/3}(t_n)S_k(t_n)+2r(t_n)}}\lambda_{\xi_{r(t_n),\epsilon(t_n)}}(z+Q_{r(t_n)+1})=0\ \text{ i.o.}\Big\}=1. \tag{5.16}$$

By (5.10), (5.13), (5.16), and by the fact that u^2/M^4 can be arbitrarily small,

$$\liminf_{n\to\infty} t_n^{-\frac{2}{k-1}}l(t_n)^{\frac{2}{3(k-1)}}\lambda_{\theta\overline{V}}\big(Q_{S_k(t_n)}\big)\le 0 \quad a.s.$$

By the fact that the principal eigenvalue $\lambda_{\theta\overline{V}}(Q_R)$ increases in R, we have completed the proof in the case $k\ge 3$.

The case $k=2$ follows from the same argument with

$$r(t)=M\big(tl(t)^{-1/3}\big)^{1/2},\quad \epsilon(t)=u^{-3}\big(t^3l(t)^{-1}\big)^{-2},$$

$$\delta(t)=\epsilon(t)^{1/3}r(t)=\Big(\frac{M}{u}\Big)\big(tl(t)^{-1/3}\big)^{-3/2}.$$

□

5.2 Upper bound for Theorem 2.1

We prove that when $\theta<16^{-1}$,

$$\mathbb{E}_0\exp\Big\{\theta\int_0^t\overline{V}(B_s)ds\Big\}<\infty \quad a.s. \tag{5.17}$$

for any $t>0$. By Hölder inequality, we may assume that $\theta>\frac{1}{24}$.

Let $l(t)\ge 0$ be a slow-varying function satisfying (5.2) and recall the notation $R_2(t)=t^3l(t)^{2/3}$. Consider the decomposition

$$\mathbb{E}_0\exp\Big\{\theta\int_0^t\overline{V}(B_s)ds\Big\}=\mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^t\overline{V}(B_s)ds\Big\};\ \tau_{Q_{R_2(t)}}\ge 2t\bigg]$$
$$+\sum_{n=1}^\infty\mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^t\overline{V}(B_s)ds\Big\};\ \tau_{Q_{R_2(2^{n-1}t)}}<2t\le\tau_{Q_{R_2(2^nt)}}\bigg].$$

Pick $p>1$ with $p\theta<16^{-1}$ and write $q=p(p-1)^{-1}$. By Hölder inequality,

$$\mathbb{E}_0\bigg[\exp\Big\{\theta\int_0^t\overline{V}(B_s)ds\Big\};\ \tau_{Q_{R_2(2^{n-1}t)}}<2t\le\tau_{Q_{R_2(2^nt)}}\bigg]$$
$$\le\Big(\mathbb{P}_0\big\{\tau_{Q_{R_2(2^{n-1}t)}}<2t\big\}\Big)^{1/q}\bigg\{\mathbb{E}_0\bigg[\exp\Big\{p\theta\int_0^t\overline{V}(B_s)ds\Big\};\ \tau_{Q_{R_2(2^nt)}}\ge 2t\bigg]\bigg\}^{1/p}.$$

Let $\delta > 0$ be a small number and condition on the event $\{\omega(B(0,\delta)) = 0\}$. Applying Lemma 3.6,

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \le X(\theta t) + Y_0(t)\exp\left\{t\lambda_{\theta\overline{V}}(Q_{R_2(t)})\right\} \tag{5.18}$$
$$+ \sum_{n=1}^\infty \Big(\mathbb{P}_0\big\{\tau_{Q_{R_2(2^{n-1}t)}} < 2t\big\}\Big)^{1/q}\Big(X(p\theta t) + Y_n(t)\exp\left\{t\lambda_{p\theta\overline{V}}(Q_{R_2(2^n t)})\right\}\Big)^{1/p},$$

where

$$X(t) = \exp\Big\{t \sup_{|x|\le\delta/2} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\},$$

$$Y_0(t) = \frac{48R_2(t)^3}{\pi\delta^3}\mathbb{E}_0\exp\Big\{\sqrt{2\delta}\theta T_1 \sup_{x\in Q_{R_2(t)}} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\}$$

and

$$Y_n(t) = \frac{48R_2(2^n t)^3}{\pi\delta^3}\mathbb{E}_0\exp\Big\{\sqrt{2\delta}p\theta T_1 \sup_{x\in Q_{R_2(2^n t)}} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\} \quad n = 1,2,\cdots.$$

Using the classical fact that there is a constant $C > 0$ such that

$$\mathbb{E}_0\exp\{bT_1\} \le \exp\{Cb^2\} \quad \forall b > 0$$

we have

$$\mathbb{E}_0\exp\Big\{\sqrt{2\delta}p\theta T_1 \sup_{x\in Q_{R_2(2^n t)}} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\} \le \exp\Big\{2\delta C(p\theta)^2\Big(\sup_{x\in Q_{R_2(2^n t)}} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big)^2\Big\}$$
$$= \exp\Big\{o\Big(\big(\log(2^n t)\big)^2\Big)\Big\} \quad a.s. \quad (n\to\infty)$$

where the last step follows from (3.11). Consequently,

$$Y_n(t) = \exp\Big\{o\big(n^2\big)\Big\} \quad a.s. \quad (n\to\infty). \tag{5.19}$$

Recall the classic fact that

$$\mathbb{P}_0\big\{\tau_{Q_{R_2(2^{n-1}t)}} < 2t\big\} = \mathbb{P}\Big\{\max_{s\le 2t} |B_s|_\infty \ge R_2(2^{n-1}t)\Big\} \tag{5.20}$$
$$= \mathbb{P}\Big\{\max_{s\le 1} |B_s|_\infty \ge (2t)^{-1/2}R_2(2^{n-1}t)\Big\} \le \exp\Big\{-C2^{6n}t^5 l(2^{n-1}t)^{4/3}\Big\}$$

for some constant $C > 0$ independent of n and t, where $|\cdot|_\infty$ is the max-norm in $\mathbb{R}^3$. By Lemma 5.1 with $k = 2$ and with θ being replaced by $p\theta$,

$$\lambda_{p\theta\overline{V}}(Q_{R_2(2^n t)}) = o\Big(\big(2^n t\big)^2 l\big(2^n t\big)^{2/3}\Big) \quad a.s. \quad (n\to\infty). \tag{5.21}$$

Combining (5.19), (5.20), (5.21) we conclude that the right hand side of (5.18) is almost surely finite. Thus, we have established (5.17) conditioning on the event $\Big\{\omega\big(B(0,\delta)\big) = 0\Big\}$. Therefore,

$$\mathbb{P}\Big\{\mathbb{E}_0 \exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\} < \infty\Big\} \geq \mathbb{P}\Big\{\omega\big(B(0,\delta)\big) = 0\Big\} = \exp\Big\{-\frac{4}{3}\pi\delta^3\Big\}.$$

Since δ can be arbitrarily small, we have completed the proof. □

5.3 Upper bound for Theorem 2.2

Consider the decomposition

$$\overline{V}(x) = \overline{V}_{1,1}(x) + \int_{\mathbb{R}^3}\frac{\alpha(|y-x|)}{|y-x|^2}\omega(dy) - \int_{\mathbb{R}^3}\frac{\alpha(|y|)}{|y|^2}dy \geq \overline{V}_{1,1}(x) - \int_{\mathbb{R}^3}\frac{\alpha(|y|)}{|y|^2}dy$$

where the notation $\overline{V}_{1,1}(x)$ comes from (3.5). We have that

$$\begin{aligned}\mathbb{E}_0 \exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\} &\geq \mathbb{E}_0\Big[\exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\};\ \tau_{B(0,t)} \geq t\Big] \\ &\geq \exp\Big\{-t\Big(\sup_{|x|\leq t}|V_{1,1}(x)| + \int_{\mathbb{R}^3}\frac{\alpha(|y|)}{|y|^2}dy\Big)\Big\}\mathbb{P}_0\Big\{\max_{s\leq t}|B_s| \leq t\Big\}.\end{aligned}$$

By (3.11) we have

$$\liminf_{t\to\infty}(t\log t)^{-1}\log \mathbb{E}_0 \exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\} \geq 0 \quad a.s. \tag{5.22}$$

To complete the proof of Theorem 2.2, therefore, all we need to show is that under the assumption (5.2),

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\Big\{\theta\int_0^t \overline{V}(B_s)ds\Big\} \leq 0 \quad a.s. \tag{5.23}$$

conditioning on the event $\Big\{\omega\big(B(0,\delta)\big) = 0\Big\}$.

In the case $1/24 < \theta < 1/16$, the bound (5.18) holds when conditioned on $\Big\{\omega\big(B(0,\delta)\big) = 0\Big\}$. By Lemma 5.1 with $k = 2$,

$$\lambda_{\theta\overline{V}}(Q_{R_2(t)}) = o\Big(t^2 l(t)^{2/3}\Big) \ \ a.s. \quad (t\to\infty).$$

The bound (5.19) can be replaced by

$$Y_n(t) = \exp\Big\{o\Big(\big(\log(2^n t)\big)^2\Big)\Big\} \quad a.s. \quad n = 0, 1, \cdots.$$

Combining these with the bound given in (5.20), (5.21), we have (5.23).

Now we consider the case $0 < \theta \le 1/24$ (so $k \ge 3$). The main reason we treat this setting separately is for it includes the critical cases when $\theta = (8k)^{-1}$, which need some special care. Similar to (5.18), for any conjugate $p, q > 1$,

$$\mathbb{E}_0 \exp\Big\{\theta \int_0^t \overline{V}(B_s)ds\Big\} \le X(\theta t) + Y_0(t)\exp\Big\{t\lambda_{\theta\overline{V}}(Q_{R_k(t)})\Big\} \tag{5.24}$$
$$+ \sum_{n=1}^{\infty}\Big(\mathbb{P}_0\big\{\tau_{Q_{R_k(2^{n-1}t)}} < 2t\big\}\Big)^{1/q}\Big(X(p\theta t) + Y_n(t)\exp\Big\{t\lambda_{p\theta\overline{V}}(Q_{R_k(2^n t)})\Big\}\Big)^{1/p},$$

where

$$X(t) = \exp\Big\{t \sup_{|x|\le \delta/2} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\},$$

$$Y_0(t) = \frac{48R_k(t)^3}{\pi\delta^3}\mathbb{E}_0\exp\Big\{\sqrt{2\delta}\theta T_1 \sup_{x\in Q_{R_k(t)}} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\},$$

$$Y_n(t) = \frac{48R_k(2^n t)^3}{\pi\delta^3}\mathbb{E}_0\exp\Big\{\sqrt{2\delta}p\theta T_1 \sup_{x\in Q_{R_k(2^n t)}} |\overline{V}_{\frac{\delta}{6},1}(x)|\Big\} \quad n = 1, 2, \cdots.$$

Similarly to (5.19) and (5.20) we get

$$Y_n(t) = \exp\Big\{o\Big(\big(\log(2^n t)\big)^2\Big)\Big\} \quad a.s. \quad n = 0, 1, \cdots$$

and

$$\mathbb{P}_0\big\{\tau_{Q_{R_k(2^{n-1}t)}} < 2t\big\} \le \exp\Big\{-C2^n(2^n t)^{\frac{k+2}{k-2}} l(2^{n-1}t)^{\frac{4}{3(k-2)}}\Big\}.$$

Due to the possibility that $\theta = (8k)^{-1}$, we can only make $p\theta < \big(8(k-1)\big)^{-1}$. So we may make $(8k)^{-1} < p\theta < \big(8(k-1)\big)^{-1}$. By the monotonicity of $\lambda_{p\theta\overline{V}}(D)$ in D,

$$\lambda_{p\theta\overline{V}}(Q_{R_k(2^n t)}) \le \lambda_{p\theta\overline{V}}(Q_{R_{k-1}(2^n t)}) = o\bigg(\Big(2^n t\Big)^{\frac{2}{k-2}} l\Big(2^n t\Big)^{\frac{2}{3(k-2)}}\bigg) \quad a.s.$$

where the second step follows from Lemma 5.1 with k being replaced by $k-1$.

Summarizing the bounds we obtained, the infinite series on the right hand side of (5.24) is asymptotically (as $t \to \infty$) and almost surely bounded by

$$C\sum_{n=1}^{\infty} \exp\Big\{-C^{-1}2^n\Big\}.$$

We now obtain desired (5.23) applying (5.24) and and the fact that

$$X(\theta t) + Y_0(t)\exp\Big\{t\lambda_{\theta\overline{V}}(Q_{R_k(t)}) = \exp\Big\{o\Big(t^{\frac{k+1}{k-1}} l(t)^{\frac{2}{3(k-1)}}\Big)\Big\} \quad a.s. \quad (t \to \infty).$$

(see Lemma 5.1). The proof is complete. □

5.4 Upper bound for Theorem 2.3

In view of (5.22), we only need to show

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \le 0 \quad a.s. \tag{5.25}$$

conditioning on the event $\left\{\omega\big(B(0,\delta)\big) = 0\right\}$.

We prove (5.25) under the extra assumption that

$$\int_1^\infty \frac{1}{t} \exp\big\{ - cl(t)\big\} dt < \infty$$

for some large constant $c > 0$, for otherwise we may consider $\tilde{l}(t) = \log\log t + l(t)$ instead of $l(t)$. Therefore, (5.9) can be assumed here.

Let $S_k(t)$ be given as in Lemma 5.2. We have that

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \qquad (5.26)\\
&\quad\le \mathbb{E}_0 \left[\exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\};\ \tau_{Q_{S_k(t)}} \ge 2t\right]\\
&\qquad + \Big(\mathbb{P}_0\big\{\tau_{Q_{S_k(t)}} < 2t\big\}\Big)^{1/q} \left(\mathbb{E}_0 \exp\left\{p\theta \int_0^t \overline{V}(B_s)ds\right\}\right)^{1/p}
\end{aligned}$$

where $p, q > 1$ are conjugate numbers.

In the case $1/24 < \theta < 1/16$ ($k = 2$), we can make p close to 1 so $p\theta < 1/16$. By the upper bound in Theorem 2.1 (with θ being replaced by $p\theta$ and $l(t) = (\log t)^2$)

$$\mathbb{E}_0 \exp\left\{p\theta \int_0^t \overline{V}(B_s)ds\right\} = \exp\left\{o\Big(t^3(\log t)^{4/3}\Big)\right\} \quad a.s. \quad (t \to \infty).$$

By the bound for Gaussian tail,

$$\mathbb{P}_0\big\{\tau_{Q_{S_2(t)}} < 2t\big\} \le \exp\Big\{ - Ct^{-1}S_2(t)^2\Big\} = \exp\Big\{ - Ct^5 l(t)^{-4/3}\Big\}.$$

Hence, the second term on the right hand side of (5.26) is negligible when $1/24 < \theta < 1/16$.

We now show that the same thing happens in the case when $0 < \theta \le 1/24$ ($k \ge 3$). In this case we can pick $p > 1$ such that $(8k)^{-1} < p\theta < \big(8(k-1)\big)^{-1}$. By Theorem 2.2 (with $l(t) = (\log t)^2$ and k being replaced by $k - 1$),

$$\mathbb{E}_0 \exp\left\{p\theta \int_0^t \overline{V}(B_s)ds\right\} = \exp\left\{o\Big(t^{\frac{k}{k-2}}(\log t)^{\frac{4}{3(k-2)}}\Big)\right\} \quad a.s. \quad (t \to \infty).$$

So our assertion follows from the Gaussian tail estimate

$$\mathbb{P}_0\Big\{\tau_{Q_{S_2(t)}} < 2t\Big\} \le \exp\Big\{ -Ct^{-1}S_k(t)^2\Big\} = \exp\Big\{ -Ct^{\frac{2k-1}{k-2}}l(t)^{-\frac{4}{3(k-2)}}\Big\}.$$

Therefore, the problem (in both $k=2$ and $k\ge 3$) has been reduced to the proof of

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}}l(t)^{\frac{2}{3(k-1)}}\log\mathbb{E}_0\left[\exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\};\ \tau_{Q_{S_k(t)}}\ge 2t\right]\le 0 \quad a.s. \tag{5.27}$$

conditioning on the event $\Big\{\omega\big(B(0,\delta)\big)=0\Big\}$.

By Lemma 3.6,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t \overline{V}(B_s)ds\right\};\ \tau_{Q_{S_k(t)}}\ge 2t\right]\le \exp\Big\{\theta t\sup_{|x|\le\delta/2}|\overline{V}_{\frac{\delta}{6},1}(x)|\Big\}\\
&\quad+\frac{6|Q_{S_k(t)}|}{\pi\delta^3}\mathbb{E}_0\exp\Big\{\sqrt{2\delta}T_1\theta\sup_{x\in Q_{S_k(t)}}|\overline{V}_{\frac{\delta}{6},1}(x)|\Big\}\exp\Big\{t\lambda_{\theta\overline{V}}(Q_{S_k(t)})\Big\}\\
&\quad=\exp\{O(t)\}+\exp\Big\{\Big(o\big(\log S_k(t)\big)^2\Big)\Big\}\exp\Big\{t\lambda_{\theta\overline{V}}(Q_{S_k(t)})\Big\}\quad a.s.\quad (t\to\infty)
\end{aligned}$$

where the last step follows from (3.11).

The required (5.27) follows from Lemma 5.2. □

6 Hardy inequality

Recall the definition of $\mathcal{F}_d(D)$ from (3.14). The family $\mathcal{F}_3$ is defined as

$$\mathcal{F}_3=\mathcal{F}_3(\mathbb{R}^3)=\Big\{g\in W^{1,2}(\mathbb{R}^3);\ \int_{\mathbb{R}^3}g^2(x)dx=1\Big\}.$$

The essential reason behind the main theorems in this paper is the Hardy's inequality. Searching in literature, we have found large amount of follow-up publication (i.e., [17] and [22]) on this subject, except Hardy's original paper. For reader's convenience, we state Hardy's inequality for $d=3$ in the following lemma and provide a short proof.

Lemma 6.1 *For any $f_\epsilon\in W^{1,2}(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3}\frac{f^2(x)}{|x|^2}dx\le 4\int_{\mathbb{R}^3}|\nabla f(x)|^2dx. \tag{6.1}$$

Further, the number 4 is the best constant in the sense that for any $\epsilon>0$ one can find a function $f_\epsilon\in W^{1,2}(\mathbb{R}^3)$ with compact support such that

$$\int_{\mathbb{R}^3}\frac{f_\epsilon^2(x)}{|x|^2}dx>(4-\epsilon)\int_{\mathbb{R}^3}|\nabla f_\epsilon(x)|^2dx. \tag{6.2}$$

Proof: Write $x = (x_1, x_2, x_3)$. Using integration by parts

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx = \int_{\mathbb{R}^3} x_j \Big[\frac{2x_i}{|x|^4} f^2(x) - \frac{2}{|x|^2} f(x) \frac{\partial f}{\partial x_j}\Big] dx \qquad j = 1, 2, 3.$$

Summing over j on the both sides

$$3\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx = 2\int_{\mathbb{R}^3} \Big[\frac{f^2(x)}{|x|^2} - \frac{\nabla f \cdot x}{|x|^2} f(x)\Big] dx.$$

Thus,

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx = -2\int_{\mathbb{R}^3} \frac{\nabla f \cdot x}{|x|} \frac{f(x)}{|x|} dx \le 2\Big(\int_{\mathbb{R}^3} \frac{|\nabla f \cdot x|^2}{|x|^2} dx\Big)^{1/2} \Big(\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx\Big)^{1/2}.$$

Therefore,

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \le 4\int_{\mathbb{R}^3} \frac{|\nabla f \cdot x|^2}{|x|^2} dx \le 4\int_{\mathbb{R}^3} |\nabla f(x)|^2 dx.$$

To establish (6.2), for each large $M > 0$, we define $g_M \in W^{1,2}(\mathbb{R}^3)$ as following:

$$g_M(x) = \begin{cases} M^{1/2} & 0 \le |x| \le M^{-1} \\ |x|^{-1/2} & M^{-1} < |x| \le M \\ \dfrac{2M - |x|}{M^{3/2}} & M < |x| \le 2M \\ 0 & |x| > 2M. \end{cases}$$

It is straightforward to exam that g_M is locally supported and

$$\int_{\mathbb{R}^3} \frac{g_M^2(x)}{|x|^2} dx = \Big\{4 - 28\Big(\frac{7}{3} + \frac{1}{2}\log M\Big)^{-1}\Big\} \int_{\mathbb{R}^3} |\nabla g_M(x)|^2 dx.$$

For each $\epsilon > 0$, take $M > 0$ sufficiently large so

$$28\Big(\frac{7}{3} + \frac{1}{2}\log M\Big)^{-1} < \epsilon$$

and let $f_\epsilon(x) = g_M(x)$.$\Box$

What has been frequently used in this paper is the following version of Hardy's inequality.

Lemma 6.2 *For any $\theta > 0$,*

$$\sup_{g \in \mathcal{F}_3} \Big\{\theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla g(x)|^2 dx\Big\} = \begin{cases} 0 & \textit{if } \theta \le 1/8, \\ \infty & \textit{if } \theta > 1/8. \end{cases} \tag{6.3}$$

Proof: By Hardy's inequality, the left hand side of (6.3) is non-positive when $\theta < 1/8$. On the other hand, it is no less than

$$-\frac{1}{2}\inf_{g\in\mathcal{F}_3}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx$$

which is equal to zero. Thus, for $\theta \le 1/8$,

$$\sup_{g\in\mathcal{F}_3}\left\{\theta\int_{\mathbb{R}^3}\frac{g^2(x)}{|x|^2}dx - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\} = 0.$$

Assume $\theta > 1/8$. By the optionality of Hardy's inequality described in (6.2),

$$H(\theta) \equiv \sup_{g\in\mathcal{F}_3}\left\{\theta\int_{\mathbb{R}^3}\frac{g^2(x)}{|x|^2}dx - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla g(x)|^2dx\right\} > 0.$$

Given $a > 0$, the substitution $g(x) = a^{3/2}f(ax)$ leads to $H(\theta) = a^2H(\theta)$. So $M(\theta) = \infty$.
□

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References

[1] Biskup, M. and König, W. (2001). Screening effect due to heavy lower tails in one-dimensional parabolic Anderson model. *J. Statist. Phys.* **102** 1253–1270.

[2] Carmona, R.A. and Molchanov, S.A. (1994). *Parabolic Anderson problem and intermittency.* Amer. Math. Soc., Providence, RI.

[3] Carmona, R.A. and Viens, F.G. (1998). Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter, Stochastics, 62, 3&4, 251 – 273

[4] Chen, X. *Random Walk Intersections: Large Deviations and Related Topics.* Math. Surv. Mono. **157**, Providence 2009.

[5] Chen, X. (2010). Quenched asymptotics for Brownian motion of renormalized Poisson potential and for the related Anderson models. Ann. Probab. (to appear)

[6] Chen, X. and Kulik, A. M. (2010). Brownian motion and parabolic Anderson model in a renormalized Poisson potential. Annales de l'Institut Henri Poincaré (to appear).

[7] Chen, X. and Kulik, A. M. Asymptotics of negative exponential moments for annealed Brownian motion in a renormalized Poisson potential. (manuscript)

[8] Cranston, M., Gauthier, D.and Mountford, T. S. (2010). On large deviations for the parabolic Anderson model. Probab. *Theory Related Fields* **147** 349–378.

[9] Dalang, R. C. and Mueller, C. (2009). Intermittency properties in a hyperbolic Anderson problem. *Annales de l'Institut Henry Poincare* **45** 1150–1164.

[10] Donsker, M. D. and Varadhan, S. R. S. (1975). Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math., 28(4)*, 525 – 565.

[11] Florescu, I. and Viens, F. (2006). Sharp estimation of the almost-sure Lyapunov exponent for the Anderson model in continuous space. *Probab. Theory Related Fields* **135** 603–644.

[12] Frank, W.M., Land, D.J. and Spector, R.M. (1971). Singular potentials. *Reviews of Modern Physics* **43** 36–98.

[13] Fukushima, R. Second order asymptotics for Brownian motion in a heavy tailed Poissonian potential. (preprint)

[14] Gärtner, J. and König, W. (2000). Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10** 192 – 217.

[15] Gärtner, J., König, W. and Molchanov, S. (2000). Almost sure asymptotics for the continuous parabolic Anderson model. *Probab. Theory Related Fields* **118** 547 – 573.

[16] Gärtner, J. and Molchanov, S. A. (1990). Parabolic problem for the Anderson model. *Comm. Math. Phys.* **132** 613-655

[17] Hardy, G., Pólya, G. and Littlewood, J. E. (1952). *Inequalities.* 2nd Edition. Combridge

[18] Komorowski, T.(2000). Brownian motion in a Poisson obstacle field. *Séminaire Bourbaki,* **1998/99** 91–111.

[19] Marcus, M. B. and Rosiński, J. (2005). Continuity and boundedness of infinitely divisible process: A Poisson point process approach. *J. Theor. Probab.* **18** 109-160.

[20] McDaniel, E.W. *Collision phenomena in ionized gases.* Wiley, New York, 1964.

[21] Newton, R.G. *Scattering theory of waves and particles.* McGraw-Hill, Inc., New York, 1966.

[22] Opic, B. and Kufner, A. (1990). Hardy-type inequalities. *Pitman Research Notes in Math.* **219** Longman.

[23] Pastur, L. A. (1977). The behavior of certain Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential. *Teoret. Mat. Fiz.,* **32(1)**, 88 – 95.

[24] Povel, T. (1999). Confinement of Brownian motion among Poissonian obstacles in $\mathbb{R}^d, d \geq 3$. *Probab. Theory Related Fields* **114** 177–205.

[25] Rajput, B. S. and Rosiński, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82** 451–487.

[26] Resnick, S.I. (1988), Association and multivariate extreme value distributions, in: C.C. Heyde, ed., *Gani Festschrift: Studies in Statistical Modelling and Statistical Science Statistical Society of Australia.*

[27] Samorodnitsky, G. (1995). Association of infinitely divisible random vectors. *Stoc. Proc. Appl* **55** 45–55.

[28] Samorodnitsky, S. and Taqqu, M.S. (1994). Lévy measures of infinitely divisible random vectors and Slepian inequalities. *Annals Probab.* 22, 1930–1956.

[29] Stolz, G. (2000). Non-monotonic random Schrödinger operators: the Anderson model. *Journal of Math. Analysis and Appl.*, 248, Issue 1, 173 – 183

[30] Sznitman, A-L. *Brownian motion, obstacles and random media.* Springer-Verlag, Berlin, 1998.

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