Math 241 (12:40-1:30pm) Quiz #5 Fall 2024 Name:

1. Find the directional derivative $D_{\mathbf{u}}f(1,1,1)$ of the function $f(x,y,z) = xy^2z^3$ in the direction of $\mathbf{v} = \langle 2,2,1\rangle$. Find the direction that leads to the maximum of $D_{\mathbf{u}}f(1,1,1)$ and the value of $D_{\mathbf{u}}f(1,1,1)$ in that direction.

Solution. $\nabla f(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ and therefore $\nabla f(1,1,1) = \langle 1,2,3\rangle$.

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

Thus,

$$D_{\mathbf{u}}f(1,1,1) = \nabla f(1,1,1)\cdot \mathbf{u} = \frac{2}{3} + 2\times\frac{2}{3} + 3\times\frac{1}{3} = \frac{7}{3}$$

The direction leading to the maximal rate is $\nabla f(1,1,1) = \langle 1,2,3\rangle$. The maximal rate is

$$\|\nabla f(1,1,1)\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

2. Find the equation of the tangent plane to the surface $x^2+y^2+z^2 = 1$ at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.

Solution. Set $F(x,y,z) = x^2+y^2+z^2$. Then $\nabla F(x,y,z) = \langle 2x, 2y, 2z\rangle$ and the normal vector $\mathbf{n} = \nabla F(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \langle \sqrt{2}, \sqrt{2}, 0\rangle$. The equation of the tangent plane is

$$\sqrt{2}(x - \frac{1}{\sqrt{2}}) + \sqrt{2}(y - \frac{1}{\sqrt{2}}) = 0 \quad \text{or} \quad x + y = \sqrt{2}$$