Parabolic Anderson model with a fractional Gaussian noise that is rough in time

Xia Chen*

March 28, 2019

Abstract

This paper concerns the parabolic Anderson equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^{d+1} W^{\mathbf{H}}}{\partial t \partial x_1 \cdots \partial x_d}$$

generated by a $(d+1)$-dimensional fractional noise with the Hurst parameter $\mathbf{H} = (H_0, H_1, \cdots, H_d)$ with special interest in the setting that some of $H_0, \cdots, H_d$ are less than half. In the recent work [9], the case of the spatial roughness has been investigated. To put the last piece of the puzzle in place, this work investigates the case when $H_0 < 1/2$ with the concern on solvability, Feynman-Kac's moment formula and intermittency of the system.

Key-words: parabolic Anderson equation, Dalang's condition, fractional, rough and critical Gaussian noises, Feynman-Kac's representation, Brownian motion, moment asymptotics

AMS subject classification (2010): 60F10, 60H15, 60H40, 60J65, 81U10.

*Research partially supported by the Simons Foundation #585506.

1 Introduction

In this paper we consider the parabolic Anderson equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + \theta \dot{W}^{\mathbf{H}}(t,x) \diamond u(t,x) & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

with the fractional Gaussian noise

$$\dot{W}^{\mathbf{H}}(t,x) = \frac{\partial^{d+1} W^{\mathbf{H}}}{\partial t \partial x_1 \cdots \partial x_d}(t, x_1, \cdots, x_d) \quad \text{where } x = (x_1, \cdots, x_d) \tag{1.2}$$

given as the formal derivative of a fractional Brownian sheet $W^{\mathbf{H}}(t,x)$ ($(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$) with the Hurst index $\mathbf{H} = (H_0, H_1, \cdots, H_d)$ ($0 < H_0, \cdots H_d < 1$) that is defined as a mean zero Gaussian field with the covariance function

$$\mathbb{E}\Big\{ W^{\mathbf{H}}(s,x) W^{\mathbf{H}}(t,y) \Big\} = Q_0(s,t) Q(x,y) \tag{1.3}$$

where

$$Q_0(s,t) = R_{H_0}(s,t) \ \text{ and } \ Q(x,y) = \prod_{j=1}^{d} R_{H_j}(x_j, y_j)$$

and

$$R_{H_j}(u,v) = \frac{1}{2}\Big\{ |u|^{2H_j} + |v|^{2H_j} - |u-v|^{2H_j} \Big\} \quad u, v \in \mathbb{R} \ \ j = 0, 1, \cdots, d.$$

In (1.1), $\theta > 0$ is a given constant and the notation "$\diamond$" represents the Wick product. For simplicity we assume the bounded initial condition

$$0 < \inf_{x \in \mathbb{R}^d} u_0(x) \le \sup_{x \in \mathbb{R}^d} u_0(x) < \infty. \tag{1.4}$$

The fractional Gaussian noise appears to be one of the most interesting Gaussian noises partially because it presents a full spectrum of very different behaviors along the change of Hurst parameter $\mathbf{H}$. The parabolic Anderson equation of fractional Gaussian noise has been extensively investigated. The case when $H_0, \cdots, H_d \ge 1/2$ is fully understood as far as question of existence/uniqueness is concerned. We cite the references [3], [6], [7], [8]. [10], [13], [17], [15], [20] and [23] for an incomplete list. In this case, the Dalang's condition

$$d - \sum_{j=1}^{d} H_j < 1 \tag{1.5}$$

gives the precise criteria as when the system (1.1) is solvable.

A recent development in literature (see, e.g.,[2], [13], [9], [11], [14], [18], [21] and [22])) is to consider the case when the fractional noise $\dot{W}^{\mathbf{H}}$ is rough, i.e., $H_j < 1/2$ for some $0 \le j \le d$.

In [9], the author shows (Theorem 1.2 and 1.3, [9]) that for $H_0 \geq 1/2$, the parabolic Anderson equation (1.1) admits a unique solution (in the sense detailed later) under the condition

$$\begin{cases} d - H < 1 \\ 4(1 - H_0) + 2(d - H) + (d_* - 2H_*) < 4. \end{cases} \tag{1.6}$$

Here and elsewhere in the paper, we adopt the notations

$$J_* = \{1 \leq j \leq d;\ H_j < 1/2\},\quad d_* = \#\{J_*\},\quad H = \sum_{j=1}^{d} H_j \ \text{ and } \ H_* = \sum_{j \in J_*} H_j.$$

The last missing piece of the puzzle in the setting of fractional Gaussian noise is the case when $H_0 < 1/2$, i.e., the noise $\dot{W}^{\mathbf{H}}$ is rough in time. It forms the main topic of this work. The relevant papers that the author is aware of are [4], [9], [14] and [19]. Theorem 1.2 in the recent paper by Deya [14] includes the setting of rough time in $(1+1)$-dimension. It should be pointed that the solution in [14] is defined in a way different from ours. As evidence, it is not hard to see that Deya's formulation (Theorem 1.2, [14]) for solvability is very different from the one given in this paper (see (1.19) below). When the time-space derivative $\dot{W}^{\mathbf{H}}$ is replaced by time-derivative noise $\partial W^{\mathbf{H}}/\partial t$, the time is allowed to be rough, as claimed in [4]. Finally, Proposition 4.4 in [19] and Proposition 1.4 in [9] suggest a possibility of rough time in our regime. On the other hand, there has not been any conclusion showing that $\dot{W}^{\mathbf{H}}$ is allowed to be rough while the equation (1.1) remains solvable in the sense of Definition 1.1 below.

To clarify what we mean by solving the equation (1.1), we introduce the following definition.

Definition 1.1 *An adapted random field $\{u(t,x);\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\}$ is a solution to the equation (1.1), if for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $u(t,x) \in \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, the process*

$$\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s);\ \ (s,y) \in \mathbb{R}^+ \times \mathbb{R}^d\}$$

is Skorokhod integrable with respect to the Gaussian differential $W^{\mathbf{H}}(\delta s, \delta y)$, and $u(t,x)$ satisfies

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u(s,y)W^{\mathbf{H}}(\delta s, \delta y) \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{1.7}$$

where $p_s(y)$ $((s,y) \in \mathbb{R}^+ \times \mathbb{R}^d)$ is the Brownian semi-group and the stochastic integral appearing on the right hand side is the Skorokhod integral.

We point the references [17] or [19] for the background of the Skorokhod integration and some other material on Malliavin calculus needed in this paper.

Once the equation (1.1) is solved, it is expected that the solution yields the Feynman-Kac moment representation

$$\begin{aligned} &\mathbb{E}u^m(t,x) \\ &= \mathbb{E}_x\left[\exp\left\{\theta^2 \sum_{1 \leq j < k \leq m} \int_0^t \int_0^t \gamma_0(s-r)\gamma\big(B_j(s) - B_k(r)\big)drds\right\} \prod_{j=1}^{m} u_0\big(B_j(t)\big)\right] \end{aligned} \tag{1.8}$$

for $x \in \mathbb{R}^d$ and $m = 2, 3, \cdots$, where "$\mathbb{E}_x$" denotes the expectation with respect to the independent d-dimensional Brownian motions $B_1(t), \cdots, B_m(t)$ with $B_1(0) = \cdots = B_m(0) = x$ and $\gamma_0(\cdot)$ and $\gamma(\cdot)$ are time and space covariance functions of the generalized Gaussian field $\dot{W}^{\mathbf{H}}$ determined by the relation

$$\mathrm{Cov}\left(\dot{W}^{\mathbf{H}}(s,x), \dot{W}^{\mathbf{H}}(t,y)\right) = \gamma_0(s-t)\gamma(x-y) \quad (s,x),(t,y) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{1.9}$$

When $H_0 \geq 1/2$, the time covariance $\gamma_0(\cdot)$ is explicitly given as

$$\begin{cases} \gamma_0(u) = H_0(2H_0-1)|u|^{-(2-2H_0)} & \text{as } \ H_0 > \frac{1}{2} \\ \\ \gamma_0(u) = \delta_0(u) & \text{as } \ H_0 = \frac{1}{2} \quad (u \in \mathbb{R}). \end{cases} \tag{1.10}$$

For the reason that the function $H_0(2H_0-1)|u|^{-(2-2H_0)}$ is no longer non-negative definite as $H_0 < 1/2$, it can not be chosen as a covariance function. The way to extend the above expressions to the setting $H_0 < 1/2$ relies on the Fourier transform

$$\gamma_0(u) = \int_{\mathbb{R}} e^{i\lambda u}\mu_0(d\lambda) \ \text{ with } \ \mu_0(d\lambda) = \frac{\Gamma(2H_0+1)\sin\pi H_0}{2\pi}|\lambda|^{1-2H_0}d\lambda \tag{1.11}$$

partially for the fact that (1.10) and (1.11) are consistent as $H_0 \geq 1/2$. When extended to the setting $H_0 < 1/2$, the constant

$$\frac{\Gamma(2H_0+1)\sin\pi H_0}{2\pi}$$

appearing in (1.11) is identified by the constraint

$$\int_0^1\int_0^1 \gamma_0(s-r)dsdr = 1, \tag{1.12}$$

the relation

$$\int_0^1\int_0^1 \gamma_0(s-r)dsdr = \int_{\mathbb{R}} \left| \int_0^1 e^{i\lambda s}ds \right|^2 \mu_0(d\lambda)$$

and the identity (6.2), Lemma 6.1 in Appendix below.

It should be pointed out that when $H_0 < 1/2$, the above Fourier transform is not point-wisely defined. Instead, $\gamma_0(\cdot)$ is given as generalized function defined by the linear form

$$\langle \gamma_0, \psi \rangle = \frac{\Gamma(2H_0+1)\sin\pi H_0}{2\pi} \int_{\mathbb{R}} |\lambda|^{1-2H_0}\mathcal{F}(\psi)(\lambda)d\lambda \quad \psi \in \mathcal{S}(\mathbb{R})$$

on the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$, where $\mathcal{F}(\psi)$ is the Fourier transform of ψ:

$$\mathcal{F}(\psi)(\lambda) = \int_{\mathbb{R}} e^{i\lambda u}\psi(u)du \quad \lambda \in \mathbb{R}.$$

As $H_0 < 1/2$, $\gamma(\cdot)$ is no longer non-negative in any reasonable sense, for the function $|\lambda|^{1-2H_0}$ is not non-negative definite. A simple but heuristic way to show it is to take $\psi \equiv 1$:

$$\int_{\mathbb{R}} \gamma_0(u)du = \langle \gamma_0, 1\rangle = \int_{\mathbb{R}} |\lambda|^{1-2H_0}\delta_0(\lambda)d\lambda = |0|^{1-2H_0} = 0.$$

Similarly, with possible roughness in space, $\gamma(\cdot)$ is formally given as

$$\gamma(x) = \int_{\mathbb{R}} e^{i\xi\cdot x}\mu(d\xi) \ \text{ with } \ \mu(d\xi) = C_H\Big(\prod_{j=1}^{d}|\xi_j|^{1-2H_j}\Big)d\xi \quad x \in \mathbb{R}^d \tag{1.13}$$

where we adopt the notation $\xi = (\xi_1, \cdots, \xi_d)$ and

$$C_H = \prod_{j=1}^{d}\frac{\Gamma(2H_j+1)\sin\pi H_j}{2\pi}.$$

In literature $\mu_0(d\lambda)$ and $\mu(d\xi)$ are called spectral measures of $\gamma_0(\cdot)$ and $\gamma(\cdot)$, respectively.

Since $\gamma_0(\cdot)$ exists as a generalized function in this paper, we seek a more comprehensive way to re-write the Feynman-Kac moment representation. Given two independent d-dimensional Brownian motion B_t and $\widetilde{B}_t$, set

$$\begin{aligned}\mathcal{Q}_t(B,\widetilde{B}) &= H_0\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\big\{s^{-(1-2H_0)} + (t-s)^{-(1-2H_0)}\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds \\ &+ \frac{H_0(1-2H_0)}{2}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\frac{\big[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}\big]\big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr\end{aligned} \tag{1.14}$$

whenever the integrals on the right hand side are properly defined.

Theorem 1.2 *Let $1/4 < H_0 < 1/2$. Under the assumption*

$$4(1-2H_0) + 2(d-H) + (d_* - 2H_*) < 2 \tag{1.15}$$

the parabolic Anderson equation (1.1) admits a unique solution $u(t,x)$ in the sense of Definition 1.1. Further, we have the Feynman-Kac moment representation

$$\mathbb{E}u^m(t,x) = \mathbb{E}_x\left[\exp\left\{\theta^2\sum_{1\le j<k\le m}\mathcal{Q}_t(B_j,B_k)\right\}\prod_{j=1}^{m}u_0\big(B_j(t)\big)\right] \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \tag{1.16}$$

for $m = 2, 3, \cdots$, where $B_1, \cdots, B_m$ are independent d-dimensional Brownian motions with $B_1(0) = \cdots = B_m(0) = x$.

One of the major ingredients of this paper is to establish the decomposition

$$\begin{aligned}&\int_0^t\int_0^t\gamma_0(s-r)\gamma\big(B_s-\widetilde{B}_r\big)drds \\ &= H_0\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\big\{s^{-(1-2H_0)} + (t-s)^{-(1-2H_0)}\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds \\ &+ \frac{H_0(1-2H_0)}{2}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\frac{\big[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}\big]\big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr\end{aligned} \tag{1.17}$$

for two independent d-dimensional Brownian motions B and $\widetilde{B}$. Consequently, (1.8) and (1.16) are consistent.

Recall (Theorem 1.2 and 1.3, [9]) that when $H_0 \geq 1/2$, the parabolic Anderson equation (1.1) admits a unique solution under the assumption (1.6). Together with Theorem 1.2, a complete picture emerges on the solvability for the parabolic Anderson equation with fractional Gaussian noise. On the other hand, by the moment representation (1.8) (with $m = 2$) and the square integrability requirement in Definition 1.1, a necessary condition for solvability is

$$\mathbb{E}_0 \exp\left\{\theta^2 \int_0^t \int_0^t \gamma_0(s-r)\gamma(B_s - \widetilde{B}_r)dsdr\right\} < \infty \quad \forall t > 0. \tag{1.18}$$

Evidence suggests that this does not happen without (1.6) when $H_0 > 1/2$ or without (1.15) when $H_0 < 1/2$. In the special case when $d = 1$, $H_0 = 1/2$ and $H < 1/2$, for instance, it is proved (p.75, [12]) that a condition necessary for

$$\mathbb{E}_0\bigg[\int_0^1 \int_0^1 \delta_0(s-r)\gamma(B_s - \widetilde{B}_s)dsdr\bigg]^2 = \mathbb{E}_0\bigg[\int_0^1 \gamma(B_s - \widetilde{B}_s)ds\bigg]^2 < \infty$$

is $H > 1/4$, which is the second part in (1.6). The argument used there strongly (but not conclusively) suggests that the conditions

$$4(1-H_0) + 2(d-H) + (d_* - 2H_*) < 4 \quad \text{and} \quad 4(1-2H_0) + 2(d-H) + (d_* - 2H_*) < 2$$

might be necessary for

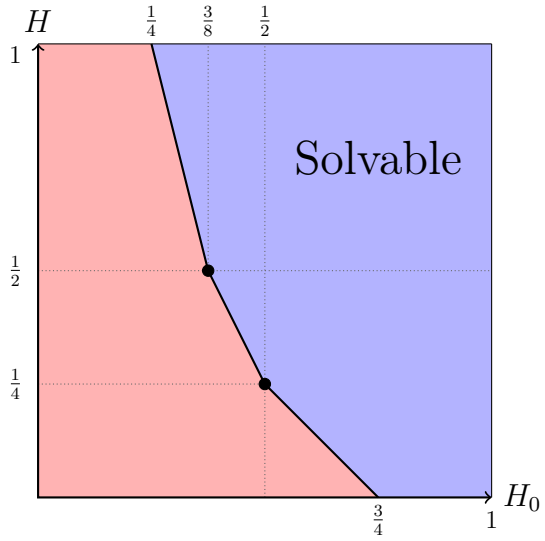
$$\mathbb{E}_0\bigg[\int_0^1 \int_0^1 \gamma_0(s-r)\gamma(B_s - \widetilde{B}_r)dsdr\bigg]^2 < \infty$$

in the settings $H_0 \geq 1/2$ and $H_0 < 1/2$, respectively.

As for the necessity of "$d - H < 1$" in (1.6), we consider the case of non-rough fractional noise with $H_0 > 1/2$ for simplicity. Given $\lambda > 0$

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\lambda \int_0^1 \int_0^1 \gamma_0(s-r)\gamma(B_s - \widetilde{B}_r)dsdr\right\} \\
&\geq \exp\left\{\lambda\gamma(2\epsilon)\int_0^1 \int_0^1 \gamma_0(s-r)dsdr\right\}\left(\mathbb{P}_0\Big\{\max_{s\leq 1}|B_s| \leq \epsilon\Big\}\right)^2 \\
&= \exp\Big\{\lambda C(H)\epsilon^{-2(d-H)}\Big\}\exp\Big\{-2\big(1+o(1)\big)C_d\epsilon^{-2}\Big\} \quad (\epsilon \to 0^+)
\end{aligned}$$

where $C(H) > 0$ depends only on H, $C_d > 0$ is the universal constant appearing in the small ball probability for d-dimensional Brownian motions, and we used (1.12) in the last step. As $\epsilon \to 0^+$, the right hand side tends to infinity for all $\lambda > 0$ as $d - H > 1$, and for large $\lambda > 0$ as $d - H = 1$. By the Brownian scaling and by the time-space homogeneities of $\gamma_0(\cdot)$ and $\gamma(\cdot)$, therefore, (1.18) is impossible when $d - H \geq 1$.



Applying the criteria (1.6) and (1.15) to the $(1+1)$-dimension (i.e., the case when $d=1$, $\mathbf{H}=(H_0,H)$), we obtain "the domain of solvability"

$$\begin{cases} H_0 \geq \dfrac{1}{2} \text{ and } H \geq \dfrac{1}{2}: & \text{automatically solvable} \\ H_0 \geq \dfrac{1}{2} \text{ and } H < \dfrac{1}{2}: & \text{solvable if } H_0 + H > \dfrac{3}{4} \\ H_0 < \dfrac{1}{2} \text{ and } H \geq \dfrac{1}{2}: & \text{solvable if } 4H_0 + H > 2 \\ H_0 < \dfrac{1}{2} \text{ and } H < \dfrac{1}{2}: & \text{solvable if } 2H_0 + H > \dfrac{5}{4} \end{cases} \tag{1.19}$$

and its graphic illustration.

The next problem is on the intermittency of the solution. Recall (Theorem 1.5, [9]) that when $H_0 \geq 1/2$, for any $m \geq 2$,

$$\lim_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}u^m(t,x) = \kappa_m(\mathbf{H})\theta^{2/H} \tag{1.20}$$

with $0 < \kappa_m(\mathbf{H}) < \infty$ under the assumption (1.6). Notice that $\frac{2H_0+H-d}{1-(d-H)} > 1$, $= 1$ and < 1 for $H_0 > 1/2$, $= 1/2$ and $< 1/2$, respectively. Does it suggest a sub-linear growth for $\log \mathbb{E}u^m(t,x)$ as $H_0 < 1/2$? The following partial result for $m=2$ (known as weak intermittency in literature) tells a different story.

Theorem 1.3 *Under the assumption of Theorem 1.2, there is a constant $\kappa(\mathbf{H}) > 0$ depending only on $\mathbf{H} = (H_0, \cdots, H_d)$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}u^2(t,x) = \kappa(\mathbf{H})\theta^{\frac{2}{2H_0+H-d}}. \tag{1.21}$$

For any $m \geq 2$, Theorem 1.3 and the Feynman-Kac moment representation (1.16) suggest the pattern of intermittency described by the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}u^m(t,x) = \kappa_m(\mathbf{H})\theta^{\frac{2}{2H_0+H-d}} \quad m = 2, 3, \cdots \tag{1.22}$$

in the setting of $H_0 < 1/2$. Establishing (1.22) with $\kappa_m(\mathbf{H})$ being identified will be an interesting problem. In addition, the high moment asymptotics

$$\log \mathbb{E}u^m(t,x) \quad (m \to \infty)$$

and related behaviors of the system need to be investigated. The comparison between (1.20) and (1.21) indicates a substantially new moment asymptotic behavior for $H_0 < 1/2$. We leave this pursue to the future study.

We now highlight some of the new ideas we introduce and new challenges we face in this paper. By a procedure through Itô-Wiener expansion (briefly reviewed in section 2), solving (1.1) and investigating its intermittency become, respectively, the exponential integrability and exponential asymptotics for the conditional covariance formally represented as

$$\begin{aligned} &\mathrm{Cov}\left(\int_0^t \dot{W}^{\mathbf{H}}(s, B_{t-s})ds, \quad \int_0^t \dot{W}^{\mathbf{H}}(s, \widetilde{B}_{t-s})ds \Big| B, \widetilde{B} \right) \\ &= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t \gamma_0(s-r) e^{i\xi\cdot(B_s - \widetilde{B}_r)} dsdr. \end{aligned} \tag{1.23}$$

By Taylor expansion, the exponential integrability is installed by sufficiently sharp bounds for the n-moment ($n = 1, 2, \cdots$) of the Brownian Hamiltonian given on the right hand side, which usually contain the factorial multiple $(n!)^{p(\mathbf{H})}$ (see (4.20) below for the ultimate bounds in our setting). To ensure the requested exponential integrability, it is important to have $p(\mathbf{H}) < 1$. Taking the risk of technicality and ignoring the fact that the time covariance $\gamma_0(\cdot)$ exists only as generalized function, we conduct the formal computation

$$\begin{aligned} &\mathbb{E}_0\left[\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t \gamma_0(s-r) e^{i\xi\cdot B_s} e^{-i\xi\cdot \widetilde{B}_r} dsdr \right]^n \\ &= \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^{2n}} \left(\prod_{k=1}^n \gamma_0(s_k - r_k) \right) \left(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B_{s_k}} \right) \left(\mathbb{E}_0 \prod_{k=1}^n e^{-i\xi_k \cdot B_{r_k}} \right) dsdr \end{aligned}$$

for any $n = 1, 2, \cdots$. Here and elsewhere in the paper, we adopt the simplified notations

$$\mu(d\xi) = \mu(d\xi_1) \cdots \mu(d\xi_n), \quad ds = ds_1 \cdots ds_n \ \text{ and } \ dr = dr_1 \cdots dr_n$$

in the context whenever it becomes obvious. To handle the Brownian expectations on the right hand side, a treatment frequently appearing in this work (and in literature as well) is the time re-arrangement. Write

$$[0,t]^n_< = \{(s_1, \cdots, s_n) \in [0,t]^n; \ \ s_1 < \cdots < s_n\}.$$

By permutation invariance, the n-moment is equal to

$$n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<\times[0,t]^n}\Big(\prod_{k=1}^n\gamma_0(s_k-r_k)\Big)\Big(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\Big)\Big(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B_{r_k}}\Big)dsdr.$$

On $[0,t]^n_<$,

$$\begin{aligned}\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}} &= \mathbb{E}_0\exp\Big\{i\sum_{k=1}^n\Big(\sum_{j=k}^n\xi_k\Big)\cdot(B_{s_k}-B_{s_{k-1}})\Big\}\\ &= \exp\Big\{-\frac{1}{2}\sum_{k=1}^n\Big|\sum_{j=k}^n\xi_k\Big|^2(s_k-s_{k-1})\Big\}\quad (s_0=0).\end{aligned}$$

So we have

$$\begin{aligned}&\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\gamma_0(s-r)e^{i\xi\cdot B_s}e^{-i\xi\cdot\widetilde{B}_r}dsdr\Big]^n\\ &= n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<\times[0,t]^n}\exp\Big\{-\frac{1}{2}\sum_{k=1}^n\Big|\sum_{j=k}^n\xi_k\Big|^2(s_k-s_{k-1})\Big\}\Big(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B_{r_k}}\Big)dsdr.\end{aligned}$$

From the above computation, we see that the factorial $n!$ appears as the cost for s-time-permutation. By a trick of time-exponentiation, the actual cost for this job is less than $n!$. Should we pay another $n!$ for getting the r-expectation evaluated? First, doing so would rule out any chance for the needed exponential integrability in the setting $H_0<1/2$ regardless how $H_1,\cdots,H_d$ are restricted. Second, the proposed payment is some what un-necessary as most of the mass concentrates near the diagonal $\{s=r\}$[1]. To a degree, therefore, re-arranging "$s_1<\cdots<s_n$" already puts extra weight on the same order "$r_1<\cdots<r_n$". The challenge is how to carry out this idea mathematically.

This is a long existing problem even for the setting $H_0>1/2$. In most publications in literature, the choice is often between the double permutation and the adoption of the obvious bound

$$0<\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot B_{r_k}}\le 1.$$

This compromised treatment may still allow for the needed exponential integrability in some cases when $H_0>1/2$ but often brings some extra restriction for solvability of (1.1). When $H_0<1/2$, the above bound is completely in-applicable as $\gamma_0(\cdot)$ is sign-switching.

In the recent paper [9], a new idea has been developed for $H_0>1/2$ which effectively lowers the cost of $n!$ by replacing the double time-integrals by a "β-multiple" but "time-free" integral with $\beta\approx 2H_0$. On the other hand, the setback in $H_0<1/2$ (Proposition 1.4, [9]) indicates a much more drastic measure is needed.

A major step in this work is the covariance decomposition given in Theorem 3.1 below which has been re-written in (1.17). In the decomposition (1.17), the first term is in a form of single

[1]For comparison, we list the easy case $H_0=1/2$ in which $\gamma_0(\cdot)=\delta_0(\cdot)$, double time-integral in (1.23) becomes a single time-integral and one permutation takes care of the computation.

time-integral to which only one time-rearrangement is needed for computing its n-moment. To bound the moment of the second term, we can lower the cost of $n!$ by using the Hölder continuity of the Brownian motions, a property that had not been made relevant to this type of the problems in which it was widely believed that Brownian motions typically take small values.

The new type of Brownian Hamiltonians appearing in the decomposition of (1.17) poses a new challenge when it comes to the problem of intermittency such as the one stated in Theorem 1.3. As the first term in the decomposition (1.17) is asymptotically negligible (Lemma 5.1), the main challenge is on the handling of the second term. A key observation (Lemma 5.2) is that the double integral of the second term is indeed one-dimensional (as far as the large t-behavior is concerned) due to high concentration of the mass near diagonal. This fact allows us to develop a strategy in connection to the spectral representation of the self-adjoint operators in establishing the limit in Theorem 1.3. To fully understand the intermittency in the case $H_0 < 1/2$, new ideas have to be developed in future study for the Brownian Hamiltonians appearing in the decomposition of Theorem 3.1.

Theorem 3.1 is partially inspired by a recent development in [4] where the time-space derivative $\dot{W}^{\mathbf{H}}$ is replaced by the time-derivative $\partial W^{\mathbf{H}}/\partial t$ with $H_0 < 1/2$. In Theorem 2.2, [4] (in connection to (2.6), [4]), the covariance decomposition

$$\begin{aligned} &\mathrm{Cov}\left(\int_0^t W^{\mathbf{H}}(ds, b_{t-s}),\quad \int_0^t W^{\mathbf{H}}(ds, \tilde{b}_{t-s})\right) \\ &= H_0 \int_0^t \left\{s^{-(1-2H_0)} + (t-s)^{-(1-2H_0)}\right\} Q(b_s, \tilde{b}_s) \\ &+ H_0(1-2H_0)\int_0^t\int_0^t \frac{Q(b_s,\tilde{b}_s) - Q(b_s,\tilde{b}_r)}{|s-r|^{2-2H_0}} dsdr \end{aligned} \tag{1.24}$$

is established for every pair of deterministic β-Hölder continuous function b_s and $\tilde{b}_s$ ($s \in [0,t]$) with $\beta H + H_0 > 1$, where $Q(x,y)$ is the space covariance function of the fractional Brownian sheet $W^{\mathbf{H}}(t,x)$ given in (1.3). We point also to [25] for its application in the intermittency for the parabolic Anderson equation with time-derivative Gaussian noise.

By Theorem 3.1, on the other hand, (1.23) can be re-written formally as

$$\begin{aligned} &\mathrm{Cov}\left(\int_0^t \dot{W}^{\mathbf{H}}(s, B_{t-s})ds,\quad \int_0^t \dot{W}^{\mathbf{H}}(s, \widetilde{B}_{t-s})ds \Big| B, \widetilde{B}\right) \\ &= H_0 \int_0^t \left\{s^{-(1-2H_0)} + (t-s)^{-(1-2H_0)}\right\} \gamma(B_s - \widetilde{B}_s) ds \\ &+ H_0(1-2H_0)\int_0^t\int_0^t \frac{\gamma(B_s-\widetilde{B}_s) - \gamma(B_s-\widetilde{B}_r)}{|s-r|^{2-2H_0}} dsdr. \end{aligned} \tag{1.25}$$

Here we recall that $\gamma(x,y) \equiv \gamma(x-y)$ is the space covariance of $\dot{W}^{\mathbf{H}}(t,x)$. The striking similarity between (1.24) and (1.25) suggests a general formula for covariance of the Gaussian integrals with rough time component. The ideas developed in connection to (1.24) and (1.25) may lead to a better understand of the parabolic Anderson equation with the general rough Gaussian noise.

Despite of the similarity in formulation, the tool for covariance decomposition in this paper is drastically different from the one developed in [4] where the proof is essentially analytic. Our argument is probabilistic and largely relies on the distributional properties of the Brownian motions.

The rest of the paper is organized as following: In section 2, a brief review of the treatment of Itô-Wiener expansion that re-formulates Theorem 1.2 and Theorem 1.3 into the problems on exponential integrability and exponential asymptotics for the Brownian Hamiltonian given in (1.23). Section 3 is devoted to the covariance decomposition (Theorem 3.1). The main topics in section 4 and section 5 are, respectively, the exponential integrability and exponential asymptotic surrounding the covariance decomposition developed in Theorem 3.1.

2 Solution in Itô-Wiener chaos expansion

In this section, we reduce the problem of solvability and Feynman-Kac moment representation to the problem on the exponential integrability for the Brownian Hamiltonian appearing in (1.23). The material presented in this section is essentially known (see, e.g., [17]). For the reader's convienience, it is briefly reviewed here without proof.

Iterating the mild equation (1.7) infinitely many times, we have formally expand the solution $u(t,x)$ (if exists) into the form

$$u(t,x)=\sum_{n=0}^{\infty} I_n\big(f_n(\cdot,t,x)\big) \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d. \tag{2.1}$$

where for each n, $I_n\big(f_n(\cdot,t,x)\big)$ is given as a n-multiple Skorokhod integral with the symmetrified integrand

$$\begin{aligned} &f_n(s_1,x_1,\cdots,s_n,x_n;t,x) \\ &=\frac{1}{n!}p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})\bigg(\prod_{k=1}^{n-1}p_{s_{\sigma(k+1)}-s_{\sigma(k)}}(x_{\sigma(k+1)}-x_{\sigma(k)})\bigg)(p_{s_{\sigma(1)}}*u_0)(x_{\sigma(1)})\mathbf{1}_{[0,t]^n}(s) \end{aligned} \tag{2.2}$$

and integration element $\theta W^{\mathbf{H}}(\delta s_1,\delta x_1)\cdots\theta W^{\mathbf{H}}(\delta s_n,\delta x_n)$, where σ denotes the permutation on $\{1,\cdots,n\}$ determined by the order $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$. By the $\mathcal{L}^2$-orthogonality of the expansion,

$$\mathbb{E}u^2(t,x)=\sum_{n=0}^{\infty} n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d. \tag{2.3}$$

A careful computation shows that for each $n\geq 0$,

$$\|I_n\big(f_n(\cdot,t,x)\big)\|^2_{\mathcal{H}^{\otimes n}}=\frac{\theta^{2n}}{(n!)^2}\mathbb{E}_x\bigg\{\bigg[\int_0^t\int_0^t\gamma_0(s-r)\gamma\big(B_s-\widetilde{B}_r\big)drds\bigg]^n u_0\big(B(t)\big)u_0\big(\tilde{B}(t)\big)\bigg\}.$$

where the notation $\mathbb{E}_x$ stands for the expectation with respect to two independent Brownian motions B and $\widetilde{B}$ with $B_0 = \widetilde{B}_0 = x$; $\gamma_0(\cdot)$ and $\gamma(\cdot)$ for the covariance functions given in (1.11) and (1.13), respectively. In connection to (2.3),

$$\mathbb{E}u^2(t,x) = \mathbb{E}_x\left[\exp\left\{\theta^2\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B(s)-\tilde{B}(r)\big)drds\right\}u_0\big(B(t)\big)u_0\big(\tilde{B}(t)\big)\right]. \quad (2.4)$$

With the bounded initial condition (1.2), the above heuristic derivation can be made into a mathematical statement that the parabolic Anderson equation (1.1) has a (unique) solution in the sense of Definition 1.1, provided

$$\mathbb{E}_0\exp\left\{C\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B_s-\widetilde{B}_r\big)drds\right\} < \infty \quad \forall C, t>0. \quad (2.5)$$

Notice that (2.4) is the Feynman-Kac moment representation (1.8) with $m=2$. In general, the Itô-Wiener chaos expansion in (2.1) morally supports the symbolic expression

$$u(t,x) = \mathbb{E}_x\left[\exp\left\{\theta\int_0^t \dot{W}^{\mathbf{H}}(s,B_{t-s})ds - \frac{\theta^2}{2}\mathrm{Var}\left(\int_0^t \dot{W}^{\mathbf{H}}(s,B_{t-s})ds\Big|B\right)\right\}u_0(B_t)\right]$$

where the Brownian motion B_t is independent of $W^{\mathbf{H}}$. By a formal use of Fubini theorem (known also *Replica* in literature)

$$\begin{aligned}
\mathbb{E}u^m(t,x) &= \mathbb{E}\otimes\mathbb{E}_x\left[\exp\left\{\theta\sum_{j=1}^m\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds\right.\right.\\
&\left.\left.-\frac{\theta^2}{2}\sum_{j=1}^m \mathrm{Var}\left(\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds\Big|B_j\right)\right\}\prod_{j=1}^n u_0(B_j(t))\right]\\
&= \mathbb{E}_x\left[\exp\left\{\frac{\theta^2}{2}\mathrm{Var}\left(\sum_{j=1}^m\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds\Big|B_1,\cdots,B_m\right)\right.\right.\\
&\left.\left.-\frac{\theta^2}{2}\sum_{j=1}^m \mathrm{Var}\left(\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds\Big|B_j\right)\right\}\prod_{j=1}^n u_0(B_j(t))\right].
\end{aligned}$$

By variance decomposition,

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{j=1}^m\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds\Big|B_1,\cdots,B_m\right)\\
&=\sum_{j=1}^m \mathrm{Var}\left(\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds\Big|B_j\right)\\
&+2\sum_{1\le j<k\le m}\mathrm{Cov}\left(\int_0^t \dot{W}^{\mathbf{H}}(s,B_j(t-s))ds,\ \int_0^t \dot{W}^{\mathbf{H}}(s,B_k(t-s))ds\Big|B_j,B_k\right)
\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Cov}\left(\int_0^t \dot{W}^{\mathbf{H}}(s, B_j(t-s))ds,\ \int_0^t \dot{W}^{\mathbf{H}}(s, B_k(t-s))ds \middle| B_j, B_k\right)\\&= \int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B_j(s) - B_k(r)\big)drds\end{aligned}$$

we have the Feynman-Kac moment representation given in (1.8). The above argument can be made mathematically rigorous, provided that the exponential integrability given in (2.5) holds. Under the assumption in Theorem 1.2, (1.8) and (1.16) are equivalent, as a consequence of Theorem 3.1.

According to the above discussion, the proof of Theorem 1.2 is reduced to the establishment of the exponential integrability given in (2.5).

Without a proper assumption, the Brownian time-integral in (2.5) does not have to make sense as $\gamma_0(\cdot)$ (and possibly $\gamma(\cdot)$) exists only as generalized function in our setting. It is defined as the $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}_0)$-limit

$$\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B_s - \widetilde{B}_r\big)drds \equiv \lim_{M,N\to\infty} \int_0^t\int_0^t \gamma_{0,N}(s-r)\gamma_M\big(B_s - \widetilde{B}_r\big)drds \tag{2.6}$$

for a sequence of "reasonable" and pointwise-defined $\gamma_{0,N}(\cdot)$ and $\gamma_M(\cdot)$, whenever the limit exists.

In view of (1.11) and (1.13), for instance, we may choose

$$\gamma_{0,N}(u) = \int_{[-N,N]} e^{i\lambda u}\mu_0(d\lambda) \ \text{ and } \ \gamma_M(x) = \int_{[-M,M]^d} e^{i\xi\cdot x}\mu(d\xi). \tag{2.7}$$

By straightforward computation, the random sequence

$$\mathcal{Q}_{M,N} = \int_0^t\int_0^t \gamma_{0,N}(s-r)\gamma_M\big(B_s - \widetilde{B}_r\big)drds$$

can be rewritten as

$$\mathcal{Q}_{M,N} = \int_{[-N,N]\times[-M,M]^d} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds$$

and satisfies

$$\begin{aligned}&\mathbb{E}_0\big[\mathcal{Q}_{M',N'} - \mathcal{Q}_{M,N}\big]^n\\&= \int_{([-N',N']\times[-M',M']^d\setminus[-N,N]\times[-M,M]^d)^n} \mu_0(\lambda)\mu(d\xi))\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k + \xi_k\cdot B(s_k))}\right)ds\right|^2\end{aligned}$$

for any $n \geq 1$, $N' > N$ and $M' > M$. Therefore, the random integrals linked by equality

$$\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B_s - \widetilde{B}_r\big)drds = \int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds \tag{2.8}$$

are well-defined as the limits in (2.6) and live in $\mathcal{L}^n(\Omega, \mathcal{A}, \mathbb{P}_0)$ for all $n \geq 1$ if and only if

$$\int_{(\mathbb{R}^{d+1})^n} \mu_0(\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2<\infty \quad n=1,2,\cdots. \tag{2.9}$$

In this case

$$\begin{aligned}&\mathbb{E}_0\bigg[\int_0^t\int_0^t \gamma_0(s-r)\gamma\big(B_s-\widetilde{B}_r\big)drds\bigg]^n \\ &=\int_{(\mathbb{R}^{d+1})^n} \mu_0(\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2 \quad n=1,2,\cdots.\end{aligned} \tag{2.10}$$

By Taylor expansion, the exponential integrability in (2.5) or the existence/uniqueness of the parabolic Anderson equation (1.1) relies on the bound for the high dimensional integral on the right hand side of (2.10). In Proposition 1.4, [9], the bound

$$\begin{aligned}&\int_{(\mathbb{R}^{d+1})^n} \mu_0(\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2 \\ &\leq (n!)^{(d-H)+(2-2H_0)}C^n t^{n(2H_0+H-d)} \quad n=1,2,\cdots\end{aligned} \tag{2.11}$$

is established under the assumption in Theorem 1.2 for a constant $C>0$ independent of t and n. Consequently, the Brownian Hamiltonians in (2.8) are well-defined and have all finite positive moments. On the other hand, the bound in (2.11) is insufficient for the needed exponential integrability (2.5) as $(d-H)+(2-2H_0)\geq 2-2H_0>1$ when $H_0<1/2$. Substantial improvement has to be made in order to establish (2.5).

3 Covariance decomposition

The requested improvement is based on the following theorem.

Theorem 3.1 *Under the assumption of Theorem 1.2,*

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds \\ &=H_0\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\big\{s^{-(1-2H_0)}+(t-s)^{-(1-2H_0)}\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds \\ &+\frac{H_0(1-2H_0)}{2}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\frac{\big[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}\big]\big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr \quad a.s.\end{aligned} \tag{3.1}$$

for every $t\geq 0$, where B_t and $\widetilde{B}_t$ are independent Brownian motions starting at 0.

Further, two integrals on the right hand side are well-defined and have all finite positive moments.

Proof: For $N > 0$, consider the decomposition

$$\int_{[-N,N]\times\mathbb{R}^d} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds$$
$$= \int_{[-N,N]\times\mathbb{R}^d} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}drds$$
$$- \frac{1}{2}\int_{[-N,N]\times\mathbb{R}^d} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)}[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}]drds$$
$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \left(\int_{-N}^N \frac{e^{i\lambda s} - e^{i\lambda(s-t)}}{i\lambda}\mu_0(d\lambda)\right)e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds$$
$$- \frac{1}{2}\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_{-N}^N e^{i\lambda(s-r)}\mu_0(d\lambda)\right)[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}]drds.$$

In view of (1.11),

$$\int_{-N}^N \frac{e^{i\lambda s} - e^{i\lambda(s-t)}}{i\lambda}\mu_0(d\lambda) = \frac{\Gamma(2H_0+1)\sin\pi H_0}{2\pi i}\int_{-N}^N \left\{e^{i\lambda s} - e^{i\lambda(s-t)}\right\}\frac{\mathrm{sgn}\,(\lambda)}{|\lambda|^{2H_0}}d\lambda$$
$$= \frac{\Gamma(2H_0+1)\sin\pi H_0}{2\pi i}\left\{\int_{-N}^N e^{i\lambda s}\frac{\mathrm{sgn}\,(\lambda)}{|\lambda|^{2H_0}}d\lambda + \int_{-N}^N e^{i\lambda(t-s)}\frac{\mathrm{sgn}\,(\lambda)}{|\lambda|^{2H_0}}d\lambda\right\}$$
$$= \frac{\Gamma(2H_0+1)\sin\pi H_0}{\pi}\left\{\int_0^N \frac{\sin(\lambda s)}{\lambda^{2H_0}}d\lambda + \int_0^N \frac{\sin(\lambda(t-s))}{\lambda^{2H_0}}d\lambda\right\}.$$

With integration by parts,

$$\int_{-N}^N e^{i\lambda(s-r)}\mu_0(d\lambda) = \frac{\Gamma(2H_0+1)\sin\pi H_0}{\pi}\int_0^N \lambda^{1-2H_0}\cos(\lambda(s-r))d\lambda$$
$$= \frac{\Gamma(2H_0+1)\sin\pi H_0}{\pi}\left\{N^{1-2H_0}\frac{\sin(N(s-r))}{(s-r)} - \frac{1-2H_0}{|s-r|}\int_0^N \frac{\sin(\lambda|s-r|)}{\lambda^{2H_0}}d\lambda\right\}.$$

Summarizing our computation,

$$\int_{[-N,N]\times\mathbb{R}^d} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds \tag{3.2}$$
$$= \frac{\Gamma(2H_0+1)\sin\pi H_0}{\pi}\left\{\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \left(\int_0^N \frac{\sin(\lambda s)}{\lambda^{2H_0}}d\lambda + \int_0^N \frac{\sin(\lambda(t-s))}{\lambda^{2H_0}}d\lambda\right)e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\right.$$
$$+ \frac{1-2H_0}{2}\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_0^N \frac{\sin(\lambda|s-r|)}{\lambda^{2H_0}}d\lambda\right)\frac{[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}]}{|s-r|}drds$$
$$\left. - \frac{1}{2}N^{1-2H_0}\int_0^t\int_0^t \frac{\sin(N(s-r))}{(s-r)}[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}]drds\right\}.$$

Consider the further decompositions

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\bigg(\int_0^N\frac{\sin(\lambda s)}{\lambda^{2H_0}}d\lambda+\int_0^N\frac{\sin(\lambda(t-s))}{\lambda^{2H_0}}d\lambda\bigg)e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds \qquad (3.3)\\
&=\bigg(\int_0^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\bigg)\int_0^t\big\{s^{-(1-2H_0)}+(t-s)^{-(1-2H_0)}\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\\
&-\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\bigg(s^{-(1-2H_0)}\int_{Ns}^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda+(t-s)^{-(1-2H_0)}\int_{N(t-s)}^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\bigg)e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\bigg(\int_0^N\frac{\sin(\lambda|s-r|)}{\lambda^{2H_0}}d\lambda\bigg)\frac{[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}]}{|s-r|}drds \qquad (3.4)\\
&=\bigg(\int_0^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\bigg)\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\frac{[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}]}{|s-r|^{2-2H_0}}drds\\
&-\int_{\mathbb{R}^d}\mu(d\xi)\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\bigg(\int_{N|s-r|}^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\bigg)\frac{[e^{i\xi\cdot B_s}-e^{i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}]}{|s-r|^{2-2H_0}}drds.
\end{aligned}$$

For any even $n\geq 2$, write

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\bigg(s^{-(1-2H_0)}\int_{Ns}^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda+(t-s)^{-(1-2H_0)}\int_{N(t-s)}^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\bigg)e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\bigg]^n\\
&=\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\bigg(\prod_{k=1}^n\big\{\varphi_N(s_k)+\varphi_N(t-s_k)\big\}\bigg)\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i\xi\cdot(B_{s_k}-\widetilde{B}_{s_k})}\bigg)ds
\end{aligned}$$

By the fact that

$$\varphi_N(s)\equiv s^{-(1-2H_0)}\int_{Ns}^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\longrightarrow 0\quad(N\to\infty),$$

by the bound $|\varphi_N(s)|\leq Cs^{-(1-2H_0)}$ and by the fact (Lemma 4.1 in the next section)

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\bigg(\prod_{k=1}^n\big\{s_k^{-(1-2H_0)}+(t-s_k)^{-(1-2H_0)}\big\}\bigg)\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i\xi\cdot(B_{s_k}-\widetilde{B}_{s_k})}\bigg)ds\\
&=\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\big\{s^{-(1-2H_0)}+(t-s)^{-(1-2H_0)}\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\bigg]^n<\infty
\end{aligned}$$

the second term in (3.3) converges to zero in $\mathcal{L}^n(\mathbb{P}_0)$ as $N\to\infty$. Consequently,

$$\begin{aligned}
&\lim_{N\to\infty}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\bigg(\int_0^N\frac{\sin(\lambda s)}{\lambda^{2H_0}}d\lambda+\int_0^N\frac{\sin(\lambda(t-s))}{\lambda^{2H_0}}d\lambda\bigg)e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds \qquad (3.5)\\
&=\bigg(\int_0^\infty\frac{\sin\lambda}{\lambda^{2H_0}}d\lambda\bigg)\int_0^t\big\{s^{-(1-2H_0)}+(t-s)^{-(1-2H_0)}\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds
\end{aligned}$$

in all positive moments.

Similarly, using Lemma 4.2 in the next section one can show that the second term in (3.4) converges to zero in $\mathcal{L}^n(\mathbb{P}_0)$ as $N \to \infty$ and therefore,

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_0^N \frac{\sin(\lambda|s-r|)}{\lambda^{2H_0}} d\lambda \right) \frac{[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}]}{|s-r|} drds \quad (3.6)$$
$$= \left(\int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda \right) \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}]}{|s-r|^{2-2H_0}} drds$$

in all positive moments.

Combining (3.2), (3.5) and (3.6) together, in view of Lemma 3.2 below,

$$\int_{\mathbb{R}^{d+1}} \mu_0(d\lambda)\mu(d\xi) \int_0^t\int_0^t e^{i\lambda(s-r)} e^{i\xi\cdot(B_s - \widetilde{B}_r)} drds$$
$$= \frac{\Gamma(2H_0+1)\sin\pi H_0}{\pi} \left(\int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda \right)$$
$$\times \left\{ \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \left\{ s^{-(1-2H_0)} + (t-s)^{-(1-2H_0)} \right\} e^{i\xi\cdot(B_s - \widetilde{B}_s)} ds \right.$$
$$\left. + \frac{1-2H_0}{2} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}]}{|s-r|^{2-2H_0}} drds \right\} \quad a.s.$$

Finally, the identity (3.1) follows from

$$\frac{\Gamma(2H_0+1)\sin\pi H_0}{\pi} \int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda = H_0$$

which is established in (6.1), Lemma 6.1 in Appendeix below. □

Lemma 3.2 *In the assumption of Theorem 1.2,*

$$\lim_{N\to\infty} N^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{\sin(N(s-r))}{(s-r)} [e^{i\xi\cdot B_s} - e^{-i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}] drds = 0 \quad (3.7)$$

in all positive moments.

Proof: Write

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{\sin(N(s-r))}{(s-r)} [e^{i\xi\cdot B_s} - e^{-i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}] drds$$
$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_{-N}^N e^{i\lambda(s-r)} d\lambda \right) [e^{i\xi\cdot B_s} - e^{-i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}] drds$$

and notice

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \left(\int_{\mathbb{R}} e^{i\lambda(s-r)} d\lambda \right) [e^{i\xi\cdot B_s} - e^{-i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}] drds$$
$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \delta_0(s-r) [e^{i\xi\cdot B_s} - e^{-i\xi\cdot B_r}][e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}] drds = 0.$$

We have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t\frac{\sin(N(s-r))}{(s-r)}[e^{i\xi\cdot B_s}-e^{-i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}]drds \qquad (3.8)\\
&=-\int_{[-N,N]^c\times\mathbb{R}^d}d\lambda\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}[e^{i\xi\cdot B_s}-e^{-i\xi\cdot B_r}][e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}]drds\\
&=2\int_{[-N,N]^c\times\mathbb{R}^d}d\lambda\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot B_s}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\\
&-2\int_{[-N,N]^c\times\mathbb{R}^d}d\lambda\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}drds.
\end{aligned}$$

Let $n\geq 2$ be an arbitrary but fixed even natural number. For the first term on the right hand side,

$$\begin{aligned}
&N^{(1-2H_0)n}\mathbb{E}_0\bigg[\int_{[-N,N]^c\times\mathbb{R}^d}d\lambda\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot B_s}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\bigg]^n\\
&=N^{(1-2H_0)n}\int_{([-N,N]^c\times\mathbb{R}^d)^n}d\lambda\mu(d\xi)\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2\\
&\leq\bigg(\frac{\Gamma(2H_0+1)\sin\pi H_0}{2\pi}\bigg)^{-n}\int_{([-N,N]^c\times\mathbb{R}^d)^n}\mu_0(d\lambda)\mu(d\xi)\bigg|\int_{[0,t]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\bigg)ds\bigg|^2\\
&\longrightarrow 0\quad(N\to\infty)
\end{aligned}$$

where the last step follows from (2.11).

As for the second term on the right hand side of (3.8), by a computation similar to the treatment for the first term in (3.2),

$$\begin{aligned}
&N^{(1-2H_0)}\int_{[-N,N]^c\times\mathbb{R}^d}d\lambda\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}drds\\
&=2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\big\{\psi_N(s)+\psi_N(t-s)\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\\
&=2\int_{\mathbb{R}^d}\mu(d\xi)\int_{N^{-1}}^{t-N^{-1}}\big\{\psi_N(s)+\psi_N(t-s)\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\\
&+2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{N^{-1}}\big\{\psi_N(s)+\psi_N(t-s)\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\\
&+2\int_{\mathbb{R}^d}\mu(d\xi)\int_{t-N^{-1}}^t\big\{\psi_N(s)+\psi_N(t-s)\big\}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds
\end{aligned}$$

with

$$\psi_N(s)=N^{1-2H_0}\int_{Ns}^\infty\frac{\sin\lambda}{\lambda}d\lambda.$$

It is easy to see that the sequence

$$\big\{\psi_N(s)+\psi_N(t-s)\big\}1_{[-N^{-1},t-N^{-1}]}(s)$$

is uniformly bounded and converges to zero pointwisely on $s \in [0,t]$ as $N \to \infty$. Hence,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_{N^{-1}}^{t-N^{-1}} \big\{\psi_N(s) + \psi_N(t-s)\big\} e^{i\xi\cdot(B_s - \widetilde{B}_s)} ds\bigg]^n \\
&= \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[-N^{-1}, t-N^{-1}]^n} \bigg(\prod_{k=1}^n \big\{\psi_N(s_k) + \psi_N(t-s_k)\big\}\bigg) \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k} - \widetilde{B}_{s_k})}\bigg) ds \\
&\longrightarrow 0 \quad (N \to \infty)
\end{aligned}$$

as ((4.1), [9])

$$\int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,t]^n} \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k} - \widetilde{B}_{s_k})}\bigg) ds \le C^n (n!)^{d-H} t^{n(H+1-d)} < \infty.$$

We now use the bound $|\{\psi_N(s) + \psi(t-s)| \le C N^{1-2H_0}$ on $[0, N^{-1}] \cup [t - N^{-1}, t]$. First,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_0^{N^{-1}} \big\{\psi_N(s) + \psi_N(t-s)\big\} e^{i\xi\cdot(B_s - \widetilde{B}_s)} ds\bigg]^n \\
&\le C^n N^{n(1-2H_0)} \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,N^{-1}]^n} \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k} - \widetilde{B}_{s_k})}\bigg) ds \\
&\le C^n (n!)^{d-H} N^{n(1-2H_0)} N^{-n(H+1-d)} \longrightarrow 0 \quad (N \to \infty)
\end{aligned}$$

where the second step follows from (4.1), [9] with $t = N^{-1}$, and the last step follows from the fact that $1 - 2H_0 < H + 1 - d$ in the assumption (1.6).

Similarly,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_{t-N^{-1}}^t \big\{\psi_N(s) + \psi_N(t-s)\big\} e^{i\xi\cdot(B_s - \widetilde{B}_s)} ds\bigg]^n \\
&\le C^n N^{n(1-2H_0)} \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[t-N^{-1}, t]^n} \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k} - \widetilde{B}_{s_k})}\bigg) ds \\
&= C^n N^{n(1-2H_0)} \int_{(\mathbb{R}^d)^n} \mu(d\xi) \exp\bigg\{-\frac{1}{2}\bigg|\sum_{k=1}^n \xi_k\bigg|(t - N^{-1})\bigg\} \int_{[0,N^{-1}]^n} \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k} - \widetilde{B}_{s_k})}\bigg) ds \\
&\le C^n N^{n(1-2H_0)} \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,N^{-1}]^n} \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k} - \widetilde{B}_{s_k})}\bigg) ds
\end{aligned}$$

where the second step follows from the independence of the Brownian increment. As we have seen, the right hand side goes to 0 as $N \to \infty$.

Summarizing our computation since (3.8), we have proved (3.7). □

4 Moment bounds in Theorem 3.1

In connection to the decomposition in (3.1), set

$$\mathcal{L}_t = \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \big\{s^{-(1-2H_0)} + (t-s)^{-(1-2H_0)}\big\} e^{i\xi\cdot(B_s - \widetilde{B}_s)} ds \tag{4.1}$$

$$\mathcal{M}_t = \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t \frac{\left[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}\right]\left[e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r}\right]}{|s-r|^{2-2H_0}} dsdr. \tag{4.2}$$

In this section we provide the moment bounds for $\mathcal{L}_t$ and $\mathcal{M}_t$ that are sufficient for (2.5) (and therefore for Theorem 1.2). First we establish

Lemma 4.1 *Under the assumption of theorem 1.2*

$$\mathbb{E}_0 \mathcal{L}_t^n \le (n!)^{(d-H)+(1-2H_0)} C^n t^{n(2H_0+H-d)} \quad t>0, \quad n=1,2,\cdots \tag{4.3}$$

where $C>0$ is a constant in dependent of t and n.

Proof: By the Brownian scaling and homogeneity of the space covariance,

$$\mathcal{L}_t \stackrel{d}{=} t^{2H_0+H-d}\mathcal{L}_1. \tag{4.4}$$

All we need is to prove (4.3) with $t=1$, i.e.,

$$\mathbb{E}_0 \mathcal{L}_1^n \le (n!)^{(d-H)+(1-2H_0)} C^n \qquad n=1,2,\cdots. \tag{4.5}$$

In the rest of the paper, we use the same notation "C" for possibly different positive constants that are independent of n and t.

Notice that

$$\left\{B_s - \widetilde{B}_s; \quad s\ge 0\right\} \stackrel{d}{=} \left\{\sqrt{2}B_s; \quad s\ge 0\right\}.$$

We have

$$\begin{aligned}
&\mathbb{E}_0\left[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)} e^{i\xi\cdot(B_s-\widetilde{B}_s)} ds\right]^n \\
&= \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\left(\prod_{k=1}^n s_k^{-(1-2H_0)}\right)\left(\mathbb{E}_0 e^{i\sqrt{2}\xi\cdot B_{s_k}}\right) ds \\
&= n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<}\left(\prod_{k=1}^n s_k^{-(1-2H_0)}\right)\left(\mathbb{E}_0 e^{i\sqrt{2}\xi\cdot B_{s_k}}\right) ds \\
&\le n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<}\left(\prod_{k=1}^n (s_k-s_{k-1})^{-(1-2H_0)}\right)\left(\mathbb{E}_0 e^{i\sqrt{2}\xi\cdot B_{s_k}}\right) ds
\end{aligned}$$

where (and elsewhere) we adopt the notations that $s_0=0$ and

$$[0,t]^n_< = \{(s_1,\cdots,s_n)\in[0,t]^n; \quad s_1<\cdots<s_n\}$$

and the last step follows from permutation invariance.

By independent Brownian increment, for any $(s_1,\cdots,s_n)\in[0,t]^n_<$

$$\begin{aligned}
\mathbb{E}_0\prod_{k=1}^n e^{i\sqrt{2}\xi\cdot B_{s_k}} &= \mathbb{E}_0 \exp\left\{i\sqrt{2}\sum_{k=1}^n\Big(\sum_{j=k}^n \xi_j\Big)\cdot(B_{s_k}-B_{s_{k-1}})\right\} \\
&= \exp\left\{-\sum_{k=1}^n\Big|\sum_{j=k}^n\xi_j\Big|^2(s_k-s_{k-1})\right\}.
\end{aligned}$$

Hence,

$$\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\Big]^n$$
$$\leq n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<}\Big(\prod_{k=1}^n(s_k-s_{k-1})^{-(1-2H_0)}\exp\Big\{-\Big|\sum_{j=k}^n\xi_j\Big|^2(s_k-s_{k-1})\Big\}\Big)ds$$

By Lemma 2.2.7, [7], therefore,

$$\begin{aligned}&\int_0^\infty e^{-t}\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\Big]^n dt \qquad (4.6)\\ &\leq n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n\int_0^\infty e^{-t}t^{-(1-2H_0)}\exp\Big\{-\Big|\sum_{j=k}^n\xi_j\Big|^2t\Big\}dt\\ &=n!\Gamma(2-2H_0)^n\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n\Big\{1+\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-2H_0}\leq n!C^n\end{aligned}$$

where the last step follows from Lemma 3.2, [9] and the fact that

$$2H_0>\frac{2(d-H)+(d_*-2H_*)}{2}$$

under the assumption in Theorem 1.2.

Similarly, with the convention $s_{n+1}=t$

$$\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t(t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\Big]^n$$
$$\leq n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n_<}\Big(\prod_{k=1}^n(s_{k+1}-s_k)^{-(1-2H_0)}\Big)\exp\Big\{-\sum_{k=1}^n\Big|\sum_{j=k}^n\xi_j\Big|^2(s_k-s_{k-1})\Big\}ds.$$

We have

$$\begin{aligned}&\int_0^\infty e^{-t}\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t(t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\widetilde{B}_s)}ds\Big]^n dt \qquad (4.7)\\ &\leq n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big(\prod_{k=2}^n\int_0^\infty e^{-t}t^{-(1-2H_0)}\exp\Big\{-\Big|\sum_{j=k}^n\xi_j\Big|^2t\Big\}dt\Big)\\ &=n!\Gamma(2-2H_0)^{n-1}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big\{1+\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-1}\prod_{k=2}^n\Big\{1+\Big|\sum_{j=1}^n\xi_j\Big|^2\Big\}^{-2H_0}\\ &\times\Big(\int_0^\infty e^{-t}\exp\Big\{-\Big|\sum_{j=k}^n\xi_j\Big|^2t\Big\}dt\Big)\Big(\int_0^\infty t^{-(1-2H_0)}e^{-t}dt\Big)\\ &=n!\Gamma(2-2H_0)^n\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n\Big\{1+\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-2H_0}\leq n!C^n.\end{aligned}$$

Combining (4.6) and (4.7)

$$\int_0^t e^{-t}\mathbb{E}_0\mathcal{L}_t^n dt \le n!C^n.$$

On the other hand, by (4.4),

$$\int_0^t e^{-t}\mathbb{E}_0\mathcal{L}_t^n dt = \mathbb{E}_0\mathcal{L}_1^n \int_0^\infty e^{-t}t^{n(2H_0+H-d)}dt = \Gamma\big(n(2H_0+H-d)+1\big)\mathbb{E}_0\mathcal{L}_1^n.$$

Finally, the bound (4.5) follows from Stirling formula. □

Lemma 4.2 *Under the assumption of Theorem 1.2, for any $\beta > 1-2H_0$ there is a constant $C>0$ such that*

$$\mathbb{E}_0\mathcal{M}_t^n \le (n!)^{(d-H)+2\beta}C^n t^{n(2H_0+H-d)} \quad t>0,\ \ n=1,2,\cdots. \tag{4.8}$$

Proof: By (1.15) we may make

$$1-2H_0 < \beta < \frac{1}{2} - \frac{2(d-H)+(d_*-H_*)}{4}. \tag{4.9}$$

Notice that

$$\mathcal{M}_t = 2\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t e^{i\xi\cdot B_s}\frac{\big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr \tag{4.10}$$

and

$$\mathcal{M}_t \overset{d}{=} t^{2H_0+H-d}\mathcal{M}_1. \tag{4.11}$$

All we need is the bound

$$\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^1\int_0^1 e^{i\xi\cdot B_s}\frac{\big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr\bigg]^n \le (n!)^{(d-H)+2\beta}C^n \tag{4.12}$$

for $n=1,2,\cdots$.

By Fubini theorem,

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^1\int_0^1 e^{i\xi\cdot B_s}\frac{\big[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr\bigg]^n\\
&=\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,1]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\bigg)\bigg\{\int_{[0,1]^n}\bigg(\prod_{k=1}^n|s_k-r_k|^{-(2-2H_0)}\bigg)\\
&\times\bigg(\mathbb{E}_0\prod_{k=1}^n\Big[e^{-i\xi_k\cdot B_{s_k}}-e^{-i\xi_k\cdot B_{r_k}}\Big]\bigg)dr\bigg\}ds\\
&\le 2^n\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,1]^n}\bigg(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\bigg)\bigg\{\int_{[0,1]^n}\bigg(\prod_{k=1}^n|s_k-r_k|^{-(2-2H_0)}\bigg)\\
&\times\bigg(\mathbb{E}_0\prod_{k=1}^n\Big|\sin\frac{\xi_k\cdot(B_{s_k}-B_{r_k})}{2}\Big|\bigg)dr\bigg\}ds
\end{aligned}$$

where the inequality follows from

$$\left|e^{-i\xi_k\cdot B_{s_k}}-e^{-i\xi_k\cdot B_{r_k}}\right|=2\left|\sin\frac{\xi_k\cdot(B_{s_k}-B_{r_k})}{2}\right|.$$

Further, pick $1-2H_0<\beta_1<\beta$. By the fact that $2\beta<1$, $|\sin(\cdot)|\le|\sin(\cdot)|^{2\beta}\le|\cdot|^{2\beta}$,

$$\begin{aligned}
&\int_{[0,1]^n}\left(\prod_{k=1}^n|s_k-r_k|^{-(2-2H_0)}\right)\left(\mathbb{E}_0\prod_{k=1}^n\left|\sin\frac{\xi_k\cdot(B_{s_k}-B_{r_k})}{2}\right|\right)dr\\
&\le\left(\frac{1}{2}\right)^n\left(\prod_{k=1}^n|\xi_k|^{2\beta}\right)\int_{[0,1]^n}\left(\prod_{k=1}^n|s_k-r_k|^{-(2-2H_0)}\right)\left(\mathbb{E}_0\prod_{k=1}^n|B_{s_k}-B_{r_k}|\right)^{2\beta}dr\\
&\le\left(\frac{1}{2}\right)^n\left(\prod_{k=1}^n|\xi_k|^{2\beta}\right)\mathbb{E}_0\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\left(\frac{|B_s-B_r|}{|s-r|^{\frac{\beta_1}{2\beta}}}\right)^{2\beta n}\int_{[0,1]^n}\left(\prod_{k=1}^n|s_k-r_k|\right)^{-((2-2H_0)-\beta_1)}dr\\
&\le C^n\left(\prod_{k=1}^n|\xi_k|^{2\beta}\right)\mathbb{E}_0\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\left(\frac{|B_s-B_r|}{|s-r|^{\frac{\beta_1}{2\beta}}}\right)^{2\beta n}
\end{aligned}$$

where the last step follows from that fact that

$$\int_{[0,1]^n}\left(\prod_{k=1}^n|s_k-r_k|\right)^{-((2-2H_0)-\beta_1)}dr=\prod_{k=1}^n\int_0^1|s_k-r|^{-((2-2H_0)-\beta_1)}dr\le C^n$$

for a $C>0$ independent of $s_1,\cdots,s_n$, as $(2-2H_0)-\beta_1<1$.

Summarizing our computation

$$\begin{aligned}
&\mathbb{E}_0\left[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^1\int_0^1e^{i\xi\cdot B_s}\frac{\left[e^{-i\xi\cdot\widetilde{B}_s}-e^{-i\xi\cdot\widetilde{B}_r}\right]}{|s-r|^{2-2H_0}}dsdr\right]^n \qquad (4.13)\\
&\le C^n\mathbb{E}_0\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\left(\frac{|B_s-B_r|}{|s-r|^{\frac{\beta_1}{2\beta}}}\right)^{2\beta n}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\left(\prod_{k=1}^n|\xi_k|^{2\beta_2}\right)\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^ne^{i\xi_k\cdot B_{s_k}}\right)ds\\
&=C^n\mathbb{E}_0\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\left(\frac{|B_s-B_r|}{|s-r|^{\frac{\beta_1}{2\beta}}}\right)^{2\beta n}\mathbb{E}_0\left[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^1e^{i\xi\cdot B_s}ds\right]^n.
\end{aligned}$$

For any $t>0$,

$$\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^te^{i\xi\cdot B_s}ds\overset{d}{=}t^{1-(d-H)-\beta}\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^1e^{i\xi\cdot B_s}ds \qquad (4.14)$$

and

$$\begin{aligned}
&\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^t e^{i\xi\cdot B_s}ds\Big]^n\\
&=\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big(\prod_{k=1}^n|\xi_k|^{2\beta}\Big)\int_{[0,t]^n}\Big(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\Big)ds\\
&=n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big(\prod_{k=1}^n|\xi_k|^{2\beta}\Big)\int_{[0,t]^n_<}\Big(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot B_{s_k}}\Big)ds\\
&=n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big(\prod_{k=1}^n|\xi_k|^{2\beta}\Big)\prod_{k=1}^n\exp\Big\{-\frac{1}{2}\Big|\sum_{j=k}^n\xi_j\Big|^2(s_k-s_{k-1})\Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty e^{-t}\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^t e^{i\xi\cdot B_s}ds\Big]^n dt\\
&=n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big(\prod_{k=1}^n|\xi_k|^{2\beta}\Big)\prod_{k=1}^n\int_0^\infty e^{-t}\exp\Big\{-\frac{1}{2}\Big|\sum_{j=k}^n\xi_j\Big|^2 t\Big\}dt\\
&=n!\int_{(\mathbb{R}^d)^n}\mu(d\xi)\Big(\prod_{k=1}^n|\xi_k|^{2\beta}\Big)\prod_{k=1}^n\Big\{1+\frac{1}{2}\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-1}.
\end{aligned}$$

Set $\eta_k=\sum_{j=k}^n\xi_j$ $(k=1,\cdots,n)$. By the fact that $0<2\beta<1$ and with the convention $\eta_{n+1}=0$

$$\begin{aligned}
&\prod_{k=1}^n|\xi_k|^{2\beta}=\prod_{k=1}^n|\eta_k-\eta_{k+1}|^{2\beta}\le\prod_{k=1}^n\big(|\eta_k|^{2\beta}+|\eta_{k+1}|^{2\beta}\big)\\
&\le\sum_l\prod_{k=1}^n|\eta_k|^{2l(k)\beta}\le 2^n\sum_l\prod_{k=1}^n\Big\{1+\frac{1}{2}|\eta_k|^2\Big\}^{2\beta}\le 2^n3^n\prod_{k=1}^n\Big\{1+\frac{1}{2}|\eta_k|^2\Big\}^{2\beta}
\end{aligned}$$

where the summation is over all possible maps l: $\{1,\cdots,n\}\longmapsto\{0,1,2\}$ and the last follows from the fact that $\#(l)\le 3^n$.

Therefore, we have the bound

$$\begin{aligned}
&\int_0^\infty e^{-t}\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^t e^{i\xi\cdot B_s}ds\Big]^n dt\\
&\le n!C^n\int_{(\mathbb{R}^d)^n}\prod_{k=1}^n\Big\{1+\frac{1}{2}\Big|\sum_{j=k}^n\xi_j\Big|^2\Big\}^{-(1-2\beta)}\le n!C^n
\end{aligned}$$

where the last step follows from Lemma 3.2, [9] and the fact (from (4.9)) that

$$1-2\beta>\frac{2(d-H)+(d_*-2H_*)}{2}.$$

On the other hand, in view of (4.14)

$$\begin{aligned}
&\int_0^\infty e^{-t}\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^t e^{i\xi\cdot B_s}ds\bigg]^n dt\\
&=\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^1 e^{i\xi\cdot B_s}ds\bigg]^n\int_0^\infty e^{-t}t^{n(2H_0+H-d)}dt\\
&=\Gamma\big(n(2H_0+H-d)\big)\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^1 e^{i\xi\cdot B_s}ds\bigg]^n.
\end{aligned}$$

By Stirling formula

$$\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)|\xi|^{2\beta}\int_0^1 e^{i\xi\cdot B_s}ds\bigg]^n\le(n!)^{(d-H)+\beta}C^n. \tag{4.15}$$

Consider the Banach space $\mathcal{C}^{\beta_1/(2\beta)}[0,1]$ of the functions f on $[0,1]$ satisfying the Hölder continuity

$$\|f\|_{\beta_1/(2\beta)}\equiv\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\frac{|f(s)-f(r)|}{|s-r|^{\beta_1/(2\beta)}}<\infty.$$

By the fact that $\beta_1/(2\beta)<1/2$, B_s ($s\in[0,1]$) is $\beta_1/(2\beta)$-Hölder continuous and therefore can be viewed as a mean-zero Gaussian random variable taking values in $\mathcal{C}^{\beta_1/(2\beta)}[0,1]$. By the Gaussian integrability (see, e.g., Corollary 3.2, [24]), there is a constant $c>0$ such that

$$\mathbb{E}_0\exp\bigg\{c\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\frac{|B_s-B_r|^2}{|s-r|^{\beta_1/\beta}}\bigg\}=\mathbb{E}_0\exp\Big\{c\|B\|^2_{\beta_1/(2\beta)}\Big\}<\infty.$$

Consequently, there is a constant $C>0$ such that

$$\mathbb{E}_0\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\bigg(\frac{|B_s-B_r|}{|s-r|^{\frac{\beta_1}{2\beta}}}\bigg)^n\le(n!)^{1/2}C^n\quad n=1,2,\cdots.$$

In particular, we have the bound

$$\mathbb{E}_0\sup_{\substack{s,r\in[0,1]\\ s\neq r}}\bigg(\frac{|B_s-B_r|}{|s-r|^{\frac{\beta_1}{2\beta}}}\bigg)^{2\beta n}\le(n!)^{\beta}C^n\quad n=1,2,\cdots. \tag{4.16}$$

Finally, the desired bound (4.12) follows from (4.13), (4.15) and (4.16). □

Notice that $(d-H)+(1-2H_0)<1$ under the assumption in Theorem 1.2. Applying Taylor expansion in Lemma 4.1,

$$\mathbb{E}_0\exp\big\{C\mathcal{L}_t\big\}<\infty\quad C,t>0. \tag{4.17}$$

By the fact that $(d-H)+2\beta<1$ under (4.9), by Lemma 4.2

$$\mathbb{E}_0\exp\big\{C\mathcal{M}_t\big\}<\infty\quad C,t>0 \tag{4.18}$$

under the assumption in Theorem 1.2.

By the covariance decomposition in Theorem 3.1,

$$\mathbb{E}_0 \exp\left\{C\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right\}<\infty. \tag{4.19}$$

Equivalently, we have (2.5) which is sufficient for validating Theorem 1.2.

Remark 4.3 *With the covariance decomposition in Theorem 3.1, the bounds we establish in this section substantially improve (2.11) obtained in [9] and more significantly, are sharp enough for the exponential integrability given in (4.19). On the other hand, they are not optimal. For the sake of possible link to the future investigation, we confirm the conjecture made in [9] stating that*

$$\begin{aligned}&\int_{(\mathbb{R}^{d+1})^n}\mu_0(\lambda)\mu(d\xi)\left|\int_{[0,t]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2 \\ &\le (n!)^{(d-H)+(1-2H_0)}C^n t^{n(2H_0+H-d)}\quad n=1,2,\cdots.\end{aligned} \tag{4.20}$$

Indeed, according to the development for moment asymptotics in the next section

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^t\int_0^t e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right\}<\infty.$$

By (2.10), in addition,

$$\mathbb{E}_0\left[\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^n\int_0^n e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right]^n\ge 0\quad n=1,2,\cdots.$$

Consequently,

$$\begin{aligned}&\frac{1}{n!}\mathbb{E}_0\left[\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^n\int_0^n e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right]^n \\ &\le \mathbb{E}_0\exp\left\{\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^n\int_0^n e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right\}\le C^n\end{aligned}$$

for some constant $C>0$ independent of $n=1,2,\cdots$. By the Brownian scaling, on the other hand,

$$\begin{aligned}&\mathbb{E}_0\left[\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^n\int_0^n e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right]^n \\ &= n^{n(2H_0+H-d)}\mathbb{E}_0\left[\int_{\mathbb{R}^{d+1}}\mu_0(d\lambda)\mu(d\xi)\int_0^1\int_0^1 e^{i\lambda(s-r)}e^{i\xi\cdot(B_s-\widetilde{B}_r)}drds\right]^n \\ &= n^{n(2H_0+H-d)}\int_{(\mathbb{R}^{d+1})^n}\mu_0(\lambda)\mu(d\xi)\left|\int_{[0,1]^n}\left(\mathbb{E}_0\prod_{k=1}^n e^{i(\lambda_k s_k+\xi_k\cdot B(s_k))}\right)ds\right|^2\end{aligned}$$

where the last step follows from (2.10). By a standard use of Stirling formula, we have (4.20) in the case $t=1$. Finally, (4.20) with the full generality follows from scaling. □

5 Moment asymptotics in Theorem 1.3

In this section we prove Theorem 1.3. By the Feynman-Kac moment representation in (1.16) with $m = 2$ and by the initial condition given in (1.4), a simple argument reduces proof to the setting $u_0(x) = 1$, in which

$$\mathbb{E}u^2(t,x) = \mathbb{E}_0 \exp\left\{\theta^2 H_0 \mathcal{L}_t + \theta^2 \frac{H_0(1-H_0)}{2}\mathcal{M}_t\right\} \tag{5.1}$$

where $\mathcal{L}_t$ and $\mathcal{M}_t$ are defined in (4.1) and (4.2).

Lemma 5.1 *For any $C > 0$*

$$\limsup_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}_0 \exp\left\{C|\mathcal{L}_t|\right\} < \infty. \tag{5.2}$$

Proof: Given the conjugate number $p, q > 1$, by Hölder inequality

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{C\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\\
&\le \left[\mathbb{E}_0 \exp\left\{Cp\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\right]^{1/p}\\
&\times \left[\mathbb{E}_0 \exp\left\{Cq\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\right]^{1/q}.
\end{aligned}$$

Notice that

$$\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds \overset{d}{=} \left(\frac{1}{2}\right)^{2H_0+H-d}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds.$$

Taking $p = 2^{2H_0+H-d}$ leads to

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{C\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\\
&\le \mathbb{E}_0 \exp\left\{Cq\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}.
\end{aligned} \tag{5.3}$$

For any integer $n \ge 1$

$$\begin{aligned}
&\mathbb{E}_0\left[\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right]^n\\
&= \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[t/2,t]^n}\left(\prod_{k=1}^n s_k\right)^{-(1-2H_0)}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k}-\tilde{B}_{s_k})}\right)ds\\
&\le \left(\frac{2}{t}\right)^{n(1-2H_0)}\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[0,t]^n}\left(\prod_{k=1}^n s_k\right)^{-(1-2H_0)}\left(\mathbb{E}_0\prod_{k=1}^n e^{i\xi_k\cdot(B_{s_k}-\tilde{B}_{s_k})}\right)ds\\
&= \left(\frac{2}{t}\right)^{n(1-2H_0)}\mathbb{E}_0\left[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right]^n.
\end{aligned}$$

By Taylor expansion,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ Cq \int_{\mathbb{R}^d} \mu(d\xi) \int_{t/2}^t s^{-(1-2H_0)} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} \\
&\le \mathbb{E}_0 \exp\left\{ Cq \Big(\frac{2}{t}\Big)^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} \\
&= \mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t^{\frac{2H_0+H-d}{1-(d-H)}}} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\}
\end{aligned} \tag{5.4}$$

where the last step follows from the Brownian scaling.

Given $t_1, t_2 > 0$, by Markov property

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_1+t_2} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} \\
&\le \mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_1} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} \\
&\times \sup_{(x,\tilde{x})} \mathbb{E}_{(x,\tilde{x})} \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_2} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\}
\end{aligned}$$

where "$\mathbb{E}_{(x,\tilde{x})}$" is the Brownian expectation with $B_0 = x$ and $\widetilde{B}_0 = \tilde{x}$.

In addition, the moment representation shows that

$$\mathbb{E}_{(x,\tilde{x})} \left[\int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_2} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right]^n \le \mathbb{E}_0 \left[\int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_2} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right]^n$$

for every $n = 1, 2, \cdots$. Applying the Taylor's expansion leads to the same comparison in exponential moment. Consequently,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_1+t_2} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} \\
&\le \mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_1} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} \\
&\times \mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^{t_2} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\}.
\end{aligned}$$

Therefore, the argument by sub-additivity leads to

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ Cq 2^{1-2H_0} \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} < \infty. \tag{5.5}$$

Applying this to (5.4) and noticing $\frac{2H_0+H-d}{1-(d-H)} > 0$,

$$\limsup_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}_0 \exp\left\{ Cq \int_{\mathbb{R}^d} \mu(d\xi) \int_{t/2}^t s^{-(1-2H_0)} e^{i\xi\cdot(B_s-\tilde{B}_s)} ds \right\} < \infty. \tag{5.6}$$

Combining this with (5.3), we conclude

$$\limsup_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}_0 \exp\left\{C\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t s^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\} < \infty. \tag{5.7}$$

Using Hölder inequality again

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{C\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\\
&\le \left[\mathbb{E}_0 \exp\left\{pC\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\right]^{1/p}\\
&\times \left[\mathbb{E}_0 \exp\left\{qC\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\right]^{1/q}.
\end{aligned}$$

By a procedure similar to the proof of (5.6)

$$\limsup_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}_0 \exp\left\{pC\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\} < \infty.$$

In addition, write

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\\
&= \int_{\mathbb{R}^d}\mu(d\xi)e^{i\xi\cdot(B_{t/2}-\tilde{B}_{t/2})}\int_0^{\frac{t}{2}} (\frac{t}{2}-s)^{-(1-2H_0)}e^{i\xi\cdot\{(B_{(t/2)+s}-B_{t/2})-(\tilde{B}(t/2)+s-\tilde{B}_{t/2})\}}ds.
\end{aligned}$$

By the independence of the Brownian increment, one can directly check that for $n = 1, 2, \cdots$,

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\bigg]^n\\
&\le \mathbb{E}\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} (\frac{t}{2}-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\bigg]^n.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{qC\int_{\mathbb{R}^d}\mu(d\xi)\int_{t/2}^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\\
&\le \mathbb{E}_0 \exp\left\{qC\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} (\frac{t}{2}-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\\
&= \mathbb{E}_0 \exp\left\{qC\Big(\frac{1}{2}\Big)^{2H_0+H-d}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}.
\end{aligned}$$

Taking $q = 2^{2H_0+H-d}$ gives

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{C\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}\\
&\le \mathbb{E}_0 \exp\left\{pC\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{t/2} (t-s)^{-(1-2H_0)}e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\}.
\end{aligned}$$

Summarizing the discussion, we have

$$\limsup_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}_0 \exp\left\{pC\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t (t-s)^{-(1-2H_0)} e^{i\xi\cdot(B_s-\tilde{B}_s)}ds\right\} < \infty. \tag{5.8}$$

Combining (5.7) and (5.8),

$$\limsup_{t\to\infty} t^{-\frac{2H_0+H-d}{1-(d-H)}} \log \mathbb{E}_0 \exp\Big\{C\mathcal{L}_t\Big\} < \infty.$$

This can be easily strengthened into (5.2) as the relation

$$\mathbb{E}_0 \exp\Big\{C|\mathcal{L}_t|\Big\} \le 2\mathbb{E}_0 \exp\Big\{C\mathcal{L}_t\Big\}. \tag{5.9}$$

Indeed, by Taylor expansion and by the fact that $\mathbb{E}_0\mathcal{L}_t^n \ge 0$ for $n =, 1, 2, \cdots$,

$$\mathbb{E}_0 \exp\Big\{-C\mathcal{L}_t\Big\} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}C^n\mathbb{E}_0\mathcal{L}_t^n \le \sum_{n=0}^{\infty}\frac{1}{n!}C^n\mathbb{E}_0\mathcal{L}_t^n = \mathbb{E}_0\exp\Big\{C\mathcal{L}_t\Big\}.$$

□

Notice that $\frac{2H_0+H-d}{1-(d-H)} < 1$ as $H_0 < 1/2$. A standard argument of exponential approximation by Hölder inequality shows that $\mathcal{L}_t$ in (5.1) does not make contribution to the asymptics stated in Theorem 1.3. By the scaling in (4.11), all we need is to show there is a constant $c_0(\mathbf{H}) > 0$ such that

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}_0 \exp\big\{\mathcal{M}_t\big\} = c_0(\mathbf{H}). \tag{5.10}$$

which leads to (1.21) with

$$\kappa(\mathbf{H}) = \Big(\frac{H_0(1-2H_0)}{2}\Big)^{\frac{1}{2H_0+H-d}} c_0(\mathbf{H}).$$

Lemma 5.2 *For any $C > 0$,*

$$\limsup_{t\to\infty}\frac{1}{t}\log \mathbb{E}_0 \exp\big\{C\mathcal{M}_t\big\} < \infty. \tag{5.11}$$

Proof: For any bounded set $D \subset \mathbb{R}^+ \times \mathbb{R}^+$, set

$$\mathcal{M}(D) = \int_{\mathbb{R}^d}\mu(d\xi)\int_D \frac{\big[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r}\big]\big[e^{-i\xi\cdot\widetilde{B}_s} - e^{-i\xi\cdot\widetilde{B}_r}\big]}{|s-r|^{2-2H_0}}dsdr. \tag{5.12}$$

In this notation, $\mathcal{M}_t = \mathcal{M}([0,t]^2)$. Notice that for $n = 1, 2, \cdots$,

$$\mathbb{E}_0\mathcal{M}^n(D) = \int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{D^n}\Big(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)}\Big)\Big|\mathbb{E}_0\prod_{k=1}^n\big(e^{i\xi\cdot B_{s_k}} - e^{i\xi\cdot B_{r_k}}\big)\Big|^2 dsdr. \tag{5.13}$$

In particular, $\mathbb{E}_0\mathcal{M}^n(D)$ is monotonic in D. Consequently,

$$\mathbb{E}_0 \exp\big\{C\mathcal{M}(D)\big\} \le \mathbb{E}_0 \exp\big\{C\mathcal{M}(D')\big\} \quad D \subset D'. \tag{5.14}$$

Therefore, it suffices to show that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_0 \exp\big\{C\mathcal{M}\big([0,4N]^2\big)\big\} < \infty. \tag{5.15}$$

Consider the decomposition

$$\begin{aligned}
&[0,4N]^2 \subset \{(s,r)|0 \le s \le 1,\ 0 \le r \le 1+s\} \cup \{(s,r)|1 \le s \le 4N,\ |r-s| \le 1\} \\
&\cup \{(s,r)|1 \le s \le 4N,\ 1+s \le r \le 4N\} \cup \{(s,r)|1 \le r \le 4N,\ 1+r \le s \le 4N\} \\
&= D_0 \cup D_1 \cup D_2 \cup D_3 \quad \text{(say)}.
\end{aligned}$$

So we have

$$\begin{aligned}
&\mathbb{E}_0 \exp\big\{C\mathcal{M}\big([0,4N]^2\big)\big\} \le \mathbb{E}_0 \exp\big\{C\mathcal{M}\big(D_0 \cup D_1 \cup D_2 \cup D_3\big)\big\} \\
&= \mathbb{E}_0 \exp\Big\{C\Big(\mathcal{M}(D_0) + \mathcal{M}(D_1) + \mathcal{M}\big(D_2\big) + \mathcal{M}\big(D_3\big)\Big)\Big\}.
\end{aligned}$$

Notice that D_0 is independent of N and by (4.18) and (5.14), $\mathcal{M}(D_0)$ exponential integrable. Notice also that $\mathcal{M}\big(D_2\big) \stackrel{d}{=} \mathcal{M}\big(D_3\big)$. By Hölder inequality, all we need is to show

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_0 \exp\big\{C\mathcal{M}(D_j)\big\} < \infty \quad j = 1, 2. \tag{5.16}$$

Write

$$D_1 = \bigcup_{k=1}^{4N-1} \{(s,r)|k \le s \le k+1,\ |r-s| \le 1\} = \bigcup_{k=1}^{4N-1} (G_k)$$

and

$$\mathcal{M}(D_1) = \sum_{k=1}^{4N-1} \mathcal{M}(G_k) = \sum_{k=1}^{N-1} \mathcal{M}(G_{4k}) + \sum_{k=0}^{N-1} \mathcal{M}(G_{4k+1}) + \sum_{k=0}^{N-1} \mathcal{M}(G_{4k+2}) + \sum_{k=0}^{N-1} \mathcal{M}(G_{4k+3}).$$

By Markov property,

$$\mathbb{E}_0 \exp\Big\{C \sum_{k=1}^{N-1} \mathcal{M}(G_{4k})\Big\} \le \Big(\sup_{(x,\tilde{x})} \mathbb{E}_{(x,\tilde{x})} \exp\big\{C\mathcal{M}(G_1)\big\}\Big)^{N-1}.$$

Notice that for each $n = 1, 2, \cdots$,

$$
\begin{aligned}
&\mathbb{E}_{(x,\tilde{x})}\mathcal{M}^n(G_1)\\
&= \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{G_1^n} \Big(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)} \Big) \Big[\mathbb{E}_0 \prod_{k=1}^n \big(e^{i\xi\cdot(x+B_{s_k})} - e^{i\xi\cdot(x+B_{r_k})} \big) \Big]\\
&\times \Big[\mathbb{E}_0 \prod_{k=1}^n \big(e^{-i\xi\cdot(\tilde{x}+B_{s_k})} - e^{-i\xi\cdot(\tilde{x}+B_{r_k})} \big) \Big] dsdr\\
&= \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{G_1^n} \Big(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)} \Big) \exp\Big\{ i\Big(\sum_{k=1}^n \xi_k \Big) \cdot (x - \tilde{x}) \Big\}\\
&\times \Big| \mathbb{E}_0 \prod_{k=1}^n \big(e^{i\xi\cdot(x+B_{s_k})} - e^{i\xi\cdot(x+B_{r_k})} \big) \Big|^2 dsdr\\
&\le \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{G_1^n} \Big(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)} \Big) \Big| \mathbb{E}_0 \prod_{k=1}^n \big(e^{i\xi\cdot(x+B_{s_k})} - e^{i\xi\cdot(x+B_{r_k})} \big) \Big|^2 dsdr\\
&= \mathbb{E}_0\mathcal{M}^n(G_1) \quad \forall x, \tilde{x} \in \mathbb{R}^d.
\end{aligned}
$$

We have

$$
\sup_{(x,\tilde{x})} \mathbb{E}_{(x,\tilde{x})} \exp\big\{ C\mathcal{M}(G_1) \big\} \le \mathbb{E}_0 \exp\big\{ C\mathcal{M}(G_1) \big\} < \infty. \tag{5.17}
$$

Hence,

$$
\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_0 \exp\Big\{ C \sum_{k=1}^{N-1} \mathcal{M}(G_{4k}) \Big\} < \infty.
$$

The same conclusion can be extended to all four summations in the decomposition of $\mathcal{M}(D_1)$. By Hölder inequality we have proved (5.16) for $j = 1$.

We now exam (5.16) for $j = 2$.

$$
\begin{aligned}
\mathcal{M}(D_2) &= \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{\big[e^{i\xi\cdot B_s} - e^{i\xi\cdot B_r} \big] \big[e^{-i\xi\cdot \widetilde{B}_s} - e^{-i\xi\cdot \widetilde{B}_r} \big]}{|s-r|^{2-2H_0}} dsdr\\
&= \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_s)}}{|s-r|^{2-2H_0}} dsdr + \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_r - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr\\
&- \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr - \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(\widetilde{B}_s - B_r)}}{|s-r|^{2-2H_0}} dsdr.
\end{aligned}
$$

The last two terms on the right hand side are identical in law. Similar to (5.9),

$$
\begin{aligned}
&\mathbb{E}_0 \exp\Big\{ C \Big| \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr \Big| \Big\}\\
&\le 2\mathbb{E}_0 \exp\Big\{ C \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr \Big\}.
\end{aligned}
$$

To establish (5.16) for $j = 2$, therefore, all we need is to show

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_0 \exp\left\{ C \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr \right\} < \infty, \tag{5.18}$$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_0 \exp\left\{ C \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_s)}}{|s-r|^{2-2H_0}} dsdr \right\} < \infty, \tag{5.19}$$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}_0 \exp\left\{ C \int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_r - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr \right\} < \infty. \tag{5.20}$$

To prove (5.18), notice that for any $n = 1, 2, \cdots$,

$$\begin{aligned}
&\mathbb{E}_0 \bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_r)}}{|s-r|^{2-2H_0}} dsdr \bigg]^n \\
&= \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,4N]^n} ds \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B_{s_k}} \bigg) \int_{[s_1+1,4N]\times\cdots\times[s_n+1,4N]} \\
&\times \bigg(\prod_{k=1}^n (r_k - s_k)^{-(2-2H_0)} \bigg) \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{-i\xi_k \cdot B_{r_k}} \bigg) dr \\
&\le \bigg(\int_1^\infty r^{-(2-2H_0)} dr \bigg)^n \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{[0,4N]^n} \bigg(\mathbb{E}_0 \prod_{k=1}^n e^{i\xi_k \cdot B_{s_k}} \bigg) ds \\
&= C^n \mathbb{E}_0 \bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_0^{4N} e^{i\xi\cdot B_s} ds \bigg]^n
\end{aligned}$$

where the inequality is costed by the actions of replacing the second $\mathbb{E}_0$-expectation by 1, and the integration domain $[s_1+1, 4N] \times \cdots \times [s_n+1, 4N]$ by $[s_1+1, \infty) \times \cdots \times [s_n+1, \infty)$; and the last step is supported by the fact that $2 - 2H_0 > 1$.

By (5.5) (with a travail notation adjustment) and the fact that $B - \widetilde{B} \stackrel{d}{=} \sqrt{2}B$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ C \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t e^{i\xi\cdot B_s} ds \right\} < \infty$$

which leads to (5.18).

The same computation leads to

$$\mathbb{E}_0 \bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_{D_2} \frac{e^{i\xi\cdot(B_s - \widetilde{B}_s)}}{|s-r|^{2-2H_0}} dsdr \bigg]^n \le C^n \mathbb{E}_0 \bigg[\int_{\mathbb{R}^d} \mu(d\xi) \int_0^{4N} e^{i\xi\cdot(B_s - \tilde{B}_s)} ds \bigg]^n.$$

So we have (5.19).

As for (5.20),

$$\begin{aligned}
&\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_{D_2}\frac{e^{i\xi\cdot(B_r-\widetilde{B}_r)}}{|s-r|^{2-2H_0}}dsdr\bigg]^n\\
&=\int_{(\mathbb{R}^d)^n}\mu(d\xi)\int_{[1,4N]^n}dr\bigg(\mathbb{E}_0\prod_{k=1}^n e^{-i\xi_k\cdot(B_{r_k}-\tilde{B}_{r_k})}\bigg)\\
&\times\int_{[0,r_1-1]\times\cdots\times[0,r_n-1]}\bigg(\prod_{k=1}^n(r_k-s_k)^{-(2-2H_0)}\bigg)ds\\
&\le\bigg(\int_1^\infty s^{-(2-2H_0)}ds\bigg)^n\mathbb{E}_0\bigg[\int_{\mathbb{R}^d}\mu(d\xi)\int_0^{4N}e^{i\xi\cdot(B_r-\tilde{B}_r)}dr\bigg]^n.
\end{aligned}$$

So (5.20) follows from (5.5). □

To complete the proof of Theorem 1.3, we prove that the limit in (5.10) exists and is positive. For any $t>0$, define the linear operator T_t on $\mathcal{L}^2(\mathbb{R}^{2d})$ as

$$T_tf(x,\tilde{x})=\mathbb{E}_{(x,\tilde{x})}\Big[\exp\big\{\mathcal{M}_t\big\}f(B_t,\widetilde{B}_t)\Big]\quad f\in\mathcal{L}^2(\mathbb{R}^{2d}).$$

The idea is to show that the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{f\in\mathcal{F}}\langle f,T_tf\rangle=c_0(\mathbf{H}) \tag{5.21}$$

exists and is positive for a sub-class $\mathcal{F}$ of $\mathcal{L}^2(\mathbb{R}^{2d})$. The unfortunate fact is that the family $\{T_t;\ \ t\ge0\}$ does not have semi-group structure (i.e., the structure defined by $T_{s+t}=T_t\circ T_s$). In the following we try to capture some useful properties of T_t that allows our technique get through.

First, we claim that for each $t>0$, T_t is bounded, i.e., there is a constant $C_t>0$ such that

$$\int_{\mathbb{R}^{2d}}[T_tf(x,\tilde{x})]^2dxd\tilde{x}\le C_t\int_{\mathbb{R}^{2d}}f^2(x,\tilde{x})dxd\tilde{x}\quad f\in\mathcal{L}^2(\mathbb{R}^{2d}). \tag{5.22}$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}[T_tf(x,\tilde{x})]^2dxd\tilde{x}=\int_{\mathbb{R}^{2d}}\bigg(\mathbb{E}_{(x,\tilde{x})}\Big[\exp\big\{\mathcal{M}_t\big\}f(B_t,\widetilde{B}_t)\Big]\bigg)^2dxd\tilde{x}\\
&\le\int_{\mathbb{R}^{2d}}\Big(\mathbb{E}_{(x,\tilde{x})}\exp\big\{2\mathcal{M}_t\big\}\Big)\Big(\mathbb{E}_{(x,\tilde{x})}f^2(B_t,\widetilde{B}_t)\Big)dxd\tilde{x}\\
&\le\Big(\mathbb{E}_0\exp\big\{2\mathcal{M}_t\big\}\Big)\int_{\mathbb{R}^{2d}}\mathbb{E}_0f^2(x+B_t,\tilde{x}+\widetilde{B}_t)dxd\tilde{x}
\end{aligned}$$

where the last step follows from the relation (5.17) (with G_1 being replaced by $[0,t]^2$). Hence, (5.22) follows from

$$\int_{\mathbb{R}^{2d}}\mathbb{E}_0f^2(x+B_t,\tilde{x}+\widetilde{B}_t)dxd\tilde{x}=\mathbb{E}_0\int_{\mathbb{R}^{2d}}f^2(x+B_t,\tilde{x}+\widetilde{B}_t)dxd\tilde{x}=\int_{\mathbb{R}^{2d}}f^2(x,\tilde{x})dxd\tilde{x}.$$

Next, we claim that for each $t > 0$, T_t is self-adjoint:

$$\langle g, T_t f\rangle = \langle T_t g, f\rangle \quad f, g \in \mathcal{L}^2(\mathbb{R}^{2d}). \tag{5.23}$$

To simply our notation in the following argument, we denote $\mathcal{M}_t$ as $\mathcal{M}_t(B, \widetilde{B})$.

$$\begin{aligned}
\langle g, T_t f\rangle &= \int_{\mathbb{R}^{2d}} g(x,\tilde{x})\mathbb{E}_0\Big[\exp\big\{\mathcal{M}_t(x+B, \tilde{x}+\widetilde{B})\big\} f(x+B_t, \tilde{x}+\widetilde{B}_t)\Big] dxd\tilde{x} \\
&= \mathbb{E}_0\bigg\{\int_{\mathbb{R}^{2d}} g(x,\tilde{x}) \exp\big\{\mathcal{M}_t(x+B, \tilde{x}+\widetilde{B})\big\} f(x+B_t, \tilde{x}+\widetilde{B}_t) dxd\tilde{x}\bigg\} \\
&= \mathbb{E}_0\bigg\{\int_{\mathbb{R}^{2d}} g(x-B_t,\tilde{x}-\widetilde{B}_t) \exp\big\{\mathcal{M}_t(x-B_t+B,\ \tilde{x}-\tilde{B}_t+\widetilde{B})\big\} f(x,\tilde{x}) dxd\tilde{x}\bigg\} \\
&= \int_{\mathbb{R}^{2d}} f(x,\tilde{x})\mathbb{E}_0\Big[\exp\big\{\mathcal{M}_t(x-B_t+B,\ \tilde{x}-\tilde{B}_t+\widetilde{B})\big\} g(x-B_t, \tilde{x}-\widetilde{B}_t)\Big] dxd\tilde{x}
\end{aligned}$$

where the third step follows from translation invariance.

Notice that

$$\begin{aligned}
&\mathcal{M}_t(x-B_t+B,\ \tilde{x}-\tilde{B}_t+\widetilde{B}) \\
&= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{\big[e^{i\xi\cdot(x+B_s-B_t)} - e^{i\xi\cdot(x+B_r-B_t)}\big]\big[e^{-i\xi\cdot(\tilde{x}+\widetilde{B}_s-\widetilde{B}_t)} - e^{-i\xi\cdot(\tilde{x}+\widetilde{B}_r-\widetilde{B}_t}\big]}{|s-r|^{2-2H_0}} dsdr \\
&= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t\int_0^t \frac{\big[e^{i\xi\cdot(x+B_{t-s}-B_t)} - e^{i\xi\cdot(x+B_{t-r}-B_t)}\big]\big[e^{-i\xi\cdot(\tilde{x}+\widetilde{B}_{t-s}-\widetilde{B}_t)} - e^{-i\xi\cdot(\tilde{x}+\widetilde{B}_{t-r}-\widetilde{B}_t}\big]}{|s-r|^{2-2H_0}} dsdr \\
&= \mathcal{M}_t(x+\beta, \tilde{x}+\tilde{\beta})
\end{aligned}$$

where

$$\beta_s = B_{t-s} - B_t \ \text{ and } \ \tilde{\beta}_s = \widetilde{B}_{t-s} - \widetilde{B}_t \quad (0 \le s \le t)$$

are two independent Brownian motions with $\beta_0 = \tilde{\beta}_0 = 0$ and $\beta_t = -B_t$, $\tilde{\beta}_t = -\widetilde{B}_t$ under $\mathbb{E}_0$. Summarizing our computation,

$$\langle g, T_t f\rangle = \int_{\mathbb{R}^{2d}} f(x,\tilde{x})\mathbb{E}_0\Big[\exp\big\{\mathcal{M}_t(x+\beta, \tilde{x}+\widetilde{\beta})\big\} g(x+\beta_t, \tilde{x}+\widetilde{\beta}_t)\Big] dxd\tilde{x} = \langle T_t g, f\rangle.$$

In connection to (5.21), set

$$\mathcal{F} = \Big\{f;\, f(x,\tilde{x}) = \bar{f}(x)\bar{f}(\tilde{x}),\ \|\bar{f}\|_{\mathcal{L}^2(\mathbb{R}^d)} = 1,\ \bar{f} \ge 0,\ \|\bar{f}\|_{\mathcal{L}(\mathbb{R}^d)} \vee \|\bar{f}\|_{\mathcal{L}^\infty(\mathbb{R}^d)} < \infty\Big\}$$

For any $D \subset [0,t]^2$, it is straightforward to check that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} f(x,\tilde{x})\mathbb{E}_{(x,\tilde{x})}\Big\{\mathcal{M}(D)^n f(B_t, \widetilde{B}_t)\Big\} dxd\tilde{x} \qquad (5.24) \\
&= \int_{(\mathbb{R}^d)^n} \mu(d\xi) \int_{D^n} \bigg(\prod_{k=1}^n |s_k - r_k|^{-(2-2H_0)}\bigg) \\
&\times \bigg|\mathcal{F}(\bar{f})\Big(\sum_{k=1}^n \xi_k\Big)\bigg(\mathbb{E}_0\bar{f}(B_t)\prod_{k=1}^n [e^{i\xi\cdot B_{s_k}} - e^{i\xi\cdot B_{r_k}}]\bigg)\bigg|^2 dsdr
\end{aligned}$$

for any $n = 1, 2, \cdots$ and $f \in \mathcal{F}$, where

$$\mathcal{F}(\bar{f})(\xi) = \int_{\mathbb{R}^d} \bar{f}(x) e^{i\xi \cdot x} dx \quad \xi \in \mathbb{R}^d.$$

By Taylor expansion, the quantity

$$\int_{\mathbb{R}^{2d}} f(x, \tilde{x}) \mathbb{E}_{(x,\tilde{x})} \Big[\exp\big\{\mathcal{M}(D)\big\} f(B_t, \widetilde{B}_t) \Big] dx d\tilde{x}$$

is monotonic in D. Taking $D = [0, t]^2$ in (5.24) one can also see that

$$\sup_{f \in \mathcal{F}} \langle f, T_t f \rangle > 1 \quad \forall t > 0. \tag{5.25}$$

We now prove that

$$\langle f, T_{mt_0} f \rangle \geq \langle f, T_{t_0} f \rangle^m \quad f \in \mathcal{F}, \;\; t_0 > 0, \;\; m = 2, 4, 6, \cdots . \tag{5.26}$$

Indeed, by the relation

$$D_m \equiv \bigcup_{j=1}^{m} [(j-1)t_0, jt_0]^2 \subset [0, mt_0]^2$$

and monotonicity

$$\langle f, T_{mt_0} f \rangle \geq \int_{\mathbb{R}^{2d}} f(x, \tilde{x}) \mathbb{E}_{(x,\tilde{x})} \Big[\exp\big\{\mathcal{M}(D_m)\big\} f(B_t, \widetilde{B}_t) \Big] dx d\tilde{x}.$$

In addition, by Markov property one can check that

$$\mathbb{E}_{(x,\tilde{x})} \Big[\exp\big\{\mathcal{M}(D_m)\big\} f(B_t, \widetilde{B}_t) \Big] dx d\tilde{x} = T_{t_0}^m f(x, \tilde{x}).$$

Thus,

$$\langle f, T_{mt_0} f \rangle \geq \langle f, T_{t_0}^m f \rangle$$

By the spectral theory for self-adjoint operator on Hilbert space, there is a measure $\mu_f(d\lambda)$ on $\mathbb{R}$ such that

$$\mu_f(\mathbb{R}) = \|f\|^2_{\mathcal{L}^2(\mathbb{R}^{2d})} = 1 \;\; and \;\; \langle f, T_{t_0} f \rangle = \int_{\mathbb{R}} \lambda \mu_f(d\lambda).$$

Further,

$$\langle f, T_{t_0}^m f \rangle = \int_{\mathbb{R}} \lambda^m \mu_f(d\lambda).$$

Notice that the function $\varphi(\lambda) = \lambda^m$ is convex on $\mathbb{R}$ when m is even. Since $\mu_f(d\lambda)$ is a probability measure, by Jensen's inequality,

$$\int_{\mathbb{R}} \lambda^m \mu_f(d\lambda) \geq \bigg(\int_{\mathbb{R}} \lambda \mu_f(d\lambda) \bigg)^m = \langle f, T_{t_0} f \rangle^m.$$

Combining our argument, we have proved (5.26).

By (5.26) and by the monotonicity of the quadratic function $\langle f, T_t f\rangle$ in t for any $f \in \mathcal{F}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \sup_{f\in\mathcal{F}} \langle f, T_t f\rangle \geq \frac{1}{t_0} \log \sup_{f\in\mathcal{F}} \langle f, T_{t_0} f\rangle. \tag{5.27}$$

Taking limsup on the right hand side over $t_0 \to \infty$ leads to the existence of the limit in (5.21) with $c_0(H)$ being a possibly extended real number. Further, taking $t_0 = 1$ in (5.27),

$$c_0(H) \geq \log \sup_{f\in\mathcal{F}} \langle f, T_1 f\rangle > 0$$

where the last step follows from (5.25) with $t = 1$.

Finally, we prove that (5.10) holds with the same $c_0(\mathbf{H})$ which automatically leads to $c_0(\mathbf{H}) < \infty$ according to Lemma 5.2.

Given $f \in \mathcal{F}$,

$$\begin{aligned}
\langle f, T_t f\rangle &= \int_{\mathbb{R}^{2d}} f(x,\tilde{x}) \mathbb{E}_{(x,\tilde{x})} \Big[\exp\big\{\mathcal{M}_t\big\} f(B_t, \widetilde{B}_t)\Big] dx d\tilde{x} \\
&\leq \|f\|_{\mathcal{L}^\infty(\mathbb{R}^{2d})} \int_{\mathbb{R}^{2d}} f(x,\tilde{x}) \mathbb{E}_{(x,\tilde{x})} \exp\big\{\mathcal{M}_t\big\} dx d\tilde{x} \\
&\leq \|f\|_{\mathcal{L}^\infty(\mathbb{R}^{2d})} \|f\|_{\mathcal{L}(\mathbb{R}^{2d})} \mathbb{E}_0 \exp\big\{\mathcal{M}_t\big\}
\end{aligned}$$

where the last step follows from (5.17) (with G_1 being replaced by $[0,t]^2$).

By (5.26) and monotonicity,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\big\{\mathcal{M}_t\big\} \geq \frac{1}{t_0} \log\langle f, T_{t_0} f\rangle \quad f \in \mathcal{F}.$$

Taking the supremum over f and then letting $t_0 \to \infty$ on the right hand side,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\big\{\mathcal{M}_t\big\} \geq c_0(\mathbf{H}). \tag{5.28}$$

On the other hand,

$$\mathbb{E}_0 \exp\big\{\mathcal{M}([1,t]^2)\big\} = \int_{\mathbb{R}^{2d}} p_1(x,\tilde{x}) \mathbb{E}_{(x,\tilde{x})} \exp\big\{\mathcal{M}_{t-1}\big\} dx d\tilde{x}$$

where $p_1(x,(\tilde{x})$ is the density of $(B_1, \widetilde{B}_1)$. Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\setminus[-t^2,t^2]^{2d}} p_1(x,\tilde{x}) \mathbb{E}_{(x,\tilde{x})} \exp\big\{\mathcal{M}_{t-1}\big\} dx d\tilde{x} \\
&\leq \mathbb{E}_0 \exp\big\{\mathcal{M}_{t-1}\big\} \int_{\mathbb{R}^{2d}\setminus[-t^2,t^2]^{2d}} p_1(x,\tilde{x}) dx d\tilde{x} \\
&= \mathbb{P}_0\Big\{ \max\{|B_1|_\infty, |\widetilde{B}_1|_\infty\} \geq t^2\Big\} \mathbb{E}_0 \exp\big\{\mathcal{M}_{t-1}\big\}.
\end{aligned}$$

Further, write

$$\begin{aligned}
&\int_{[-t^2,t^2]^{2d}} p_1(x,\tilde{x})\mathbb{E}_{(x,\tilde{x})}\exp\big\{\mathcal{M}_{t-1}\big\}dxd\tilde{x}\\
&=\int_{[-t^2,t^2]^{2d}} p_1(x,\tilde{x})\mathbb{E}_{(x,\tilde{x})}\Big[\exp\big\{\mathcal{M}_{t-1}\big\}1_{[-t^2,t^2]^2}(B_t,\widetilde{B}_t)\Big]dxd\tilde{x}\\
&+\int_{[-t^2,t^2]^{2d}} p_1(x,\tilde{x})\mathbb{E}_{(x,\tilde{x})}\Big[\exp\big\{\mathcal{M}_{t-1}\big\}1_{\mathbb{R}^{2d}\setminus[-t^2,t^2]^{2d}}(B_t,\widetilde{B}_t)\Big]dxd\tilde{x}\\
&\le (2\pi)^{-d}\int_{[-t^2,t^2]^{2d}}\mathbb{E}_{(x,\tilde{x})}\Big[\exp\big\{\mathcal{M}_{t-1}\big\}1_{[-t^2,t^2]^{2d}}(B_t,\widetilde{B}_t)\Big]dxd\tilde{x}\\
&+\Big(\mathbb{E}_0\exp\big\{2\mathcal{M}_{t-1}\big\}\Big)^{1/2}\Big(\mathbb{P}_0\Big\{\max\{|B_t|_\infty,|\widetilde{B}_t|_\infty\ge t^2\Big\}\Big)^{1/2}.
\end{aligned}$$

Notice that $f_t(x,\tilde{x})=(2t)^{-2d}1_{[-t^2,t^2]^d}(x)1_{[-t^2,t^2]^d}(\tilde{x})$ is in $\mathcal{F}$. Summarizing our estimate,

$$\begin{aligned}
&\mathbb{E}_0\exp\big\{\mathcal{M}([1,t]^2)\big\}\le(2t)^{2d}\sup_{f\in\mathcal{F}}\langle f,T_tf\rangle\\
&+\Big(\mathbb{E}_0\exp\big\{2\mathcal{M}_{t-1}\big\}\Big)^{1/2}\Big(\mathbb{P}_0\Big\{\max\{|B_t|_\infty,|\widetilde{B}_t|_\infty\ge t^2\Big\}\Big)^{1/2}\\
&+\mathbb{P}_0\Big\{\max\{|B_1|_\infty,|\widetilde{B}_1|_\infty\}\ge t^2\Big\}\mathbb{E}_0\exp\big\{\mathcal{M}_{t-1}\big\}.
\end{aligned}$$

By (5.21), by Lemma 5.2 and by the Gaussian tail,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\big\{\mathcal{M}([1,t]^2)\big\}.\le c_0(\mathbf{H}).$$

Given the conjugate numbers $p,q>1$

$$\mathbb{E}_0\exp\big\{p^{-1}\mathcal{M}_t\big\}\le\Big(\mathbb{E}_0\exp\big\{\mathcal{M}([1,t]^2)\big\}\Big)^{1/p}\Big(\mathbb{E}_0\exp\big\{(p-1)^{-1}\mathcal{M}([0,1]^2)\big\}\Big)^{1/q}.$$

So we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\big\{p^{-1}\mathcal{M}_t\big\}\le p^{-1}c_0(\mathbf{H}).$$

By scaling,

$$p^{-1}\mathcal{M}_t\overset{d}{=}\mathcal{M}_{t_p}\quad\text{ where }t_p=tp^{-\frac{1}{2H_0+H-d}}.$$

By variable substitution

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\big\{\mathcal{M}([0,t]^2)\big\}\le p^{-1}p^{\frac{1}{2H_0+H-d}}c_0(\mathbf{H}).$$

Letting $p\to1^+$ on the right hand side leads to the upper bound for (5.10). □

6 Appendix

Here we establish two identities on gamma functions that have been used in this paper.

Lemma 6.1 *For any* $0 < H_0 < 1$,

$$\int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda = \frac{\pi H_0}{\Gamma(2H_0+1)\sin(H_0\pi)}, \tag{6.1}$$

$$\int_{\mathbb{R}} \left| \int_0^1 e^{i\lambda s} ds \right|^2 |\lambda|^{1-2H_0} d\lambda = \frac{2\pi}{\Gamma(2H_0+1)\sin(H_0\pi)}. \tag{6.2}$$

Proof: To prove (6.1), we begin with Hankel's representation ([16])

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty \lambda^{z-1} e^{-s\lambda} d\lambda$$

with $z \neq 0, -1, -2, \cdots$ and $|\arg(s)| \leq \pi$. Here we point out the gamma function $\Gamma(z)$ can be extended analytically into $\mathbb{C} \setminus \{0, -1, -2, \cdots\}$ by the limit (eq. 6.1.2. [1])

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{n^z}{n!} z(z+1)\cdots(z+n).$$

Taking $s = -i$ and $z = 1 - 2H_0$ in Hankel's representation gives

$$\exp\Big\{i\frac{1-2H_0}{2}\pi\Big\}\Gamma(1-2H_0) = \int_0^\infty \frac{e^{i\lambda}}{\lambda^{2H_0}} d\lambda.$$

Comparing the imaginary part,

$$\int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda = \sin\Big(\frac{1-2H_0}{2}\pi\Big)\Gamma(1-2H_0) = \cos(\pi H_0)\Gamma(1-2H_0).$$

Recall Euler's reflection formula (eq. 6.1.17, [1])

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad z \notin \mathbb{Z}.$$

Taking $z = 2H_0$,

$$\int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda = \frac{\pi\cos(\pi H_0)}{\Gamma(2H_0)\sin(2\pi H_0)} = \frac{\pi}{2\Gamma(2H_0)\sin(\pi H_0)} = \frac{\pi}{\Gamma(2H_0+1)\sin(\pi H_0)}$$

where the last step follows from the fact that $\Gamma(2H_0+1) = 2H_0\Gamma(2H_0)$.

As for (6.2)

$$\int_{\mathbb{R}} \left| \int_0^1 e^{i\lambda s} ds \right|^2 |\lambda|^{1-2H_0} d\lambda = \int_{\mathbb{R}} \left| \frac{e^{i\lambda} - 1}{i\lambda} \right|^2 |\lambda|^{1-2H_0} d\lambda$$
$$= 4 \int_{\mathbb{R}} \frac{\sin^2(\lambda/2)}{|\lambda|^{2H_0+1}} d\lambda = 8 \int_0^\infty \frac{\sin^2(\lambda/2)}{\lambda^{2H_0+1}} d\lambda$$
$$= \frac{4}{H_0} \int_0^\infty \frac{1}{\lambda^{2H_0}} \Big(\frac{d}{d\lambda} \sin^2(\lambda/2) \Big) d\lambda = \frac{2}{H_0} \int_0^\infty \frac{\sin\lambda}{\lambda^{2H_0}} d\lambda$$
$$= \frac{2}{H_0} \frac{\pi}{\Gamma(2H_0+1)\sin(\pi H_0)} = \frac{2\pi}{\Gamma(2H_0+1)\sin(\pi H_0)}$$

where the fourth equality follows from integration by parts and the sixth equality from (6.1). □

References

[1] Abramowitz, M. and Stegun, I. A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* Dover, New York, 1965.

[2] Balan, R., Jolis, M. and Quer-Sardanyons, L. SPDEs with fractional noise in space with index $H < 1/2$ *Statist. Probab. Lett.* **119** (2016) 310–316

[3] Carmona, R. A. and Molchanov. S. A. Parabolic Anderson model and intermittency. *Mem. Amer. Math. Soc.* **108** (1994)

[4] Chen, L., Hu, Y. Z., Kalbasi, K. and Nualart, D. Intermittency for the stochastic heat equation driven by a rough time fractional Gaussian noise. *Probab. Theor. Rel. Fields* **171** (2018) 431–457.

[5] Chen, X. *Random Walk Intersections: Large Deviations and Related Topics.* Mathematical Surveys and Monographs, **157**. American Mathematical Society, Providence 2009.

[6] Chen, X. Precise intermittency for the parabolic Anderson equation with an $(1 + 1)$-dimensional time-space white noise. *Annales de l'Institut Henri Poincare* **51** (2015) 1486-1499.

[7] Chen, X. Spatial asymptotics for the parabolic Anderson models with generalized time-space Gaussian noise. *Ann. Probab.* **44** (2016) 1535-1598.

[8] Chen, X. Moment asymptotics for parabolic Anderson equation with fractional time-space noise: in Skorokhod regime PDF. *Annales de l'Institut Henri Poincare* **53** (2017) 819-841.

[9] Chen, X. Parabolic Anderson model with rough or critical Gaussian noise *Annales de l'Institut Henri Poincare* (to appear).

[10] Chen, X, Hu, Y. Z., Song, S. and Xing, F. Exponential asymptotics for time-space Hamiltonians. *Annales de l'Institut Henri Poincare* **51** (2015) 1529-1561.

[11] Chen, X., Hu, Y. Z., Nualart, D. and Tindel, S. Spatial asymptotics for the parabolic Anderson model driven by a Gaussian rough noise PDF *EJP* **22** (2017) 1-38.

[12] Chen, X. and Phan, T. V. Free energy in a mean field of Brownian particles. *Discrete and Continuous Dynamical Systems* **39** (2019) 747-769.

[13] Dalang, R. C. Extending martingale measure stochastic integral with applications to spatially homogeneous S.P.D.E's. *Electron. J. Probab.* **4** (1999) 1-29.

[14] Deya, A. On a modelled rough heat equation. *Probab. Theor. Rel. Fields* **166** (2016) 1-65.

[15] Hairer, M. Labbé, C. A simple construction of the continuum parabolic Anderson model on $\mathbb{R}^2$. *Electron. Commun. Probab.* **20** (2015), No. 43, 11 pp.

[16] Hankel, H. Die Euler'schen Integrale bei unbeschränkter Variabilität des Arguments. *Z. Math. Phys.* **9** (1864) 1–21.

[17] Hu, Y. Z., Huang, J., Nualart, D. and Tindel, D. Stochastic heat equations with general multiplicative Gaussian noise: Hölder continuity and intermittency. *Electron. J. Probab.* **20** (2015) No. 55, 50 pp.

[18] Hu, Y. Z., Huang, J., Le, K., Nualart, D. and Tindel, S. Stochastic heat equation with rough dependence in space. *Ann. Probab.* **45** (2017) 4561-4616.

[19] Hu, Y. Z., and Nualart, D. Stochastic heat equation driven by fractional noise and local time. *Probab. Theor. Rel. Fields* **143** (2009) 285-328.

[20] Hu, Y. Z., Nualart, D. and Song, J. Feynman-Kac formula for heat equation driven by fractional white noise. *Ann. Probab.* **39** (2011) 291-326.

[21] Hu, Y. Z., Huang, J., Nualart, D. and Tindel, S. Parabolic Anderson model with rough dependence in space. *Proceedings of the Abel Conference* (to appear).

[22] Huang, J., Lê, K. and Nualart, D. Large time asymptotics for the parabolic Anderson model driven by space and time correlated noise. *Stoch. Partial Differ. Equ. Anal. Comput.* **5** (2017) 614–651.

[23] Lê, K. A remark on a result of Xia Chen. *Statistics & Probability Letters* **118** (2016) 124-126.

[24] Ledoux, M. and Talagrand, M. *Probability in Banach Spaces.* Springer, Berlin, 1991.

[25] Heyu Li and Chen, X. Precise moment asymptotics for the stochastic heat equation of a time-derivative Gaussian noise PDF. *Acta Mathematica Scientia (English ser)* (to appear)

Xia Chen
Department of Mathematics
University of Tennessee
Knoxville TN 37996, USA
xchen@math.utk.edu