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Small Deviations for the Mutual Intersection Local Time of Brownian Motions

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Abstract
In this note, we establish the bounds

$$c\varepsilon^{\frac{2}{3}} \le P\left\{ \int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le \varepsilon \right\} \le C\varepsilon^{\frac{2}{3}}$$

for the mutual intersection local time of two independent 1-dimensional Brownian motions B and $\tilde{B}$.

Keywords Brownian motion · Random walk · Intersection local time · Small ball probability

Mathematics Subject Classification (2020) 60J55 · 60F05 · 60B12 · 60J65

1 Introduction

Intersection local times (including self-intersection local time and mutual intersection local time) of stochastic processes are a fundamental concept in stochastic analysis, which also play an important role in the study of the quantum field theory, stochastic partial differential equations and models of polymers. Our goal in this note is to establish the small deviation for the mutual intersection local time

$$\int_0^t\int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r, \qquad t > 0$$

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driven by two independent 1-dimensional Brownian motions B_s and $\tilde{B}_s$ with $B_0 = \tilde{B}_0 = 0$, where $\delta_0(\cdot)$ is the Dirac delta function on $\mathbb{R}$ (see, e.g., [1, Chapter 2] for the detailed construction of intersection local times). The mutual intersection local time measures the intensity of path intersection between two independent Brownian trajectories. The following is the main result of this paper.

Theorem 1.1 *There exists constants $0 < c \le C < \infty$ such that for small $\varepsilon > 0$,*

$$c\varepsilon^{\frac{2}{3}} \le P\left\{ \int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)dsdr \le \varepsilon \right\} \le C\varepsilon^{\frac{2}{3}}. \tag{1.1}$$

The small deviation (or, small ball probability) is an interesting subject for stochastic models. An interested reader is referred to [5] for a survey of general development in this direction. The goal is to find the decay rate of the probability $P\{\|G\| \le \varepsilon\}$ as $\varepsilon \to 0^+$, where $G(\cdot)$ is a stochastic process or a random field that is properly embedded in a Banach space endowed with the norm $\|\cdot\|$. In our setting,

$$\int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r = \int_{-\infty}^{\infty} L(1,x)\tilde{L}(1,x)\mathrm{d}x \stackrel{\Delta}{=} \|G\|,$$

where $L(t,x)$ and $\tilde{L}(t,x)$ $((t,x) \in \mathbb{R}^+ \times \mathbb{R})$ are local times of B an $\tilde{B}$, respectively; and the random field $G(x) = L(1,x)\tilde{L}(1,x)$ $(x \in \mathbb{R})$ is embedded in the space $L^1(\mathbb{R})$. Due to technical limitations, the random fields $G(\cdot)$ in the literature of small ball probability are predominantly Gaussian. Theorem 1.1 represents a rare setting for the non-Gaussian situation. Another similar example of the non-Gaussian is the small deviation for the self-intersection local time (see (1.3) below).

In addition, this work is also motivated by a practical matter arising from the area of stochastic partial differential equations: The integrability

$$\mathbb{E}\left[\int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r\right]^{-p} < \infty \text{ for some } p > 0, \tag{1.2}$$

plays an important rule when studying the regularity of the probability distribution for the solution of 1-dimensional parabolic Anderson equation (driven by a space-white Gaussian noise) in [7]. For possible quantification of the smoothness of the solution density in the future, it might be desirable to find the critical value of p. The following result is a direct consequence of (1.1):

Corollary 1.2 *The negative-moment-integrability (1.2) holds if and only if $p < \frac{2}{3}$.*

The large and small deviations are two research subjects concerning the probabilities of the rare events representing two extreme behaviors of the stochastic system: the random variable takes unusually large values and the random variable takes unusually small values. For the purpose of comparison in terms of large and small deviations, we introduce the self-intersection local time

$$\int_0^t\int_0^t \delta_0(B_s - B_r)\mathrm{d}s\mathrm{d}r, \qquad t > 0.$$

By Theorem 1.1 in [2] ($m = 1$, $p = 2$ for self-intersection, and $m = 2$, $p = 1$ for mutual intersection), we have the large deviations for both self and mutual intersections:

$$\lim_{\varepsilon\to 0^+} \varepsilon^2 \log P\left\{\int_0^1\int_0^1 \delta_0(B_s - B_r)\mathrm{d}s\mathrm{d}r \ge \varepsilon^{-1}\right\} = -\frac{3}{2};$$
$$\lim_{\varepsilon\to 0^+} \varepsilon^2 \log P\left\{\int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \ge \varepsilon^{-1}\right\} = -3.$$

It is noticeable that the large deviations for two different types of intersections take the forms close to each other. The story behind is the following deterministic relation:

$$\begin{aligned}\int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r &= \int_{-\infty}^{\infty} L(1,x)\tilde{L}(1,x)\mathrm{d}x \le \frac{1}{2}\int_{-\infty}^{\infty} L^2(1,x)\mathrm{d}x \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} \tilde{L}^2(1,x)\mathrm{d}x \\ &= \frac{1}{2}\int_0^1\int_0^1 \delta_0(B_s - B_r)\mathrm{d}s\mathrm{d}r + \frac{1}{2}\int_0^1\int_0^1 \delta_0(\tilde{B}_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r\end{aligned}$$

where $L(t, x)$ and $\tilde{L}(t, x)$ are, respectively, the local times of B_t and $\tilde{B}_t$. In the game of large deviations, the Brownian trajectories that make mutual intersection large are those that make $L(1, \cdot)$ and $\tilde{L}(1, \cdot)$ close to each other so that "$\le$" is replaced by "$\approx$" in the above deterministic relation.

The small ball probability for the self-intersection local time [6] takes the form

$$\lim_{\varepsilon\to 0^+} \varepsilon^2 \log P\left\{\int_0^1\int_0^1 \delta_0(B_s - B_r)\mathrm{d}s\mathrm{d}r \le \varepsilon\right\} = -c \tag{1.3}$$

with constant $c > 0$. A striking difference between (1.3) and (1.1) lies in the disparity between exponential decay and power decay. The typical paths that the Brownian motion takes for maximizing self-avoiding are the ones that are close to straight lines, i.e., $B_t \approx c\epsilon^{-1}t$, $t \in [0, 1]$. For the mutual intersection local time, the small deviation means a totally different game: Two independent Brownian paths tend to avoid meeting each other right after they set out from the same starting point.

In the following, we outline the idea in our proof: By the Brownian scaling property and the relation $\delta_0(\lambda x) = \lambda^{-1}\delta_0(x)$ for any $\lambda > 0$, we have

$$\int_0^t\int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \overset{d}{=} t^{3/2}\int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r, \qquad \forall t > 0, \tag{1.4}$$

where "$\overset{d}{=}$" means equality in distribution. Therefore, the proof of (1.1) is reduced to establishing the following bound

$$c\frac{1}{t} \le P\left\{\int_0^t\int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le a\right\} \le C\frac{1}{t} \tag{1.5}$$

for large t and for some fixed constants $c, C, a > 0$ which are independent of t.

In the proof of the lower bound, we separate the Brownian paths $B[1,t]$ and $\tilde{B}[1,t]$ into two disjoint and distanced half-lines in a way that leads to $B[0,t] \cap \tilde{B}[1,t] = \emptyset$ and $B[1,t] \cap \tilde{B}[0,t] = \emptyset$. Consequently, the only unavoidable intersections are the ones happening on the time square $[0, 1]^2$.

In other words, the strategy lies in the comparison between the probability of "low intersection" and the probability of "no intersection". Indeed, the steps taken in (2.2) below morally suggests the relation

$$P\{\text{low intersection on } [0,t]^2\} \geq cP\{\text{no intersection on } [1,t]^2\}. \qquad (1.6)$$

Since the probability on the right-hand side is not hard to compute (only for 1-dimensional Brownian motions, of course), the proof of the lower bound (see Sect. 2) is the easier one, in comparison with the proof of the upper bound (see Sect. 3). As for the upper bound, the real challenge is to fill the gap between "low intersection" and "no intersection" suggested by (1.6).

We end the discussion with a remark on the possible multi-dimensional extensions in the future. It is well-known that the mutual intersection local time between two independent d-dimensional Brownian motions exists for $d = 1, 2, 3$. Therefore, it makes sense to raise the same question in these dimensions. In view of the heuristic comparison (1.6),[1] it brings the famous theorem of intersection exponent by Lawler, Schramm and Werner (Theorem 1.1, [4]) in which it claims that when $d = 2$, the right-hand side of (1.6) is of the order $t^{-(5/8)+o(1)}$ as $t \to \infty$. When $d = 2$, noting that

$$\int_0^t \int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \overset{d}{=} t \int_0^1 \int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r, \qquad \forall t > 0,$$

we conjecture that

$$P\left\{\int_0^1 \int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \leq \varepsilon\right\} = \varepsilon^{(5/8)+o(1)}, \qquad (\varepsilon \to 0^+). \qquad (1.7)$$

2 Lower Bound

In this section, all we need is to prove the lower bound in (1.5) with $a = 1$, i.e.,

$$P\left\{\int_0^t \int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \leq 1\right\} \geq \frac{c}{t} \qquad (2.1)$$

[1] We believe it remains morally true in multi-dimensional settings.

for large t. First, we claim that for $t > 1$,

$$\begin{aligned} &P\left\{ \int_0^t \int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le 1 \right\} \\ &\ge P\left\{ \int_0^1 \int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le 1, \ \min_{0\le s\le 1} B_s \ge -1, \ \max_{0\le s\le 1} \tilde{B}_s \le 1, \right. \\ &\left. \min_{1\le s\le t} B_s > 1, \ \max_{1\le s\le t} \tilde{B}_s < -1 \right\} \\ &= \mathbb{E}\left[\mathbf{1}_A \cdot P_{B_1}\left\{ \min_{0\le s\le t-1} B_s > 1 \right\} P_{\tilde{B}_1}\left\{ \max_{0\le s\le t-1} \tilde{B}_s < -1 \right\} \right], \end{aligned} \tag{2.2}$$

where

$$A = \left\{ \int_0^1 \int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le 1, \ \min_{0\le s\le 1} B_s \ge -1, \ \max_{0\le s\le 1} \tilde{B}_s \le 1 \right\},$$

and the last step follows from the Markov property and the independence between B and $\tilde{B}$.

Indeed, the first step is justified by the fact that on the event

$$\left\{ \min_{0\le s\le 1} B_s \ge -1, \ \max_{0\le s\le 1} \tilde{B}_s \le 1, \ \min_{1\le s\le t} B_s > 1, \ \max_{1\le s\le t} \tilde{B}_s < -1 \right\},$$

$B[0,t] \cap \tilde{B}[1,t] = \emptyset$ and $B[1,t] \cap \tilde{B}[0,t] = \emptyset$ which leads to

$$\int_0^t \int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r = \int_0^1 \int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r.$$

On $\{B_1 \ge 2\}$, by (2.5) in Lemma 2.1 we have

$$\begin{aligned} P_{B_1}\left\{ \min_{0\le s\le t-1} B_s > 1 \right\} &\ge P\left\{ 1 + \min_{0\le s\le t-1} B_s > 0 \right\} \\ &= P\left\{ |B_{t-1}| \le 1 \right\} \sim \frac{1}{\sqrt{2\pi t}} \qquad (t \to \infty), \end{aligned} \tag{2.3}$$

and similarly, using (2.6) in Lemma 2.1 we conclude that on $\{\tilde{B}_1 \le -2\}$,

$$P_{\tilde{B}_1}\left\{ \max_{0\le s\le t-1} \tilde{B}_s < -1 \right\} \ge P\{|B_{t-1}| \le 1\} \sim \frac{1}{\sqrt{2\pi t}} \qquad (t \to \infty). \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we get that when t is sufficiently large, there exists a constant $c > 0$ such that

$$P\left\{\int_0^t\int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le 1\right\}$$
$$\ge \frac{c}{t} P\left\{\int_0^1\int_0^1 \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le 1,\ \min_{0\le s\le 1} B_s \ge -1,\ B_1 \ge 2,\right.$$
$$\left.\max_{0\le s\le 1} \tilde{B}_s \le 1,\ \tilde{B}_1 \le -2\right\},$$

where the probability on the right-hand side is clearly positive. The desired lower bound (2.1) is proved.

Lemma 2.1 *For any* $x > 0$ *and* $t > 0$*, we have*

$$P\{x + B_s > 0;\ \forall s \le t\} = P\{|B_t| \le x\}, \tag{2.5}$$

and

$$P\{-x + B_s < 0;\ \forall s \le t\} = P\{|B_t| \le x\}. \tag{2.6}$$

Proof Denote $\chi_x \overset{\Delta}{=} \inf\{s \ge 0;\ \ B_s = x\}$. Then,

$$P\{x + B_s > 0;\ \forall s \le t\} = P\{\chi_{-x} \ge t\} = P\{\chi_x \ge t\}$$
$$= P\left\{\max_{0\le s\le t} B_s \le x\right\} = P\{|B_t| \le x\},$$

where the last equality follows from the reflection principle $\max_{0\le s\le t} B_s \overset{d}{=} |B_t|$. This yields (2.5). Equation (2.6) can be obtained by replacing B by $-B$ in (2.5).

3 Upper Bound

In this section, we prove the upper bound of (1.5) which can be slightly rephrased as

$$P\left\{\int_0^t\int_0^t \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \le a\right\} = O\left(\frac{1}{t}\right) \quad (t \to \infty) \tag{3.1}$$

for some constant $a > 0$. We shall specify the constant $a > 0$ later.

As we mentioned earlier, the central piece of the proof is to compare the "low intersection" probability in (3.1) with the probability of "no intersection" in a direction opposite to the one in (1.6). More precisely, we compare the probability of low intersection for B and $\tilde{B}$ with the probability of no intersection for two independent simple random walks. We refer to (3.25) and (3.31) below for the implementation of this idea.

To this end, we generate two independent simple random walks $\{S_k, k \geq 0\}$ and $\{\tilde{S}_k, k \geq 0\}$ by Brownian motions in a way that the trajectory $\{S_k;\ 0 \leq k \leq n\}$ is intimately close to the Brownian path $\{B_s;\ 0 \leq t \leq \tau_n\}$ up to the stopping time τ_n that is given below (same thing is expected between $\tilde{S}_k$ and $\tilde{B}_t$):

We first define a sequence of stopping times $\{\tau_k\}_{k\geq 0}$ related to the Brownian motion B_t:

$$\tau_0 = 0;\ \ \tau_k = \inf\{t \geq \tau_{k-1};\ \ |B_t - B_{\tau_{k-1}}| = 1\} \text{ for } k = 1, 2, \dots \tag{3.2}$$

Then $\{S_k = B_{\tau_k}, k \geq 0\})$ is a simple random walk with $S_0 = B_0$. The stopping times $\{\tilde{\tau}_k, k \geq 0\}$ and the simple random walk $\{\tilde{S}_k, k \geq 0\}$ are generated from $\{\tilde{B}_t, t \geq 0\}$ in the same way.

Clearly, $\{\tau_k - \tau_{k-1}\}_{k\geq 1}$ is a sequence of i.i.d. random variables with the same distribution as τ_1. Here we point out that the distribution of τ_1 does not depend on the starting point of B.[2] The exact distribution of τ_1 is known (p. 342, [3]). An instructive link [5] to the integrability of τ_1 is given as follows: By the definition of τ_1,

$$\begin{aligned} P\{\tau_1 \geq t\} &= P\Big\{\max_{s\leq t} |B_s| \leq 1\Big\} = P\Big\{\max_{s\leq 1} |B_s| \leq \frac{1}{\sqrt{t}}\Big\} \\ &= \exp\Big\{-\Big(\frac{\pi^2}{8} + o(1)\Big)t\Big\} \quad (t \to \infty), \end{aligned}$$

where the last step is the classic result on the small ball probability for Brownian motions (see, e.g., (1.3), [5]). In particular, $\mathbb{E}e^{\theta\tau_1} < \infty$ for $\theta < \pi^2/8$. By a standard application of Chebyshev's inequality to the sum of independent random variables, we get

$$P\Big\{|\tau_n - n\mathbb{E}\tau_1| \geq n\delta\Big\} \leq \frac{1}{n\delta^2}\mathrm{Var}(\tau_1). \tag{3.3}$$

The mutual intersection local time Q_n of the random walks S and $\tilde{S}$ is given by

$$Q_n \triangleq \sum_{j,k=1}^{n} \mathbf{1}_{\{S_j = \tilde{S}_k\}}. \tag{3.4}$$

For $n \in \mathbb{N}$,

$$\begin{aligned} \int_0^{\tau_{n+1}}\int_0^{\tilde{\tau}_{n+1}} \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r &= \sum_{j,k=0}^{n} \int_{\tau_j}^{\tau_{j+1}}\int_{\tilde{\tau}_k}^{\tilde{\tau}_{k+1}} \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \\ &\geq \sum_{j,k=1}^{n} \int_{\tau_j}^{\tau_{j+1}}\int_{\tilde{\tau}_k}^{\tilde{\tau}_{k+1}} \delta_0(B_s - \tilde{B}_r)\mathrm{d}s\mathrm{d}r \end{aligned}$$

[2] We mention this as we allow $B_0 \neq 0$ and $\tilde{B}_0 \neq 0$ in later discussion.

$$\geq \sum_{j,k=1}^{n} \mathbf{1}_{\{S_j=\tilde{S}_k\}} \int_{\tau_j}^{\tau_{j+1}} \int_{\tilde{\tau}_k}^{\tilde{\tau}_{k+1}} \delta_0(B_s - \tilde{B}_r) \mathrm{d}s \mathrm{d}r$$
$$= \sum_{j,k=1}^{n} \mathbf{1}_{\{S_j=\tilde{S}_k\}} \xi_{j,k} = H_n \qquad \text{(say)}, \tag{3.5}$$

where we denote

$$\xi_{j,k} \triangleq \int_{\tau_j}^{\tau_{j+1}} \int_{\tilde{\tau}_k}^{\tilde{\tau}_{k+1}} \delta_0(B_s - \tilde{B}_r) \mathrm{d}s \mathrm{d}r \quad \text{for } j, k = 1, 2, \ldots$$

For a technical reason, we allow the Brownian motions to start from somewhere different from 0 and use $P_{(x,\tilde{x})}(\cdot)$ for the probability with $B_0 = x$ and $\tilde{B}_0 = \tilde{x}$. We follow the convention $P(\cdot) = P_{(0,0)}(\cdot)$. Recall the standard notation used in the textbook on Markov process:

$$P_\mu(\cdot) = \int_{\mathbb{R}^2} \mu(\mathrm{d}x, \mathrm{d}\tilde{x}) P_{(x,\tilde{x})}(\cdot)$$

for any probability measure μ on $\mathbb{R}^2$. In the following discussion, we set

$$\mu = \frac{1}{4}\Big\{\delta_{(-1,-1)} + \delta_{(1,1)} + \delta_{(-1,1)} + \delta_{(1,-1)}\Big\}.$$

Despite the fact that (0, 0) is not listed as a starting point of the 2-dimensional Brownian motion $(B_t, \tilde{B}_t)$ under the distribution μ, noting that

$$\begin{cases} P_{(-1,-1)}\{H_n \leq a\} = P_{(1,1)}\{H_n \leq a\} = P\{H_n \leq a\}, \\ P_{(-1,1)}\{H_n \leq a\} = P_{(1,-1)}\{H_n \leq a\} \qquad n = 1, 2, \ldots, \end{cases} \tag{3.6}$$

we have

$$P_\mu\{H_n \leq a\} = \frac{1}{2} P\{H_n \leq a\} + \frac{1}{2} P_{(1,-1)}\{H_n \leq a\}. \tag{3.7}$$

Here, we recall the notation from (3.5),

$$H_n = \sum_{j,k=1}^{n} \mathbf{1}_{\{S_j=\tilde{S}_k\}} \xi_{j,k}.$$

The proof of (3.1) can be reduced to establishing the bound

$$P_\mu\{H_n \leq a\} = O\Big(\frac{1}{n}\Big) \qquad (n \to \infty). \tag{3.8}$$

Indeed, by (3.7) we have $P\{H_n \le a\} \le 2P_\mu\{H_n \le a\}$. Therefore, (3.5) leads to

$$P\left\{\int_0^{\tau_{n+1}}\int_0^{\tilde\tau_{n+1}} \delta_0(B_s-\tilde B_r)\mathrm{d}s\mathrm{d}r \le a\right\} = O\Big(\frac{1}{n}\Big) \quad (n\to\infty).$$

Consequently, for any $0<\delta<1$,

$$\begin{aligned}
&P\left\{\int_0^t\int_0^t \delta_0(B_s-\tilde B_r)\mathrm{d}s\mathrm{d}r \le a\right\}\\
&\le P\left\{\int_0^{\tau_{[(1-\delta)t/\mathbb{E}\tau_1]+1}}\int_0^{\tilde\tau_{[(1-\delta)t/\mathbb{E}\tau_1]+1}} \delta_0(B_s-\tilde B_r)\mathrm{d}s\mathrm{d}r \le a\right\}\\
&\quad + 2P\left\{\tau_{[(1-\delta)t/\mathbb{E}\tau_1]+1} \ge t\right\}\\
&\le O\Big(\frac{1}{t}\Big) + \frac{C}{t} = O\Big(\frac{1}{t}\Big) \quad (t\to\infty),
\end{aligned}$$

where the second inequality partially follows from (3.3).

The remaining of the section is devoted to the proof of (3.8). Set the random set on $\mathbb{Z}_+^2$:

$$\Lambda(A) \triangleq \{(j,k)\in A;\ \ S_j = \tilde S_k\}, \qquad A\subset \mathbb{Z}_+^2.$$

Notice that $\#(\Lambda([1,n]^2) = Q_n$ where Q_n is the number of intersections given in (3.4). We introduce the following stopping time

$$\sigma \triangleq \min\left\{n\ge 1;\ \ \Lambda([1,n]^2)\ne\emptyset\right\} = \min\left\{n\ge 1;\, Q_n>0\right\}. \tag{3.9}$$

By the law of total probability, we have

$$\begin{aligned}
P_\mu\{H_n\le a\} &= \sum_{l=1}^n P_\mu\{\sigma=l,\ \ H_n\le a\} + P_\mu\{\sigma>n\}\\
&= \sum_{l=1}^n P_\mu\{\sigma=l,\ \ H_n\le a\} + P_\mu\{Q_n=0\}.
\end{aligned} \tag{3.10}$$

On the event $\{\sigma=l\}$, the intersection $\{S_j=\tilde S_k\}$ happens either on $[1,l]\times\{l\}$ or on $\{l\}\times[1,l]$, or on both. An intersection $\{S_j=\tilde S_k\}$ is called an early intersection, if there is no $(j_1,k_1)\ne(j,k)$ such that $j_1\le j, k_1\le k$, and $S_{j_1}=\tilde S_{k_1}$. In general, there may be multiple early intersections due to the multi-dimension of time. On $\{\sigma=l\}$, early intersections happen on $[1,l]\times\{l\}$ or on $\{l\}\times[1,l]$, or on both, i.e.,

$$\begin{aligned}
\{\sigma=l\} = &\left\{\Lambda([0,l-1]^2)=\emptyset,\ \text{early intersection happens on } [1,l]\times\{l\}\right\}\\
&\bigcup\left\{\Lambda([0,l-1]^2)=\emptyset,\ \text{early intersection happens on } \{l\}\times[1,l]\right\}.
\end{aligned}$$

Therefore, for $l = 1, \dots, n$,

$$\begin{aligned} &P_\mu\{\sigma = l,\ \ H_n \le a\} \\ &\quad \le P_\mu\left\{\Lambda([0, l-1]^2) = \emptyset,\ \text{ early intersection happens on } [1, l] \times \{l\},\ \ H_n \le a\right\} \\ &\qquad + P_\mu\left\{\Lambda([0, l-1]^2) = \emptyset,\ \text{ early intersection happens on } \{l\} \times [1, l],\ \ H_n \le a\right\} \\ &\quad = 2P_\mu\left\{\Lambda([0, l-1]^2) = \emptyset,\ \text{ early intersection happens on } [1, l] \times \{l\},\ \ H_n \le a\right\}. \end{aligned} \tag{3.11}$$

On the event $\left\{\Lambda([0, l-1]^2) = \emptyset,\ \text{ early intersection happens on } [1, l] \times \{l\}\right\}$, there is a unique early intersection on $[1, l] \times \{l\}$ happening at (ρ, l) for some $1 \le \rho \le l$, described by the event

$$F_{\rho,l} \triangleq \left\{\Lambda\Big(\big([1, \rho] \times [1, l]\big) \setminus (\rho, l)\Big) = \emptyset,\ \ S_\rho = \tilde{S}_l\right\}, \qquad \rho = 1, \dots, l. \tag{3.12}$$

Thus, we have

$$\left\{\Lambda([0, l-1]^2) = \emptyset,\ \text{ early intersection happens on } [1, l] \times \{l\}\right\} \subset \bigcup_{\rho=1}^{l} F_{\rho,l}.$$

Here we mention that, when $\rho = l$, the fact that "$S_l = \tilde{S}_l$" is an early intersection requires, in addition to "$S_1 \ne \tilde{S}_l, \dots, S_{l-1} \ne \tilde{S}_l$", that "$S_l \ne \tilde{S}_1, \dots, S_l \ne \tilde{S}_{l-1}$" (and therefore $\Lambda\left(([1, l] \times [1, l]) \setminus (l, l)\right) = \emptyset$). Thus,

$$\begin{aligned} &P_\mu\left\{\Lambda([1, l-1]^2) = \emptyset,\ \text{ early intersection happens on } [1, l] \times \{l\},\ \ H_n \le a\right\} \\ &\quad \le \sum_{\rho=1}^{l} P_\mu\big\{F_{\rho,l} \cap \{H_n \le a\}\big\} = \sum_{\rho=1}^{l} P_\mu\big\{F_{\rho,l} \cap \{\xi_{\rho,l} \le a\} \cap \{H_n \le a\}\big\}. \end{aligned}$$

Combining this with (3.11) yields

$$P_\mu\{\sigma = l,\ \ H_n \le a\} \le 2\sum_{\rho=1}^{l} P_\mu\big\{F_{\rho,l} \cap \{\xi_{\rho,l} \le a\} \cap \{H_n \le a\}\big\}, \qquad l = 1, \dots, n. \tag{3.13}$$

When $1 \le l \le n-2$, we have on $F_{\rho,l}$,

$$
\begin{aligned}
H_n &\geq \sum_{j=\rho+2}^{n} \sum_{k=l+2}^{n} \mathbf{1}_{\{S_j=\tilde{S}_k\}} \xi_{j,k} \\
&= \sum_{j=1}^{n-\rho-1} \sum_{k=1}^{n-l-1} \mathbf{1}_{\{(S_{\rho+1+j}-S_{\rho+1})+(S_{\rho+1}-S_\rho)=(\tilde{S}_{l+1+k}-\tilde{S}_{l+1})+(\tilde{S}_{l+1}-\tilde{S}_l)\}} \xi_{\rho+1+j,l+1+k} \\
&\geq \sum_{j,k=1}^{n-l-1} \mathbf{1}_{\{(S_{\rho+1+j}-S_{\rho+1})+(S_{\rho+1}-S_\rho)=(\tilde{S}_{l+1+k}-\tilde{S}_{l+1})+(\tilde{S}_{l+1}-\tilde{S}_l)\}} \xi_{\rho+1+j,l+1+k} \\
&= \bar{H}_{n-l-1} \quad \text{(say)},
\end{aligned} \tag{3.14}
$$

where the equality is due to $S_\rho = \tilde{S}_l$ on $F_{\rho,l}$. Note that

$$
\begin{aligned}
\xi_{\rho+1+j,l+1+k} &= \int_{\tau_{\rho+1+j}}^{\tau_{\rho+2+j}} \int_{\tilde{\tau}_{l+1+k}}^{\tilde{\tau}_{l+2+k}} \delta_0(B_s - \tilde{B}_r) \mathrm{d}s \mathrm{d}r \\
&= \int_{\tau_{\rho+1+j}-\tau_{\rho+1}}^{\tau_{\rho+2+j}-\tau_{\rho+1}} \int_{\tilde{\tau}_{l+1+k}-\tilde{\tau}_{l+1}}^{\tilde{\tau}_{l+2+k}-\tilde{\tau}_{l+1}} \delta_0\Big(B_{\tau_{\rho+1}+s} - \tilde{B}_{\tilde{\tau}_{l+1}+r}\Big) \mathrm{d}s \mathrm{d}r \\
&= \int_{\tau_{\rho+1+j}-\tau_{\rho+1}}^{\tau_{\rho+2+j}-\tau_{\rho+1}} \int_{\tilde{\tau}_{l+1+k}-\tilde{\tau}_{l+1}}^{\tilde{\tau}_{l+2+k}-\tilde{\tau}_{l+1}} \delta_0\Big(\big((B_{\tau_{\rho+1}+s} - B_{\tau_{\rho+1}}) + (S_{\rho+1} - S_\rho)\big) \\
&\quad - \big((\tilde{B}_{\tilde{\tau}_{l+1}+r} - \tilde{B}_{\tilde{\tau}_{l+1}}) + (\tilde{S}_{l+1} - \tilde{S}_l)\big)\Big) \mathrm{d}s \mathrm{d}r.
\end{aligned}
$$

Thus, given $(S_{\rho+1}-S_\rho, \tilde{S}_{l+1}-\tilde{S}_l) = (z_1, z_2)$, the distribution of $\bar{H}_{n-l-1}$ is the same as that of H_{n-l-1} under $P_{(z_1,z_2)}$, and is independent of $\{(B_s, \tilde{B}_r);\ 0 \leq s \leq \tau_{\rho+1}$ and $0 \leq r \leq \tilde{\tau}_{l+1}\}$, and is therefore independent of $F_{\rho,l}$ and $\xi_{\rho,l}$. By (3.14) and the Markov property

$$
\begin{aligned}
&P_\mu \left\{F_{\rho,l} \cap \{\xi_{\rho,l} \leq a\} \cap \{H_n \leq a\}\right\} \leq P_\mu \left\{F_{\rho,l} \cap \{\xi_{\rho,l} \leq a\} \cap \{\bar{H}_{n-l-1} \leq a\}\right\} \\
&\quad = \mathbb{E}_\mu \left[\mathbf{1}_{F_{\rho,l}} \mathbf{1}_{\{\xi_{\rho,l} \leq a\}} P_{(S_{\rho+1}-S_\rho, \tilde{S}_{l+1}-\tilde{S}_l)} \{H_{n-l-1} \leq a\}\right].
\end{aligned} \tag{3.15}
$$

By the fact that $S_{\rho+1} - S_\rho = \pm 1$ and $\tilde{S}_{l+1} - \tilde{S}_l = \pm 1$, and by (3.6), the right-hand side of (3.15) is bounded from above by

$$
\begin{aligned}
&P_{(1,1)}\{H_{n-l-1} \leq a\} \mathbb{E}_\mu \left[\mathbf{1}_{F_{\rho,l}} \mathbf{1}_{\{\xi_{\rho,l} \leq a\}} \mathbf{1}_{\{S_{\rho+1}-S_\rho=\tilde{S}_{l+1}-\tilde{S}_l)}\right] \\
&\qquad + P_{(1,-1)}\{H_{n-l-1} \leq a\} \mathbb{E}_\mu \left[\mathbf{1}_{F_{\rho,l}} \mathbf{1}_{\{\xi_{\rho,l} \leq a\}} \mathbf{1}_{\{S_{\rho+1}-S_\rho \neq \tilde{S}_{l+1}-\tilde{S}_l)}\right] \\
&\quad \leq \left(P_{(1,1)}\{H_{n-l-1} \leq a\} + P_{(1,-1)}\{H_{n-l-1} \leq a\}\right) P_\mu \left\{F_{\rho,l} \cap \{\xi_{\rho,l} \leq a\}\right\} \\
&\quad = 2 P_\mu\{H_{n-l-1} \leq a\} P_\mu \left\{F_{\rho,l} \cap \{\xi_{\rho,l} \leq a\}\right\}.
\end{aligned} \tag{3.16}
$$

On $F_{\rho,l}$,

$$\xi_{\rho,l} = \int_0^{\tau_{\rho+1}-\tau_\rho} \int_0^{\tilde{\tau}_{l+1}-\tilde{\tau}_l} \delta_0\Big((B_{\tau_\rho+s} - B_{\tau_\rho}) - (\tilde{B}_{\tilde{\tau}_l+r} - \tilde{B}_{\tilde{\tau}_l})\Big)\mathrm{d}s\mathrm{d}r$$

and the right-hand side has the same distribution as $\xi_{1,1}$ (under the law $P(\cdot)$), and is independent of $\big\{(B_s, \tilde{B}_r);\ \ s \le \tau_\rho \ \text{ and } \ r \le \tilde{\tau}_l\big\}$, and therefore of $F_{\rho,l}$. Hence,

$$P_\mu\left\{F_{\rho,l} \cap \{\xi_{\rho,l} \le a\}\right\} = P_\mu\{F_{\rho,l}\}P\{\xi_{1,1} \le a\}. \tag{3.17}$$

In summary, combining (3.13), (3.15), (3.16) and (3.17), we get for $1 \le l \le n-2$,

$$P_\mu\{\sigma = l,\ \ H_n \le a\} \le 4P\{\xi_{1,1} \le a\}\bigg(\sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\}\bigg)P_\mu\{H_{n-l-1} \le a\}. \tag{3.18}$$

When $l \in \{n-1, n\}$, by (3.13) and (3.17) we have the bound

$$P_\mu\{\sigma = l,\ \ H_n \le a\} \le 2P\{\xi_{1,1} \le a\}\bigg(\sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\}\bigg). \tag{3.19}$$

Now, combining (3.10), (3.18), and (3.19), we get for $n \ge 3$,

$$\begin{aligned} &P_\mu\{H_n \le a\} \\ &\quad\le 4P\{\xi_{1,1} \le a\}\sum_{l=1}^{n-2}\bigg(\sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\}\bigg)P_\mu\{H_{n-l-1} \le a\} \\ &\qquad + 2P\{\xi_{1,1} \le a\}\bigg(\sum_{l\in\{n-1,n\}}\sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\}\bigg) + P_\mu\{Q_n = 0\}. \end{aligned} \tag{3.20}$$

Therefore, for any $\theta \in (0, 1)$,

$$\begin{aligned} &\sum_{n=3}^{\infty} \theta^n P_\mu\{H_n \le a\} \\ &\quad\le \sum_{n=3}^{\infty} \theta^n P_\mu\{Q_n = 0\} \\ &\qquad +4P\{\xi_{1,1} \le a\}\sum_{n=3}^{\infty}\theta^n\sum_{l=1}^{n-2}\bigg(\sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\}\bigg)P_\mu\{H_{n-l-1} \le a\} \\ &\qquad +2P\{\xi_{1,1} \le a\}\sum_{n=3}^{\infty}\theta^n\bigg(\sum_{\rho=1}^{n-1} P_\mu\{F_{\rho,n-1}\}\bigg) \end{aligned}$$

$$+2P\{\xi_{1,1} \le a\} \sum_{n=3}^{\infty} \theta^n \Big(\sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \Big). \tag{3.21}$$

Notice that

$$\begin{aligned} &\sum_{n=3}^{\infty} \theta^n \sum_{l=1}^{n-2} \Big(\sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\} \Big) P_\mu\{H_{n-l-1} \le a\} \\ &= \theta \sum_{n=2}^{\infty} \sum_{l=1}^{n-1} \Big(\theta^l \sum_{\rho=1}^{l} P_\mu\{F_{\rho,l}\} \Big) \theta^{n-l} P_\mu\{H_{n-l} \le a\} \\ &= \theta \Big(\sum_{n=1}^{\infty} \theta^n \sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \Big) \Big(\sum_{n=1}^{\infty} \theta^n P_\mu\{H_n \le a\} \Big). \end{aligned}$$

Plugging this into (3.21) and noting $\theta \in (0, 1)$, we get

$$\begin{aligned} \sum_{n=3}^{\infty} \theta^n P_\mu\{H_n \le a\} &\le \sum_{n=3}^{\infty} \theta^n P_\mu\{Q_n = 0\} + 12 \sum_{n=1}^{\infty} \sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \\ &+ 4P\{\xi_{1,1} \le a\} \Big(\sum_{n=1}^{\infty} \sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \Big) \sum_{n=3}^{\infty} \theta^n P_\mu\{H_n \le a\}. \end{aligned} \tag{3.22}$$

We now install the summability of $\{P(F_{\rho,l});\ 1 \le \rho \le l < \infty\}$. Let (ρ, l) be fixed for a while. We start by tracking the possible site $z \in \mathbb{Z}$ where the early intersection "$S_\rho = S_l = z$" (given in the definition of $F_{\rho,l}$) occurs. Given that $S_0 = \pm 1$ and $\tilde{S}_0 = \pm 1$ under P_μ, we must have that $z \in [-2, 2]$. Hence,

$$P_\mu(F_{\rho,l}) = \sum_{z \in [-2,2]} P_\mu(F_{\rho,l} \cap \{S_\rho = \tilde{S}_l = z\}).$$

Define the hitting times

$$T_z \triangleq \min\{n \ge 1;\ S_n = z\} \quad \text{and} \quad \tilde{T}_z \triangleq \min\{n \ge 1;\ \tilde{S}_n = z\}.$$

By the fact that "$S_\rho = S_l$" is an early intersection, we have

$$P_\mu(F_{\rho,l} \cap \{S_\rho = \tilde{S}_l = z\}) \le P_\mu\{T_z = \rho,\ \tilde{T}_z = l\} = P_\mu\{T_z = \rho\} P_\mu\{T_z = l\},$$

where the last step follows from the independence between S and $\tilde{S}$. In summary,

$$P_\mu(F_{\rho,l}) \le \sum_{z \in [-2,2]} P_\mu\{T_z = \rho\} P_\mu\{T_z = l\}, \qquad (1 \le \rho \le l < \infty). \tag{3.23}$$

Consequently,

$$\sum_{l=1}^{\infty}\sum_{\rho=1}^{l} P_\mu(F_{\rho,l}) \le \sum_{z\in[-2,2]}\sum_{l=1}^{\infty}\sum_{\rho=1}^{l} P_\mu\{T_z=\rho\}P_\mu\{T_z=l\} \le \sum_{z\in[-2,2]} 1 = 5. \tag{3.24}$$

Choose $a>0$ sufficiently small, so that

$$P\{\xi_{1,1}\le a\} < \frac{1}{20}.$$

By (3.22) and (3.24),

$$\sum_{n=3}^{\infty}\theta^n P_\mu\{H_n\le a\} \le \Big(1-20P\{\xi_{1,1}\le a\}\Big)^{-1}\Big\{60+\sum_{n=3}^{\infty}\theta^n P_\mu\{Q_n=0\}\Big\}. \tag{3.25}$$

Lemma 3.1 *There is a constant $C>0$ such that*

$$P_\mu\{Q_n=0\} \le \frac{C}{n}, \qquad n=1,2,\ldots, \tag{3.26}$$

and

$$\max_{z\in[-2,2]} P_\mu\{T_z\ge n\} \le \frac{C}{\sqrt{n}}, \qquad n=1,2,\ldots. \tag{3.27}$$

Proof Under P_μ, $S_0=\pm1$ and $\tilde S_0=\pm1$. When $S_0\neq\tilde S_0$,

$$\{Q_n=0\}\subset\Big\{\min_{1\le k\le n} S_k\ge -S_0,\ \max_{1\le k\le n}\tilde S_k\le-\tilde S_0\Big\}$$
$$\bigcup\Big\{\min_{1\le k\le n}\tilde S_k\ge-\tilde S_0,\ \max_{1\le k\le n} S_k\le -S_0\Big\}.$$

When $S_0=\tilde S_0$,

$$\{Q_n=0\}\subset\Big\{\min_{1\le k\le n} S_k\ge S_0,\ \max_{1\le k\le n}\tilde S_k\le\tilde S_0\Big\}\bigcup\Big\{\min_{1\le k\le n}\tilde S_k\ge\tilde S_0,\ \max_{1\le k\le n} S_k\le S_0\Big\}.$$

By sub-additivity, symmetry and independence, we get

$$\begin{aligned}
&P_\mu\{Q_n=0\}\\
&\le P_1\Big\{\min_{1\le k\le n} S_k\ge-1\Big\}P_{-1}\Big\{\max_{1\le k\le n} S_k\le 1\Big\}+P_1\Big\{\min_{1\le k\le n} S_k\ge 1\Big\}P_1\Big\{\max_{1\le k\le n} S_k\le 1\Big\}\\
&=\Big(P\Big\{\min_{1\le k\le n} S_k\ge-2\Big\}\Big)^2+\Big(P\Big\{\min_{1\le k\le n} S_k\ge 0\Big\}\Big)^2\le 2\Big(P\Big\{\min_{1\le k\le n} S_k\ge-2\Big\}\Big)^2.
\end{aligned}$$

As for the hitting time, we have

$$
\begin{aligned}
P_\mu\{T_z \ge n\} &= \frac{1}{2}P\{T_{z-1} \ge n\} + \frac{1}{2}P\{T_{z+1} \ge n\} \\
&= \frac{1}{2}P\{T_{-|z-1|} \ge n\} + \frac{1}{2}P\{T_{-|z+1|} \ge n\} \\
&\le \frac{1}{2}P\Big\{\min_{1\le k\le n} S_k \ge -|z-1|\Big\} + \frac{1}{2}P\Big\{\min_{1\le k\le n} S_k \ge -|z+1|\Big\},
\end{aligned}
$$

where the second equality follows from symmetry.

Therefore, both (3.26) and (3.27) are reduced to the proof for the bound of the form

$$
P\Big\{\min_{1\le k\le n} S_k \ge -z\Big\} \le \frac{C_z}{\sqrt{n}}.
$$

The bound like this looks classic and must exist somewhere in literature. For the reader's convenience, we give a short proof here. Since the probability on the left is non-decreasing in z, we may assume that $z \ge 2$. By our construction of the simple random walk, $|B_t - S_k| \le 1$ for $\tau_k \le t \le \tau_{k+1}$. Therefore,

$$
\begin{aligned}
P\Big\{\min_{1\le k\le n} S_k \ge -z\Big\} &\le P\Big\{\min_{1\le s\le \tau_n} B_s \ge -(z-1)\Big\} \\
&\le P\Big\{\min_{1\le s\le (1-\delta)n\mathbb{E}\tau_1} B_s \ge -(z-1)\Big\} + P\{\tau_n \le (1-\delta)n\mathbb{E}\tau_1\}, \\
&\le P\Big\{|B_{(1-\delta)n\mathbb{E}\tau_1}| \le z-1\Big\} + \frac{1}{n\delta^2}\mathbb{E}\tau_1^2 \le \frac{C_z}{\sqrt{n}}
\end{aligned}
$$

where the third step follows from Lemma 2.1 and the inequality (3.3). □

Bringing (3.26) to the bound (3.25) we have

$$
\sum_{n=3}^{\infty} \theta^n P_\mu\{H_n \le a\} \le C\Big\{1 + \sum_{n=1}^{\infty}\frac{\theta^n}{n}\Big\} = C\Big\{1 + \log\frac{1}{1-\theta}\Big\}.
$$

Therefore (with possibly different $C > 0$),

$$
n\theta^n P_\mu\{H_n \le a\} \le \sum_{k=1}^{n} \theta^k P\{H_k \le k\} \le C\log\frac{1}{1-\theta} \qquad (\theta \to 1^-).
$$

Taking $\theta = 1 - \frac{1}{n}$ for large n, we have

$$
n\Big(1 - \frac{1}{n}\Big)^n P_\mu\{H_n \le a\} \le C\log n.
$$

So we have the bound

$$
P_\mu\{H_n \le a\} = O\Big(\frac{\log n}{n}\Big) \qquad (n \to \infty). \tag{3.28}
$$

This bound is weaker than the desired estimate (3.8) and therefore needs to be strengthened.

Under the re-index $l \mapsto n-l-1$ in the second term on the right-hand side, (3.20) becomes

$$P_\mu\{H_n \le a\} \le P_\mu\{Q_n = 0\} + 4P\{\xi_{1,1} \le a\} \sum_{l=1}^{n-2} \left(\sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \right) P_\mu\{H_l \le a\}$$
$$+ 2P\{\xi_{1,1} \le a\} \left(\sum_{\rho=1}^{n-1} P_\mu\{F_{\rho,n-1}\} \right) + 2P\{\xi_{1,1} \le a\} \left(\sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \right).$$

For large integer k, therefore,

$$\begin{aligned}
\sum_{n=2^k+1}^{2^{k+1}} P_\mu\{H_n \le a\} &\le \sum_{n=2^k+1}^{2^{k+1}} P_\mu\{Q_n = 0\} + 2P\{\xi_{1,1} \le a\} \left(\sum_{n=2^k+1}^{2^{k+1}} \sum_{\rho=1}^{n-1} P_\mu\{F_{\rho,n-1}\} \right) \\
&\quad + 2P\{\xi_{1,1} \le a\} \left(\sum_{n=2^k+1}^{2^{k+1}} \sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \right) \\
&\quad + 4P\{\xi_{1,1} \le a\} \sum_{n=2^k+1}^{2^{k+1}} \sum_{l=1}^{n-2} \left(\sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \right) P_\mu\{H_l \le a\} \\
&\le 20 + \sum_{n=2^k+1}^{2^{k+1}} P_\mu\{Q_n = 0\} \\
&\quad + 4P\{\xi_{1,1} \le a\} \sum_{n=2^k+1}^{2^{k+1}} \sum_{l=1}^{n-2} \left(\sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \right) P_\mu\{H_l \le a\},
\end{aligned} \tag{3.29}$$

where the last step follows from (3.24).

For the summation in the last term on the right-hand side,

$$\begin{aligned}
&\sum_{n=2^k+1}^{2^{k+1}} \sum_{l=1}^{n-2} \left(\sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \right) P_\mu\{H_l \le a\} \\
&\le \sum_{l=1}^{2^{k+1}} P_\mu\{H_l \le a\} \sum_{n=\max\{l+1,2^k\}+1}^{2^{k+1}} \sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \\
&\le \sum_{l=1}^{2^k} P_\mu\{H_l \le a\} \sum_{n=2^k+1}^{2^{k+1}} \sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \\
&\quad + \sum_{l=2^k+1}^{2^{k+1}} P_\mu\{H_l \le a\} \sum_{n=l+2}^{2^{k+1}} \sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\}.
\end{aligned} \tag{3.30}$$

For the first term on the right-hand side, by (3.23) and (3.28),

$$\begin{aligned}
&\sum_{l=1}^{2^k} P_\mu\{H_l \le a\} \sum_{n=2^k+1}^{2^{k+1}} \sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \\
&\quad \le C \sum_{l=1}^{2^k} \frac{\log l}{l} \sum_{n=2^k+1}^{2^{k+1}} \sum_{\rho=1}^{n-1-l} \sum_{z\in[-2,2]} P_\mu\{T_z=\rho\} P_\mu\{T_z = n-1-l\} \\
&\quad \le C \sum_{z\in[-2,2]} \sum_{l=1}^{2^k} \frac{\log l}{l} \sum_{n=2^k+1}^{\infty} P_\mu\{T_z = n-1-l\} \\
&\quad \le C \sum_{z\in[-2,2]} \sum_{l=1}^{2^k} \frac{\log l}{l} P_\mu\{T_z \ge 2^k - l\} \\
&\quad \le C \sum_{l=1}^{2^k} \frac{\log l}{l} \frac{1}{\sqrt{2^k - l + 1}},
\end{aligned}$$

where the last step follows from (3.27), Lemma 3.1. We claim that the summation on the right-hand side is bounded in k. Indeed,

$$\begin{aligned}
&\sum_{l=1}^{2^k} \frac{\log l}{l} \frac{1}{\sqrt{2^k - l + 1}} \le 2^{-\frac{k-1}{2}} \sum_{l=1}^{2^{k-1}} \frac{\log l}{l} + \frac{\log 2^{k-1}}{2^{k-1}} \sum_{l=2^{k-1}+1}^{2^k} \frac{1}{\sqrt{2^k - l + 1}} \\
&\le C \left\{ 2^{-\frac{k-1}{2}} (\log 2^{k-1})^2 + \frac{\log 2^{k-1}}{2^{k-1}} 2^{\frac{k-1}{2}} \right\},
\end{aligned}$$

and clearly the right-hand side is bounded in k.

As for the second term on the right-hand side of (3.30), we use the easy bound

$$\sum_{n=l+2}^{2^{k+1}} \sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \le \sum_{n=1}^{\infty} \sum_{\rho=1}^{n} P_\mu\{F_{\rho,n}\} \le 5,$$

where the last step follows from (3.24).

Combining the steps after (3.30), we get

$$\sum_{n=2^k+1}^{2^{k+1}} \sum_{l=1}^{n-2} \left(\sum_{\rho=1}^{n-1-l} P_\mu\{F_{\rho,n-l-1}\} \right) P_\mu\{H_l \le a\} \le C + 5 \sum_{n=2^k+1}^{2^{k+1}} P_\mu\{H_n \le a\}.$$

Plugging this into (3.29), we have

$$\sum_{n=2^k+1}^{2^{k+1}} P_\mu\{H_n \le a\} \le C + \sum_{n=2^k+1}^{2^{k+1}} P_\mu\{Q_n = 0\} + 20 P\{\xi_{1,1} \le a\} \sum_{n=2^k+1}^{2^{k+1}} P_\mu\{H_n \le a\},$$

or equivalently,

$$\sum_{n=2^k+1}^{2^{k+1}} P_\mu\{H_n \le a\} \le \Big(1 - 20P\{\xi_{1,1} \le a\}\Big)^{-1} \Big\{C + \sum_{n=2^k+1}^{2^{k+1}} P_\mu\{Q_n = 0\}\Big\}. \quad (3.31)$$

In view of (3.26) in Lemma 3.1, the sequence

$$\sum_{n=2^k+1}^{2^{k+1}} P_\mu\{Q_n = 0\} \le \sum_{n=2^k+1}^{2^{k+1}} \frac{C}{n} \le C\left(\log 2^{k+1} - \log 2^k\right) \le C$$

is uniformly bounded in k. Therefore, (3.31) yields that there exists a constant C independent of k such that

$$\sum_{n=2^k+1}^{2^{k+1}} P_\mu\{H_n \le a\} \le C \quad \text{for all } k = 1, 2, \ldots,$$

which leads to, noting the monotonicity of $P_\mu\{H_n \le a\}$ in n,

$$P_\mu\{H_{2^{k+1}} \le a\} \le C\frac{1}{2^k} \quad \text{for all } k = 1, 2, \ldots$$

By the monotonicity of the random sequence $\{H_n\}$, this leads to (3.8). □

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Declarations

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