Feynman-Kac formula for fractional heat equation driven by fractional white noise

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Abstract

In this paper we obtain a Feynman-Kac formula for the solution of a fractional stochastic heat equation driven by fractional noise. One of the main difficulties is to show the exponential integrability of some singular nonlinear functionals of symmetric stable Lévy motion. This difficulty will be overcome by a technique developed in the framework of large deviation. This Feynman-Kac formula is applied to obtain the Hölder continuity and moment formula of the solution.

1 Introduction

Let $0 < \alpha < 2$ and let $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ be the Laplacian. In this paper, we shall obtain a Feynman-Kac type formula for the following stochastic equation driven by fractional noise

$$\begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}} u + u \dfrac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d} \\ u(0,x) = f(x)\,, \end{cases} \tag{1.1}$$

where $W(t,x)$ is a fractional Brownian sheet with Hurst parameters H_0 in time and $(H_1, \dots, H_d)$ in space, respectively. More specifically, for the solution $u(t,x)$ to the above equation, we can write down the following Feynman-Kac formula

$$u(t,x) = E^X \left[f(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(X_{t-r}^x - y) W(dr,dy) \right) \right] , \tag{1.2}$$

where E^X denotes the expectation with respect to the symmetric α-stable Lévy motion X_t^x, and δ denotes the Dirac delta function.

When $\dfrac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$ in (1.1) is replaced by a (deterministic) continuous function $c(t,x)$, this is the classical Feynman-Kac formula, which has been widely studied (see [7]).

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In our above stochastic case, if $-(-\Delta)^{\alpha/2}$ in (1.1) is replaced by the classical Laplacian Δ (the case $\alpha = 2$), the corresponding Feynman-Kac formula was studied first in [11] in the case that all Hurst parameters $H_0, H_1, \cdots, H_d$ are greater than or equal to $1/2$ and then in [9] in the case $H_0 < 1/2$. In both papers, the symmetric α-stable Lévy motion is replaced by the standard Brownian motion.

In [11], the main difficulties to overcome are the following: The first one is to show the exponential integrability of some functionals of the Brownian motion so that (1.2) is well-defined. The second one is to show that (1.2) is a solution to (1.1). In [9], even the existence of the stochastic integral in (1.2) becomes a big challenge and was dealt with great care. We expect the similar difficulties will appear in our current situation.

We shall follow the approach of [11] and [9] to approximate the noise in (1.1) by smooth ones. However, to show the exponential integrability of the stochastic integral in (1.2) we shall use a technique developed in [4]. This will largely simplify our computation even in the standard Brownian motion case. We shall also use Malliavin calculus to show that (1.2) is indeed a weak solution to (1.1).

It is straightforward to extend our results in the following two directions. First, $-(-\Delta)^{\alpha/2}$ in (1.1) can be replaced by a generator of more general Lévy process or more general Markov process. Secondly, $W(t, x)$ in (1.1) can be replaced by a more general Gaussian field. We restrict ourselves to the specific case of (1.1) is to make our presentation simple and to present our idea and approach in a clear way.

The paper is organized as follows. In Section 2, we very briefly present some preliminary material on Malliavin calculus and stable Lévy motion that we need to fix some notations. In Section 3, we give a definition to the stochastic integral in (1.2). This is necessary since the integrand involves the Dirac delta function. We also show the exponential integrability of the mentioned stochastic integral. In Section 4, we use Malliavin calculus to prove that (1.2) is a solution to (1.1) in a weak sense. Section 5 is an application of Feynman-Kac formula (1.2). We prove that the solution $u(t, x)$ given by (1.1) has a Hölder continuous version. We also find the Hölder exponent. The main part of the paper assumes that the stochastic integral involved in (1.1) is in the sense of Stratonovich. In Section 6, we also discussed the Feynman-Kac formula where the stochastic integral in (1.1) is in the sense of Itô-Skorohod. We also obtain a Feynman-Kac formula to represent the Itô-Wiener chaos coefficients of the solution. A formula for the moments of the solution is also given. Finally in Section 7, we present some lemmas used in this paper.

2 Preliminaries

2.1 Fractional Brownian motion and Malliavin calculus

Fix a vector of Hurst parameters $H = (H_0, H_1, \ldots, H_d)$, where $H_i \in \left(\frac{1}{2}, 1\right)$. Suppose that $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a zero mean Gaussian random field with the covariance function

$$E(W(t,x)W(s,y)) = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i),$$

where for any $H \in (0,1)$ we denote by $R_H(s,t)$, the covariance function of the fractional Brownian motion with Hurst parameter H, that is,

$$R_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

In other words, W is a fractional Brownian sheet with Hurst parameters H_0 in time variable and H_i in space variables, $i = 1, \ldots, d$.

Denote by $\mathcal{E}$ the linear span of the indicator functions of rectangles of the form $(s,t] \times (x,y]$ in $\mathbb{R}_+ \times \mathbb{R}^d$, where $(x,y] = (x_1, y_1] \times \cdots \times (x_d, y_d]$. Consider in $\mathcal{E}$ the inner product defined by

$$\langle I_{(0,s]\times(0,x]}, I_{(0,t]\times(0,y]} \rangle_{\mathcal{H}} = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i).$$

In the above formula, if $x_i < 0$ we assume by convention that $I_{(0,x_i]} = -I_{[x_i,0)}$. We denote by $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to this inner product. The mapping $W : I_{(0,t]\times(0,x]} \to W(t,x)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. We will denote this isometry by

$$W(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,x) W(dt,dx),$$

if $\phi \in \mathcal{H}$. Notice that if ϕ and ψ are functions in $\mathcal{E}$, then

$$\begin{aligned} &E\left(W(\phi)W(\psi)\right) = \langle \phi, \psi \rangle_{\mathcal{H}} = \alpha_H \\ &\times \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \phi(s,x)\psi(t,y)|s-t|^{2H_0-2} \prod_{i=1}^{d} |x_i - y_i|^{2H_i-2} ds dt dx dy, \quad (2.1) \end{aligned}$$

where $\alpha_H = \prod_{i=0}^{d} H_i(2H_i - 1)$. Furthermore, $\mathcal{H}$ contains the class of measurable functions ϕ on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} |\phi(s,x)\phi(t,y)||s-t|^{2H_0-2} \prod_{i=1}^{d} |x_i - y_i|^{2H_i-2} ds dt dx dy < \infty. \quad (2.2)$$

We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \ldots, W(\phi_n)),$$

$\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathcal{H}}^2)}.$$

We denote by δ the adjoint of the derivative operator, determined by duality formula

$$E(\delta(u)F) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \tag{2.3}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathcal{H})$ in the domain of δ. The operator δ is also called the Skorohod integral because in the case of the Brownian motion it coincides with an extension of the Itô integral introduced by Skorohod. We refer to Nualart [16] for a detailed account on the Malliavin calculus with respect to a Gaussian process. If DF and u are almost surely measurable functions on $\mathbb{R}_+ \times \mathbb{R}^d$ verifying condition (2.2), then the duality formula (2.3) can be written using the expression of the inner product in $\mathcal{H}$ given in (2.1):

$$\begin{aligned} &E\left(\delta(u)F\right) = \alpha_H \\ &\times E\left(\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} D_{s,x}Fu(t,y)|s-t|^{2H_0-2}\prod_{i=1}^{d}|x_i - y_i|^{2H_i-2}dsdtdxdy\right). \end{aligned}$$

We recall the following formula, which will be used in the paper

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi\rangle_{\mathcal{H}}, \tag{2.4}$$

for any $\phi \in \mathcal{H}$ and any random variable F in the Sobolev space $\mathbb{D}^{1,2}$.

2.2 Symmetric α-stable Lévy motion

In this section we recall symmetric stable distribution and symmetric α-stable Lévy motion. For more general and detailed result about stable processes, we refer to [18].

A random variable X is said to be symmetric α-stable if there is parameters $0 < \alpha \leq 2$ and $\sigma \geq 0$ such that its characteristic function

$$Ee^{i\theta X} = e^{-\sigma^\alpha|\theta|^\alpha},$$

and we will denote that $X \sim S(\alpha, \sigma)$.

Notice that when $\alpha = 2, X$ is a Gaussian random variable. When $\alpha \in (0, 2)$, we have $E|X|^p < \infty$ if $-1 < p < \alpha$ and $E|X|^p = \infty$ if $p \geq \alpha$.

A stochastic process $\{X(t), t \geq 0\}$ is called symmetric α-stable Lévy motion if

(1) $X(0) = 0$ a.s..

(2) X has independent increments.

(3) $X(t) - X(s) \sim S(\alpha, (t-s)^{\frac{1}{\alpha}})$ for any $0 \le s < t < \infty$ and for some $0 < \alpha \le 2$.

Throughout the paper C will denote a positive constant which may vary from one formula to another one.

3 Definition and exponential integrability of the generalized stochastic convolution

For any $\varepsilon > 0$ we denote by $p_\varepsilon(x)$ the d-dimensional heat kernel

$$p_\epsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\epsilon}},\ x \in \mathbb{R}^d.$$

On the other hand, for any $\delta > 0$ we define the function

$$\varphi_\delta(x) = \frac{1}{\delta} I_{[0,\delta]}(x).$$

Then, $\varphi_\delta(t)p_\varepsilon(x)$ provides an approximation of the Dirac delta function $\delta(t,x)$ as ε and δ tend to zero. We denote by $W^{\epsilon,\delta}$ the approximation of the fractional Brownian sheet $W(t,x)$ defined by

$$W^{\epsilon,\delta}(t,x) = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s)p_\epsilon(x-y)W(s,y)dsdy\,. \tag{3.1}$$

Fix $x \in \mathbb{R}^d$ and $t > 0$. Suppose that $X = \{X_t, t \ge 0\}$ is a d-dimensional symmetric α-stable Lévy motion independent of W. We denote by $X_t^x = X_t + x$ the symmetric α-stable Lévy motion starting at the point x. We are going to define the random variable $\int_0^t \int_{\mathbb{R}^d} \delta(X_{t-r}^x - y)W(dr,dy)$ by approximating the Dirac delta function $\delta(X_{t-r}^x - y)$ by

$$A_{t,x}^{\epsilon,\delta}(r,y) = \int_0^t \varphi_\delta(t-s-r)p_\varepsilon(X_s^x - y)ds. \tag{3.2}$$

We will show that for any $\varepsilon > 0$ and $\delta > 0$ the function $A_{t,x}^{\epsilon,\delta}$ belongs to the space $\mathcal{H}$ almost surely, and the family of random variables

$$V_{t,x}^{\varepsilon,\delta} = \int_0^t \int_{\mathbb{R}^d} A_{t,x}^{\epsilon,\delta}(r,y)W(dr,dy)\,. \tag{3.3}$$

converges in L^2 as ε and δ tend to zero.

The specific approximation chosen here will allow us in Section 4 to construct an approximate Feynman-Kac formula with the random potential

$\dot{W}^{\epsilon,\delta}(t,x)$ given in (4.1). Moreover, this approximation has the useful properties proved in Lemmas 7.4 and 7.5. We may use other types of approximation schemes with similar results. Also, we can restrict ourselves to the special case $\delta = \varepsilon$, but the slightly more general case considered here does not need any additional effort.

Along the paper we denote by $E^X(\Phi(X,W))$ (resp. by $E^W(\Phi(X,W))$) the expectation of a functional $\Phi(X,W)$ with respect to X (resp. with respect to W). We will use E for the composition $E^X E^W$, and also in case of a random variable depending only on X or W.

Theorem 3.1 *Suppose that* $2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i - 2) > 1$. *Then, for any* $\varepsilon > 0$ *and* $\delta > 0$, $A_{t,x}^{\varepsilon,\delta}$ *defined in (3.2) belongs to* $\mathcal{H}$ *and the family of random variables* $V_{t,x}^{\epsilon,\delta}$ *defined in (3.3) converges in* L^2 *to a limit denoted by*

$$V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(X_{t-r}^x - y) W(dr, dy)\,. \tag{3.4}$$

Conditional to X, $V_{t,x}$ *is a Gaussian random variable with mean* 0 *and variance*

$$\mathrm{Var}^W(V_{t,x}) = \alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d \left|X_r^i - X_s^i\right|^{2H_i-2} drds\,. \tag{3.5}$$

Proof Fix ε, ε', δ and $\delta' > 0$. Let us compute the inner product

$$\begin{aligned}\left\langle A_{t,x}^{\varepsilon,\delta}, A_{t,x}^{\varepsilon',\delta'}\right\rangle_{\mathcal{H}} &= \alpha_H \int_{[0,t]^4}\int_{\mathbb{R}^{2d}} p_\epsilon(X_s^x - y) p_{\epsilon'}(X_r^x - z) \\ &\quad \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v) \\ &\quad \times |u-v|^{2H_0-2}\prod_{i=1}^d |y_i - z_i|^{2H_i-2} dydzdudvdsdr. \end{aligned} \tag{3.6}$$

By Lemmas 7.4 and 7.5 we have the estimate

$$\begin{aligned} &\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} p_\epsilon(X_s^x - y) p_{\epsilon'}(X_r^x - z) \\ &\times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2}\prod_{i=1}^d |y_i - z_i|^{2H_i-2} dydzdudv \\ \le\ & C|s-r|^{2H_0-2}\prod_{i=1}^d \left|X_s^i - X_r^i\right|^{2H_i-2}, \end{aligned} \tag{3.7}$$

where and in what follows $C > 0$ denotes a constant independent of ϵ and δ.

The expectation of this random variable is integrable in $[0,t]^2$ because

$$
\begin{aligned}
& E^X \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d \left|X_s^i - X_r^i\right|^{2H_i-2} dsdr \\
= & \prod_{i=1}^d E|X_1|^{2H_i-2} \int_0^t \int_0^t |s-r|^{2H_0+\sum_{i=1}^d \frac{1}{\alpha}(2H_i-2)-2} dsdr \\
= & \frac{2\prod_{i=1}^d E|\xi|^{2H_i-2} t^{\kappa+1}}{\kappa(\kappa+1)} < \infty, \qquad (3.8)
\end{aligned}
$$

where

$$
\kappa = 2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i-2) - 1 > 0. \qquad (3.9)
$$

and ξ is a standard symmetric α-stable random variable.

As a consequence, taking the mathematical expectation with respect to X in Equation (3.6), letting $\varepsilon = \varepsilon'$ and $\delta = \delta'$ and using the estimates (3.7) and (3.8) yields

$$
E^X \left\| A_{t,x}^{\varepsilon,\delta} \right\|_{\mathcal{H}}^2 \le C.
$$

This implies that almost surely $A_{t,x}^{\varepsilon,\delta}$ belongs to the space $\mathcal{H}$ for all ε and $\delta > 0$. Therefore, the random variables $V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta})$ are well defined and we have

$$
E^X E^W (V_{t,x}^{\epsilon,\delta} V_{t,x}^{\epsilon',\delta'}) = E^X \left\langle A_{t,x}^{\varepsilon,\delta}, A_{t,x}^{\varepsilon',\delta'} \right\rangle_{\mathcal{H}}.
$$

For any $s \neq r$ and $X_s \neq X_r$, as ε, ε', δ and δ' tend to zero, the left-hand side of the inequality (3.7) converges to $|s-r|^{2H_0-2}\prod_{i=1}^d \left|X_s^i - X_r^i\right|^{2H_i-2}$. Therefore, by dominated convergence theorem we obtain that $E^X E^W (V_{t,x}^{\epsilon,\delta} V_{t,x}^{\epsilon',\delta'})$ converges to Σ_t as ε, ε', δ and δ' tend to zero, where

$$
\Sigma_t = \frac{2\alpha_H \prod_{i=1}^d E|\xi|^{2H_i-2} t^{\kappa+1}}{\kappa(\kappa+1)}.
$$

Thus we obtain

$$
E\left(V_{t,x}^{\varepsilon,\delta} - V_{t,x}^{\varepsilon',\delta'}\right)^2 = E\left(V_{t,x}^{\varepsilon,\delta}\right)^2 - 2E\left(V_{t,x}^{\varepsilon,\delta} V_{t,x}^{\varepsilon',\delta'}\right) + E\left(V_{t,x}^{\varepsilon',\delta'}\right)^2 \to 0\,.
$$

This implies that $V_{t,x}^{\epsilon_n,\delta_n}$ is a Cauchy sequence in L^2 for all sequences ε_n and δ_n converging to zero. As a consequence, $V_{t,x}^{\epsilon_n,\delta_n}$ converges in L^2 to a limit denoted by $V_{t,x}$, which does not depend on the choice of the sequences ε_n and δ_n. Finally, by a similar argument we show (3.5). ■

Proposition 3.2 *Suppose* $2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i-2) \le 1$. *Then, conditionally to* X *the family* $V_{t,x}^{\varepsilon,\delta}$ *does not converge in probability as* ε *and* δ *tend to zero, for a non-zero set of trajectories of* X.

Proof We will prove by contradiction. Suppose $V_{t,x}^{\varepsilon,\delta}$ converges to $V_{t,x}$ in probability as ε and δ tend to zero given X for almost all trajectories of X. Since given X, $V_{t,x}^{\varepsilon,\delta}$ is Gaussian, so $E^W(V_{t,x}^2) = \lim_{\varepsilon\to 0,\delta\to 0} E^W(V_{t,x}^{\varepsilon,\delta})^2 = \int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr < \infty$ for almost all trajectories of X. As in the Step 2 in the proof of the Theorem 3.3, we can prove that $\left(\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr\right)^{\frac{1}{2}}$ is sub-additive and it's also finite almost surely, hence we have

$$E\exp\left(\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr\right)^{\frac{1}{2}} < \infty,$$

implying that $E\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr < \infty$ which is a contradiction. ■

The next result provides the exponential integrability of the random variable $V_{t,x}$ defined in (3.4).

Theorem 3.3 *Suppose that* $2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i - 2) > 1$. *Then, for any* $\lambda \in \mathbb{R}$, *we have*

$$E\exp\left(\lambda\int_0^t\int_{\mathbb{R}^d} \delta(X_{t-r}^x - y)W(dr,dy)\right) < \infty\,. \tag{3.10}$$

Proof The proof will be done in several steps.

Step 1 From (3.5) we obtain

$$Ee^{\lambda V_{t,x}} = E^X\exp\left(\frac{\lambda^2}{2}\alpha_H\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr\right),$$

and the scaling property of the stable Lévy motion yields

$$Ee^{\lambda V_{t,x}} = Ee^{\mu Y}, \tag{3.11}$$

where $\mu = \frac{\lambda^2}{2}\alpha_H t^{\kappa+1}$, where κ has been defined in (3.9), and

$$Y = \int_0^1\int_0^1 |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr. \tag{3.12}$$

Then, it suffices to show that the random variable Y has exponential moments of all orders.

Step 2 Let

$$Z_t = \left(\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^i - X_r^i|^{2H_i-2}dsdr\right)^{\frac{1}{2}}.$$

We use the the identity

$$|s-r|^{2H_0-2} = C_0 \int_{\mathbb{R}} |s-u|^{\frac{2H_0-3}{2}} |r-u|^{\frac{2H_0-3}{2}} du$$

$$|X_s^i - X_r^i|^{2H_i-2} = C_i \int_{\mathbb{R}} |X_s^i - x|^{\frac{2H_i-3}{2}} |X_r^i - x|^{\frac{2H_i-3}{2}} dx \qquad i = 1, \cdots, d$$

where C_i only depends on H_i for $i = 0, 1, \cdots, d$.
We have

$$Z_t = \Big(\int_{\mathbb{R}\times\mathbb{R}^d} \xi_t^2(u, x_1, \cdots, x_d) du dx_1 \cdots dx_d \Big)^{1/2}$$

where

$$\xi_t(u, x_1, \cdots, x_d) = \Big(\prod_{i=0}^d C_i \Big) \int_0^t |s-u|^{\frac{2H_0-3}{2}} \prod_{i=1}^d |X_s^i - x_i|^{\frac{2H_i-3}{2}} ds$$

For $t_1, t_2 > 0$, by triangular inequality

$$Z_{t_1+t_2} \le Z_{t_1} + \Big(\int_{\mathbb{R}\times\mathbb{R}^d} \Big[\xi_{t_1+t_2}(u, x_1, \cdots, x_d) - \xi_{t_1}(u, x_1, \cdots, x_d) \Big]^2 du dx_1 \cdots dx_d \Big)^{1/2}$$

Write $\widetilde{X}_s^i = X_{t_1+s}^i - X_{t_1}^i$.

$$\begin{aligned} &\xi_{t_1+t_2}(u, x_1, \cdots, x_d) - \xi_{t_1}(u, x_1, \cdots, x_d) \\ =&\Big(\prod_{i=0}^d C_i \Big) \int_{t_1}^{t_1+t_2} |s-u|^{\frac{2H_0-3}{2}} \prod_{i=1}^d |X_s^i - x_i|^{\frac{2H_i-3}{2}} ds \\ =&\Big(\prod_{i=0}^d C_i \Big) \int_0^{t_2} |s+t_1-u|^{\frac{2H_0-3}{2}} \prod_{i=1}^d |\widetilde{X}_s^i + X_{t_1}^i - x_i|^{\frac{2H_i-3}{2}} ds \end{aligned}$$

By translation invariance,

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}^d} \Big[\xi_{t_1+t_2}(u, x_1, \cdots, x_d) - \xi_{t_1}(u, x_1, \cdots, x_d) \Big]^2 \\ =&\int_{\mathbb{R}\times\mathbb{R}^d} \widetilde{\xi}_{t_2}^2(u, x_1, \cdots, x_d) du dx_1 \cdots dx_d \end{aligned}$$

where

$$\widetilde{\xi}_{t_2}^2(u, x_1, \cdots, x_d) = \Big(\prod_{i=0}^d C_i \Big) \int_0^{t_2} |s-u|^{\frac{2H_0-3}{2}} \prod_{i=1}^d |\widetilde{X}_s^i - x_i|^{\frac{2H_i-3}{2}} ds$$

Therefore, the process Z_t is sub-additive, which means that for any $t_1, t_2 > 0$, $Z_{t_1+t_2} \leq Z_{t_1} + \widetilde{Z}_{t_2}$, where $\widetilde{Z}_{t_2}$ is independent of $\{Z_s;\ \ s \leq t_1\}$ and has a distribution same as Z_{t_2}.

Step 3 Notice $Z_t \geq 0$ is non-decreasing and path-wise continuous. By Theorem 1.3.5 in [4], for any $\theta > 0$ and $t > 0$

$$E \exp\big\{\theta Z_t\big\} < \infty$$

and the limit

$$\lim_{t\to\infty} \frac{1}{t} \log E \exp\big\{\theta Z_t\big\} = \Psi(\theta)$$

where $0 \leq \Psi(\theta) < \infty$. By the scaling $Z_t \overset{d}{=} t^{\gamma/2} Z_1$ with

$$\gamma = 2H_0 + \frac{1}{\alpha} \sum_{i=1}^{d} (2H_i - 2), \tag{3.13}$$

we have

$$\lim_{t\to\infty} \frac{1}{t} \log E \exp\big\{\theta Z_t\big\} = \lim_{t\to\infty} \frac{1}{t\theta^{\frac{2}{\gamma}}} \log E \exp\big\{\theta Z_{t\theta^{\frac{2}{\gamma}}}\big\} \theta^{\frac{2}{\gamma}} = \Psi(1)\theta^{\frac{2}{\gamma}}$$

Using Chebyshev inequality, we have $e^{\theta t}\mathbb{P}\big\{Z_t \geq t\big\} \leq Ee^{\theta Z_t}$. Hence we have $\theta t + \log \mathbb{P}\big\{Z_t \geq t\big\} \leq \log Ee^{\theta Z_t}$, and then $\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big\{Z_t \geq t\big\} \leq \lim_{t\to\infty} \frac{1}{t} \log Ee^{\theta Z_t} - \theta = \Psi(\theta) - \theta = \theta^{\frac{2}{\gamma}}\Psi(1) - \theta$ which is strictly negative when we choose $\theta > 0$ sufficiently small. Hence there exists $C > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big\{Z_t \geq t\big\} \leq -C.$$

So we have the bound

$$\mathbb{P}\Big\{Z_1 \geq t^{\frac{2-\gamma}{2}}\Big\} \leq \exp\{-Ct\}$$

Since for random variable $X \geq 0, Ee^X = E\int_0^X e^y dy + 1 = \int_0^\infty \mathbb{P}\{X \geq y\} e^y dy + 1$, we have

$$\begin{aligned}
&Ee^{\theta Z_1^{\frac{2}{2-r}}} \\
\leq &\int_0^\infty \mathbb{P}\{\theta Z_1^{\frac{2}{2-r}} \geq y\} e^y dy + 1 \\
\leq &\sum_{K=0}^\infty \mathbb{P}\{\theta Z_1^{\frac{2}{2-r}} \geq K\} e^{K+1} + 1 \\
\leq &\sum_{K=0}^\infty e^{-\frac{C}{\theta}K} e^{K+1} + 1
\end{aligned}$$

This give the critical integrability

$$E\exp\left\{\theta Z_1^{\frac{2}{2-\gamma}}\right\}<\infty$$

for some $\theta > 0$, which implies that $E\exp\left\{\lambda Z_1^2\right\} < \infty$ for all $\lambda > 0$ since $1 < \gamma < 2$.

■

4 Feynman-Kac formula

We recall that W is a fractional Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}^d$ with Hurst parameters $(H_0, H_1, \dots, H_d)$ where $H_i \in (\frac{1}{2}, 1)$ for $i = 0, \dots, d$. For any ε, $\delta > 0$ we define

$$\dot{W}^{\epsilon,\delta}(t,x) := \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s)p_\epsilon(x-y)W(ds,dy). \tag{4.1}$$

In order to give a notion of solution for the heat equation with fractional noise (1.1) we need the following definition of the Stratonovitch integral, which is equivalent to that of Russo-Vallois in [17].

Definition 4.1 *Given a random field $v = \{v(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that*

$$\int_0^T \int_{\mathbb{R}^d} |v(t,x)|dxdt < \infty$$

almost surely for all $T > 0$, the Stratonovitch integral $\int_0^T \int_{\mathbb{R}^d} v(t,x)W(dt,dx)$ is defined as the following limit in probability if it exists

$$\lim_{\epsilon,\delta\downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t,x)\dot{W}^{\epsilon,\delta}(t,x)dxdt.$$

We are going to consider the following notion of solution for Equation (1.1).

Definition 4.2 *A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a weak solution to Equation (1.1) if for any C^∞ function φ with compact support on $\mathbb{R}^d$, we have*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)dx - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)(-\Delta)^{\frac{\alpha}{2}}\varphi(x)dxds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx),\end{aligned}$$

almost surely, for all $t \geq 0$, where the last term is a Stratonovitch stochastic integral in the sense of Definition 4.1.

The following is the main result of this section.

Theorem 4.3 *Suppose that* $2H_0+\frac{1}{\alpha}\sum_{i=1}^d(2H_i-2)>1$ *and that* f *is a bounded measurable function. Then process*

$$u(t,x)=E^X\left(f(X_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(X_{t-r}^x-y)W(dr,dy)\right)\right) \tag{4.2}$$

is a weak solution to Equation (1.1).

Proof Consider the approximation of the Equation (1.1) given by the following heat equation with a random potential

$$\begin{cases}\dfrac{\partial u^{\varepsilon,\delta}}{\partial t}=-(-\Delta)^{\frac{\alpha}{2}}u^{\varepsilon,\delta}+u^{\varepsilon,\delta}\dot{W}_{t,x}^{\epsilon,\delta}\\ u^{\varepsilon,\delta}(0,x)=f(x).\end{cases} \tag{4.3}$$

¿From the classical Feynman-Kac formula we know that

$$u^{\varepsilon,\delta}(t,x)=E^X\left(f(X_t^x)\exp\left(\int_0^t\dot{W}^{\epsilon,\delta}(t-s,X_s^x)ds\right)\right),$$

where X^x is a d-dimensional symmetric α-stable Lévy motion independent of W starting at x. By Fubini's theorem we can write

$$\begin{aligned}\int_0^t\dot{W}^{\varepsilon,\delta}(t-s,X_s^x)ds &= \int_0^t\left(\int_0^t\int_{\mathbb{R}^d}\varphi_\delta(t-s-r)p_\varepsilon(X_s^x-y)W(dr,dy)\right)ds\\ &= \int_0^t\int_{\mathbb{R}^d}\left(\int_0^t\varphi_\delta(t-s-r)p_\varepsilon(X_s^x-y)ds\right)W(dr,dy)\\ &= V_{t,x}^{\varepsilon,\delta},\end{aligned}$$

where $V_{t,x}^{\varepsilon,\delta}$ is defined in (3.3). Therefore,

$$u^{\varepsilon,\delta}(t,x)=E^X\left(f(X_t^x)\exp\left(V_{t,x}^{\varepsilon,\delta}\right)\right).$$

Step 1 We will prove that for any $x\in\mathbb{R}^d$ and any $t>0$, we have

$$\lim_{\varepsilon,\delta\downarrow 0}\quad E^W|u^{\varepsilon,\delta}(t,x)-u(t,x)|^p=0, \tag{4.4}$$

for all $p\ge 2$, where $u(t,x)$ is defined in (4.2). Notice that

$$\begin{aligned}E^W|u^{\varepsilon,\delta}(t,x)-u(t,x)|^p &= E^W\left|E^B\left(f(B_t^x)\left[\exp\left(V_{t,x}^{\varepsilon,\delta}\right)-\exp\left(V_{t,x}\right)\right]\right)\right|^p\\ &\le \|f\|_\infty^p\,E\left|\exp\left(V_{t,x}^{\varepsilon,\delta}\right)-\exp\left(V_{t,x}\right)\right|^p,\end{aligned}$$

where $V_{t,x}$ is defined in (3.4). Since $\exp\left(V_{t,x}^{\varepsilon,\delta}\right)$ converges to $\exp\left(V_{t,x}\right)$ in probability by Theorem 3.1, to show (4.4) it suffices to prove that for any $\lambda\in\mathbb{R}$

$$\sup_{\epsilon,\delta}E\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right)<\infty. \tag{4.5}$$

The estimate (4.5) follows from (3.3), (3.7), and (3.10):

$$\begin{aligned} E\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right) &= E\exp\left(\frac{\lambda^2}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right) \\ &\le E\exp\left(\frac{\lambda^2}{2}C\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d \left|X_r^i - X_s^i\right|^{2H_i-2} drds\right) \\ &< \infty. \end{aligned} \tag{4.6}$$

Step 2 Now we prove that $u(t,x)$ is a weak solution to Equation (1.1) in the sense of Definition 4.2. Suppose φ is a smooth function with compact support. We know that,

$$\begin{aligned} \int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)dx - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)(-\Delta)^{\frac{\alpha}{2}}\varphi(x)dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx. \end{aligned} \tag{4.7}$$

Therefore, it suffices to prove that

$$\lim_{\varepsilon,\delta\downarrow 0}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx = \int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx),$$

in probability. From (4.7) and (4.4) it follows that $\int_0^t\int_{\mathbb{R}^d} u^{\epsilon,\delta}(s,x)\varphi(x)\dot{W}^{\epsilon,\delta}(s,x)dsdx$ converges in L^2 to the random variable

$$G = \int_{\mathbb{R}^d} u(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(t,x)(-\Delta)^{\frac{\alpha}{2}}\varphi(x)dxds$$

as ε and δ tend to zero. Hence, if

$$B_{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}^d} (u^{\epsilon,\delta}(s,x) - u(s,x))\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx$$

converges in L^2 to zero, then

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}dsdx = \int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}dsdx - B_{\varepsilon,\delta}$$

converges to G in L^2. Thus $u(s,x)\varphi(x)$ will be Stratonovich integrable and

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx) = G\,,$$

which will complete the proof. In order to show the convergence to zero of $B_{\varepsilon,\delta}$, we will express the product $(u^{\varepsilon,\delta}(s,x) - u(s,x))\dot{W}^{\varepsilon,\delta}(s,x)$ as the sum of a divergence integral plus a trace term (see (2.4))

$$
\begin{aligned}
&(u^{\varepsilon,\delta}(s,x)-u(s,x))\dot{W}^{\varepsilon,\delta}(s,x)\\
&\quad=\int_0^t\int_{\mathbb{R}^d}(u^{\varepsilon,\delta}(s,x)-u(s,x))\varphi_\delta(s-r)p_\varepsilon(x-z)\delta W_{r,z}\\
&\qquad+\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
B_{\varepsilon,\delta} &= \int_0^t\int_{\mathbb{R}^d}\phi^{\epsilon,\delta}_{r,z}\delta W_{r,z}\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\varphi(x)\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}dsdx\\
&= B^1_{\varepsilon,\delta}+B^2_{\varepsilon,\delta}, \qquad (4.8)
\end{aligned}
$$

where

$$
\phi^{\varepsilon,\delta}_{r,z}=\int_0^t\int_{\mathbb{R}^d}(u^{\varepsilon,\delta}(s,x)-u(s,x))\varphi(x)\varphi_\delta(s-r)p_\varepsilon(x-z)dsdx,
$$

and $\delta(\phi^{\varepsilon,\delta})=\int_0^t\int_{\mathbb{R}^d}\phi^{\varepsilon,\delta}_{r,z}\delta W_{r,z}$ denotes the divergence or the Skorohod integral of $\phi^{\varepsilon,\delta}$.

Step 3 For the term $B^1_{\varepsilon,\delta}$ we use the following L^2 estimate for the Skorohod integral

$$
E[(B^1_{\varepsilon,\delta})^2]\le E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}})+E(\|D\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}\otimes\mathcal{H}})\,. \qquad (4.9)
$$

The first term in (4.9) is estimated as follows

$$
\begin{aligned}
E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}}) &= \int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}E\left[(u^{\varepsilon,\delta}(s,x)-u(s,x))(u^{\varepsilon,\delta}(r,y)-u(r,y))\right]\\
&\quad\times\varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot),\varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}}dsdxdrdy. \qquad (4.10)
\end{aligned}
$$

Using Lemmas 7.4 and 7.5 we can write

$$
\begin{aligned}
&\langle\varphi_\delta(s-\cdot)p_\epsilon(x-\cdot),\varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}}\\
&= \alpha_H\left(\int_{[0,t]^2}\varphi_\delta(s-\sigma)\varphi_\delta(r-\tau)|\sigma-\tau|^{2H_0-2}d\sigma d\tau\right)\\
&\quad\times\left(\int_{\mathbb{R}^{2d}}p_\varepsilon(x-z)p_\varepsilon(y-w)\prod_{i=1}^d|z_i-w_i|^{2H_i-2}dzdw\right)\\
&\le C|s-r|^{2H_0-2}\prod_{i=1}^d|x-y|^{2H_i-2}, \qquad (4.11)
\end{aligned}
$$

for some constant $C>0$. As a consequence, the integrand on the right-hand side of Equation (4.10) converges to zero as ε and δ tend to zero for any s, r,

x, y due to (4.4). From (4.6) we get

$$\sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le t} E\left(u^{\epsilon,\delta}(s,x)\right)^2 \le \|f\|_\infty^2 \sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le t} E\exp\left(2V_{s,x}^{\varepsilon,\delta}\right) < \infty. \tag{4.12}$$

Hence, from (4.11) and (4.12) we get that the integrand on the right-hand side of Equation (4.10) is bounded by $C|s-r|^{2H_0-2}\prod_{i=1}^d |x_i-y_i|^{2H_i-2}$, for some constant $C>0$. Therefore, by dominated convergence we get that $E(\|\phi^{\varepsilon,\delta}\|_{\mathcal{H}}^2)$ converges to zero as ε and δ tend to zero.

Step 4 On the other hand, we have

$$D(u^{\varepsilon,\delta}(t,x)) = E^X\left[f(X_t^x)\exp(V_{t,x}^{\varepsilon,\delta})A_{t,x}^{\varepsilon,\delta}\right],$$

where $A_{t,x}^{\varepsilon,\delta}$ is defined in (3.2). Therefore,

$$\begin{aligned} & E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathcal{H}} \\ = \; & E^W E^X\Big(f(X_t^1+x)f(X_t^2+x) \\ & \times\exp(V_{t,x}^{\varepsilon,\delta}(X^1)+V_{t,x}^{\varepsilon,\delta}(X^2))\langle A_{t,x}^{\varepsilon,\delta}(X^1), A_{t,x}^{\varepsilon',\delta'}(X^2)\rangle_{\mathcal{H}}\Big), \end{aligned} \tag{4.13}$$

where X^1 and X^2 are two independent d-dimensional symmetric α-stable Lévy motions, and here E^X denotes the expectation with respect to (X^1, X^2). Then from the previous results it is easy to show that

$$\begin{aligned} & \lim_{\varepsilon,\delta\downarrow 0} E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathcal{H}} \\ = \; & E\Big[f(X_t^1+x)f(X_t^2+x) \\ & \times\exp\left(\frac{\alpha_H}{2}\sum_{j,k=1}^2\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^{j,i}-X_r^{k,i}|^{2H_i-2}dsdr\right) \\ & \times\alpha_H\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^{1,i}-X_r^{2,i}|^{2H_i-2}dsdr\Big]. \end{aligned} \tag{4.14}$$

This implies that $u^{\varepsilon,\delta}(t,x)$ converges in the space $\mathbb{D}^{1,2}$ to $u(t,x)$ as $\delta\downarrow 0$ and $\varepsilon\downarrow 0$. Letting $\varepsilon'=\varepsilon$ and $\delta'=\delta$ in (4.13) and using the same argument as for (4.12), we obtain

$$\sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le t} E\left\|D(u^{\varepsilon,\delta}(s,x))\right\|_{\mathcal{H}}^2 < \infty.$$

Then

$$\begin{aligned} E\|D\phi^{\varepsilon,\delta}\|_{\mathcal{H}\otimes\mathcal{H}}^2 \; = \; & \int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}} E\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)), D(u^{\varepsilon,\delta}(r,y)-u(r,y))\rangle_{\mathcal{H}} \\ & \times\varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}}dsdxdrdy \end{aligned}$$

converges to zero as ε and δ tend to zero. Hence, by (4.9) $B^1_{\varepsilon,\delta}$ converges to zero in L^2 as ε and δ tend to zero.

Step 5 The second summand in the right-hand side of (4.8) can be written as

$$\begin{aligned}
B^2_{\varepsilon,\delta} &= \int_0^t \int_{\mathbb{R}^d} \varphi(x)\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} dsdx \\
&= \int_0^t \int_{\mathbb{R}^d} \varphi(x)E^X\left(f(X^x_s)\exp\left(V^{\varepsilon,\delta}_{s,x}\right)\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\epsilon(x-\cdot)\rangle_{\mathcal{H}}\right) dsdx \\
&\quad - \int_0^t \int_{\mathbb{R}} \varphi(x)E^X\left(f(X^x_s)\exp\left(V_{s,x}\right)\langle \delta(X^x_{s-\cdot}-\cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}\right) dsdx \\
&= B^3_{\varepsilon,\delta} - B^4_{\varepsilon,\delta}
\end{aligned}$$

where

$$\begin{aligned}
\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\epsilon(x-\cdot)\rangle_{\mathcal{H}} &= \alpha_H \int_{[0,s]^3}\int_{\mathbb{R}^{2d}} |r-v|^{2H_0-2}\prod_{i=1}^d |y_i-z_i|^{2H_i-2} \\
&\quad \times \varphi_\delta(s-r)p_\varepsilon(X^x_r-y) \\
&\quad \times \varphi_\delta(s-v)p_\varepsilon(x-z)dydyzdrdrdv,
\end{aligned}$$

and

$$\begin{aligned}
&\langle \delta(X^x_{s-\cdot}-\cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \\
= &\ \alpha_H \int_{[0,s]^2}\int_{\mathbb{R}^d} v^{2H_0-2}\prod_{i=1}^d |X^{x_i}_r - y_i|^{2H_i-2}\varphi_\delta(r-v)p_\varepsilon(x-y)dydvdr.
\end{aligned}$$

Lemma 7.4 and Lemma 7.5 imply that

$$\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \le C\int_0^s r^{2H_0-2}\prod_{i=1}^d |X^i_r|^{2H_i-2}dr, \tag{4.15}$$

and

$$\langle \delta(B^x_{s-\cdot}-\cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \le C\int_0^s r^{2H_0-2}\prod_{i=1}^d |X^i_r|^{2H_i-2}dr, \tag{4.16}$$

for some constant $C>0$. Then, from (4.15) and (4.16) and from the fact that the random variable $\int_0^s r^{2H_0-2}\prod_{i=1}^d |X^i_r|^{2H_i-2}dr$ is square integrable because of Lemma 7.6, we can apply the dominated convergence theorem and get that $B^3_{\varepsilon,\delta}$ and $B^4_{\varepsilon,\delta}$ converge both in L^2 to

$$\alpha_H \int_0^t \int_{\mathbb{R}^d} \varphi(x)E^B\left(f(X^x_s)\exp\left(V_{s,x}\right)\int_0^s r^{2H_0-2}\prod_{i=1}^d |X^i_r|^{2H_i-2}dr\right) dsdx,$$

as ε and δ tend to zero. Therefore $B^2_{\varepsilon,\delta}$ converges in L^2 to zero as ε and δ tend to zero. This completes the proof. ■

Remark. The uniqueness of the solution remains to be investigated in a future work. The definition of the Stratonovich integral as a limit in probability makes the uniqueness problem nontrivial, and it is not clear how to proceed.

As a corollary of Theorem 4.3 we obtain the following result.

Corollary 4.4 *Suppose* $2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i - 2) > 1$. *Then the solution* $u(t,x)$ *given by (4.2) has finite moments of all orders. Moreover, for any positive integer* p*, we have*

$$E\left(u(t,x)^p\right) = E\Bigg(\prod_{j=1}^p f(X_t^j + x) \tag{4.17}$$

$$\times \exp\left[\frac{\alpha_H}{2}\sum_{j,k=1}^p \int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^d |X_s^{j,i} - X_r^{k,i}|^{2H_i-2} ds dr\right]\Bigg),$$

where $X_1, \dots, X_p$ *are independent* d*-dimensional standard symmetric* α*-stable Lévy motions.*

5 Hölder continuity of the solution

In this section, we study the Hölder continuity of the solution to Equation (1.1). The main result of this section is the following theorem.

Theorem 5.1 *Suppose that* $2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i - 2) > 1$ *and let* $u(t,x)$ *be the solution of Equation (1.1). Then* $u(t,x)$ *has a continuous modification such that for any* $\rho \in \left(0, \frac{\kappa}{2}\right)$ *(where* κ *has been defined in (3.9)), and any compact rectangle* $I \subset \mathbb{R}_+ \times \mathbb{R}^d$ *there exists a positive random variable* K_I *such that almost surely, for any* $(s,x), (t,y) \in I$ *we have*

$$|u(t,y) - u(s,x)| \le K_I(|t-s|^\rho + |y-x|^{\alpha\rho}).$$

Proof The proof will be done in several steps.

Step 1 Recall that $V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(X_{t-r}^x - y) W(dr, dy)$ denotes the random variable introduced in (3.4) and

$$u(t,x) = E^X\left(f(X_t^x)\exp\left(V_{(t,x)}\right)\right).$$

Set $V = V_{s,x}$ and $\tilde V = V_{t,y}$. Then we can write

$$\begin{aligned} E^W|u(s,x) - u(t,y)|^p &= E^W|E^X(e^V - e^{\tilde V})|^p \\ &\le E^W\left(E^X[|\tilde V - V|e^{\max(V,\tilde V)}]\right)^p \\ &\le E^W\left[\left(E^X e^{2\max(V,\tilde V)}\right)^{p/2}\left(E^X(\tilde V - V)^2\right)^{p/2}\right] \\ &\le \left[E^W E^X e^{2p\max(V,\tilde V)}\right]^{\frac12}\left[E^W\left(E^X(\tilde V - V)^2\right)^p\right]^{\frac12}. \end{aligned}$$

Applying Minkowski's inequality, the equivalence between the L^2 norm and the L^p norm for a Gaussian random variable, and using the exponential integrability property (3.10) we obtain

$$
\begin{aligned}
E^W|u(s,x)-u(t,y)|^p &\le C\big[E^W\big(E^X(\tilde{V}-V)^2\big)^p\big]^{\frac{1}{2}} \\
&\le C_p\big[E^X E^W|\tilde{V}-V|^2\big]^{p/2}. \qquad (5.1)
\end{aligned}
$$

In a similar way to (3.5) we can deduce the following formula for the conditional variance of $\tilde{V}-V$

$$
\begin{aligned}
E^W|\tilde{V}-V|^2 = \alpha_H E^X\Bigg(& \int_0^s\int_0^s |r-v|^{2H_0-2}\prod_{i=1}^d |X^i_{s-r}-X^i_{s-v}|^{2H_i-2}drdv \\
&+\int_0^t\int_0^t |r-v|^{2H_0-2}\prod_{i=1}^d |X^i_{t-r}-X^i_{t-v}|^{2H_i-2}drdv \\
&-2\int_0^s\int_0^t |r-v|^{2H_0-2}\prod_{i=1}^d |X^i_{s-r}-X^i_{t-v}+x_i-y_i|^{2H_i-2}drdv\Bigg) \\
&:= \alpha_H C(s,t,x,y). \qquad (5.2)
\end{aligned}
$$

Step 2 Fix $1\le j\le d$. Let us estimate $C(s,t,x,y)$ when $s=t$, and $x_i=y_i$ for all $i\neq j$. We can write

$$
C(t,t,x,y)=2\int_0^t\int_0^t |r-v|^{\kappa-1}\prod_{i\neq j}^d E\big(|\xi|^{2H_i-2}\big)E\big(|\xi|^{2H_j-2}-|z+\xi|^{2H_j-2}\big)drdv, \qquad (5.3)
$$

where $z=\dfrac{x_j-y_j}{|r-v|^{\frac{1}{\alpha}}}$ and ξ is a standard α-stable variable. By Lemma 7.7 the factor $E\big(|\xi|^{2H_j-2}-|z+\xi|^{2H_j-2}\big)$ can be bounded by a constant if $|r-v|\le(x_j-y_j)^\alpha$, and it can be bounded by $C|x_j-y_j|^2|r-v|^{-\frac{2}{\alpha}}$ if $|r-s|>(x_j-y_j)^\alpha$. In this way we obtain

$$
\begin{aligned}
C(t,t,x,y) &\le C\int_{\{0<r,v<t,|r-v|\le(x_j-y_j)^\alpha\}} |r-v|^{\kappa-1}drdv \\
&+C|x_j-y_j|^2\int_{\{0<r,v<t,|r-v|>(x_j-y_j)^\alpha\}} |r-v|^{\kappa-1-\frac{2}{\alpha}}drdv \\
&\le C|x_j-y_j|^{\alpha\kappa}.
\end{aligned}
$$

So, from (5.1) we have

$$
E^W|u(t,x)-u(t,y)|^p\le C|x_j-y_j|^{\frac{\alpha}{2}\kappa p}. \qquad (5.4)
$$

Step 3 Suppose now that $s < t$, and $x = y$. Set $\delta = \frac{1}{\alpha}\sum_{i=1}^{d}(2H_i - 2)$. We have

$$\begin{aligned} &C(s,t,x,x) \\ =&C\Bigg[\int_s^t \int_s^t |r-v|^{\kappa-1} dr dv \\ &+ \int_0^s \int_0^t |r-v|^{2H_0-2}\big(|r-v|^{\delta} - |r-v+t-s|^{\delta}\big) dr dv\Bigg]. \end{aligned}$$

The first integral is $O((t-s)^{\kappa+1})$, when $t-s$ is small. For the second integral we use the change of variable $\sigma = v - r, v = \tau$, and we have

$$\begin{aligned} &\int_0^s \int_0^t |r-v|^{2H_0-2}\big(|r-v|^{\delta} - |r-v+t-s|^{\delta}\big) dr dv \\ \leq &\int_0^s d\tau \int_{-t}^{s} |\sigma|^{2H_0-2}\big||\sigma|^{\delta} - |\sigma+t-s|^{\delta}\big| d\sigma \\ =&t\Bigg[\int_0^s \sigma^{2H_0-2}\big(\sigma^{\delta} - (\sigma+t-s)^{\delta}\big) d\sigma \\ &+ \int_{-t}^{s-t} (-\sigma)^{2H_0-2}\big((-\sigma-t+s)^{\delta} - (-\sigma)^{\delta}\big) d\sigma \\ &+ \int_{s-t}^{0} (-\sigma)^{2H_0-2}\big|(-\sigma)^{\delta} - (\sigma+t-s)^{\delta}\big| d\sigma\Bigg] \\ =&t[A' + B' + C']. \end{aligned}$$

For the first term in the above decomposition we can write

$$\begin{aligned} A' &= (t-s)^{\kappa} \int_0^{\frac{s}{t-s}} \sigma^{2H_0-2}\big(\sigma^{\delta} - (\sigma+1)^{\delta}\big) d\sigma \\ &\leq (t-s)^{\kappa} \int_0^{\infty} \sigma^{2H_0-2}\big(\sigma^{\delta} - (\sigma+1)^{\delta}\big) d\sigma \\ &\leq C(t-s)^{\kappa}, \end{aligned}$$

because $2H_0 - 2 + \delta - 1 < -1$. Similarly we can get that

$$B' \leq (t-s)^{\kappa} \int_1^{\infty} \sigma^{2H_0-2}\big(\sigma^{\delta} - (\sigma+1)^{\delta}\big) d\sigma.$$

At last,

$$C' \leq \int_0^{t-s} \sigma^{2H_0-2}\big(\sigma^{\delta} + (t-s-\sigma)^{\delta}\big) d\sigma = C(t-s)^{\kappa}.$$

So we have

$$E^W |u(s,x) - u(t,y)|^p \leq C(t-s)^{\frac{\kappa}{2}p}. \tag{5.5}$$

Step 4 Combining Equation 5.4 and Equation 5.5 with the estimates (5.1) and (5.2), the result of this theorem now can be concluded from Theorem 1.4.1 in Kunita [13] if we choose p large enough. ■

6 Skorohod type equations and chaos expansion

In this section we consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}} u + u \diamond \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W \\ u(0,x) = f(x) \,. \end{cases} \tag{6.6}$$

The difference between the above equation and Equation (1.1) is that here we use the Wick product $\diamond$ (see [12], for example). This equation is studied in [10] in the case $H_1 = \cdots = H_d = 1/2$. As in that paper, we can define the following notion of mild solution.

Definition 6.1 *An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that $E(u^2(t,x)) < \infty$ for all (t,x) is a mild solution to Equation (6.6) if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{q_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s), s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds*

$$u(t,x) = q_t f(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y) u(s,y) \delta W_{s,y}, \tag{6.7}$$

where $q_t(x)$ denotes the density function of X_t and $q_t f(x) = \int_{\mathbb{R}^d} q_t(x-y) f(y) dy$.

As in the paper [10] the mild solution $u(t,x)$ to (6.6) admits the following Wiener chaos expansion

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)), \tag{6.8}$$

where I_n denotes the multiple stochastic integral with respect to W and $f_n(\cdot,t,x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$, defined explicitly as

$$\begin{aligned} & f_n(s_1, y_1, \ldots, s_n, y_n, t, x) = \frac{1}{n!} \\ & \times q_{t-s_{\sigma(n)}}(x - y_{\sigma(n)}) \cdots q_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)} - y_{\sigma(1)}) q_{s_{\sigma(1)}} f(y_{\sigma(1)}). \end{aligned} \tag{6.9}$$

In the above equation σ denotes a permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Moreover, the solution if it exist, it will be unique because the kernels in the Wiener chaos expansion are uniquely determined.

The following is the main result of this section.

Theorem 6.2 *Suppose that $2H_0 + \frac{1}{\alpha} \sum_{i=1}^d (2H_i - 2) > 1$ and that f is a bounded measurable function. Then the process*

$$\begin{aligned} u(t,x) &= E^X \Bigg[f(X_t^x) \exp\Bigg(\int_0^t \int_{\mathbb{R}^d} \delta(X_{t-r}^x - y) W(dr, dy) \\ & \quad - \frac{1}{2}\alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d \left| X_r^i - X_s^i \right|^{2H_i-2} drds \Bigg) \Bigg] \end{aligned} \tag{6.10}$$

is the unique mild solution to Equation (1.1).

Proof ¿From Theorem 3.3, we obtain that the expectation E^X in Equation (6.10) is well defined. Then, it suffices to show that the random variable $u(t,x)$ has the Wiener chaos expansion (6.8). This can be easily proved by expanding the exponential and then taken the expectation with respect to X.

Theorem 3.1 implies that almost surely $\delta(X^x_{t-\cdot} - \cdot)$ is and element of $\mathcal{H}$ with a norm given by (3.4). As a consequence, almost surely with respect to the stable Lévy motion X, we have the following chaos expansion for the exponential factor in Equation (6.10)

$$\exp\Bigg(\int_0^t \int_{\mathbb{R}^d} \delta(X^x_{t-r} - y) W(dr, dy)$$
$$-\frac{1}{2}\alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^d \left|X^i_r - X^i_s\right|^{2H_i-2} drds\Bigg) = \sum_{n=0}^\infty I_n(g_n)\,,$$

where g_n is the symmetric element in $\mathcal{H}^{\otimes n}$ given by

$$g_n(s_1, y_1, \dots, s_n, y_n, t, x) = \frac{1}{n!}\, \delta(X^x_{t-s_1} - y_1) \cdots \delta(X^x_{t-s_n} - y_n)\,. \tag{6.11}$$

Thus the right hand side of (6.10) admits the following chaos expansion

$$u(t,x) = \sum_{n=0}^\infty \frac{1}{n!} I_n(h_n(\cdot, t, x))\,, \tag{6.12}$$

with

$$h_n(t,x) = E^X \left[f(X^x_t) \delta(X^x_{t-s_1} - y_1) \cdots \delta(X^x_{t-s_n} - y_n)\right]\,. \tag{6.13}$$

This can be regarded as a Feynman-Kac formula for the coefficients of chaos expansion of the solution of (6.6). To compute the above expectation we shall use the following

$$\begin{aligned} E^X \left[f(X^x_t) \delta(X^x_t - y) \middle| \mathcal{F}_s\right] &= \int_{\mathbb{R}^d} q_{t-s}(X^x_s - z) f(z) \delta(z-y) dz \\ &= q_{t-s}(X^x_s - y) f(y)\,. \end{aligned} \tag{6.14}$$

Assume that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ for some permutation σ of $\{1, 2, \cdots, n\}$. Then conditioning with respect to $\mathcal{F}_{t-s_{\sigma(1)}}$ and using the Markov property of the Lévy motion we have

$$\begin{aligned} h_n(t,x) &= E^X \big\{ E^X \big[\delta(X^x_{t-s_{\sigma(n)}} - y_{\sigma(n)}) \\ &\qquad \times \cdots \delta(X^x_{t-s_{\sigma(1)}} - y_{\sigma(1)}) f(X^x_t) \big| \mathcal{F}_{t-s_{\sigma(1)}} \big] \big\} \\ &= E^X \left[\delta(X^x_{t-s_{\sigma(n)}} - y_{\sigma(n)}) \cdots \delta(X^x_{t-s_{\sigma(1)}} - y_{\sigma(1)}) q_{s_{\sigma(1)}} f(X^x_{t-s_{\sigma(1)}}) \right]\,. \end{aligned}$$

Conditioning with respect to $\mathcal{F}_{t-s_{\sigma(2)}}$ and using (6.14), we have

$$\begin{aligned}
h_n(t,x) &= E^B\big\{E^X\big[\delta(X_{t-s^x_{\sigma(n)}}-y_{\sigma(n)})\\
&\quad\times\delta(X^x_{t-s_{\sigma(1)}}-y_{\sigma(1)})q_{s_{\sigma(1)}}f(X^x_{t-s_{\sigma(1)}})\big]\big|\mathcal{F}_{t-s_{\sigma(2)}}\big\}\\
&= E^X\Bigg\{\delta(X_{t-s^x_{\sigma(n)}}-y_{\sigma(n)})\cdots\delta(X^x_{t-s_{\sigma(2)}}-y_{\sigma(2)})\\
&\qquad\times E^X\left[\delta(X^x_{t-s_{\sigma(1)}}-y_{\sigma(1)})q_{s_{\sigma(1)}}f(X^x_{t-s_{\sigma(1)}})\big|\mathcal{F}_{t-s_{\sigma(2)}}\right]\Bigg\}\\
&= E^X\Bigg[\delta(X_{t-s^x_{\sigma(n)}}-y_{\sigma(n)})\cdots\delta(X^x_{t-s_{\sigma(2)}}-y_{\sigma(2)})\\
&\qquad\times p_{s_{\sigma(2)}-s_{\sigma(1)}}(X^x_{t-s_{\sigma(2)}}-y_{\sigma(1)})q_{s_{\sigma(1)}}f(y_{\sigma(1)})\Bigg].
\end{aligned}$$

Continuing this way we shall find out that

$$h_n(t,x)=q_{t-s_{\sigma(n)}}(x-y_{\sigma(n)})\cdots q_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)}-y_{\sigma(1)})q_{s_{\sigma(1)}}f(y_{\sigma(1)})$$

which is the same as (6.9). ■

Remark. The method of this section can be applied to obtain a Feynman-Kac formula for the coefficients of the chaos expansion of the solution to Equation (1.1):

$$u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h_n(\cdot,t,x)),$$

with

$$\begin{aligned}
h_n(t,x) &= E^X\Bigg[f(X^x_t)\delta(X^x_{t-s_1}-y_1)\cdots\delta(X^x_{t-s_n}-y_n)\\
&\qquad\times\exp\left(\frac{1}{2}\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d\left|X^i_r-X^i_s\right|^{2H_i-2}drds\right)\Bigg].
\end{aligned} \tag{6.15}$$

From the Feyman-Kac formula we can derive the following formula for the moments of the solution analogous to (4.17).

$$\begin{aligned}
&E\left(u(t,x)^p\right)=E\Bigg(\prod_{j=1}^p f(X^j_t+x)\\
&\times\exp\left[\alpha_H\sum_{j,k=1,j<k}^p\int_0^t\int_0^t|s-r|^{2H_0-2}\prod_{i=1}^d|X^{j,i}_s-X^{k,i}_r|^{2H_i-2}dsdr\right]\Bigg),
\end{aligned}$$

where $p\geq 1$ is an integer, and $X^j, 1\leq j\leq d$, are independent d-dimensional stable Lévy motions.

7 Appendix

Lemma 7.1 *For any deterministic sub-additive function* $a(t), t \in \mathbb{R}^+$*, the e-quality*

$$\lim_{t\to\infty} t^{-1}a(t) = \inf_{s>0} s^{-1}a(s)$$

holds in the extended real line $[-\infty, \infty)$*.*

Lemma 7.2 *Suppose* $0 < \beta < 1, \epsilon > 0, x > 0$*, and that* X *is a standard normal random variable. Then there is a constant* C *independent of* x *and* ϵ *(it may depend on* β*) such that*

$$E|x + \epsilon X|^{-\beta} \le C \min(\epsilon^{-\beta}, x^{-\beta}) \,.$$

Proof It is straightforward to check that $K = \sup_{z\ge 0} E|z + X|^{-\beta} < \infty$. Thus

$$E|x + \epsilon X|^{-\beta} = \epsilon^{-\beta} E|\frac{x}{\epsilon} + X|^{-\beta} \le K\epsilon^{-\beta} \,. \tag{7.16}$$

On the other hand,

$$\begin{aligned} E|x + \epsilon X|^{-\beta} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x + \epsilon y|^{-\beta} e^{-\frac{y^2}{2}} \, dy \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{\{|x+\epsilon y| > \frac{x}{2}\}} |x + \epsilon y|^{-\beta} e^{-\frac{y^2}{2}} \, dy \right. \\ &\quad \left. + \int_{\{|x+\epsilon y| \le \frac{x}{2}\}} |x + \epsilon y|^{-\beta} e^{-\frac{y^2}{2}} \, dy \right) . \end{aligned}$$

It is easy to see that the first integral is bounded by $Cx^{-\beta}$ for some constant C. The second integral, denoted by B is bounded as follows.

$$\begin{aligned} B &= C\frac{1}{\epsilon} \int_{|z|<\frac{x}{2}} |z|^{-\beta} e^{-\frac{(z-x)^2}{2\epsilon^2}} \, dz \le C\frac{1}{\epsilon} \int_{|z|<\frac{x}{2}} |z|^{-\beta} e^{-\frac{x^2}{8\epsilon^2}} \, dz \\ &= C\frac{x}{\epsilon} e^{-\frac{x^2}{8\epsilon^2}} x^{-\beta} \le Cx^{-\beta} \,. \end{aligned}$$

Thus we have $E|x + \epsilon X|^{-\beta} \le C|x|^{-\beta}$. Combining this with (7.16), we obtain the lemma. ∎

Lemma 7.3 *Suppose* $0 < \beta < 1, \epsilon > 0, a > 0$*, and that* Y *is a standard symmetric* α*-stable distributed random variable. Then there is a constant* C *independent of* x *and* ϵ *(it may depend on* β*) such that*

$$E|x + \epsilon Y|^{-\beta} \le C\epsilon^{-\beta}.$$

Proof

$$
\begin{aligned}
& E|x+\epsilon Y|^{-\beta} \\
=& \int_{\mathbb{R}} \mathcal{F}\{|x+\epsilon \cdot|^{-\beta}\}(\xi) e^{-|\xi|^{\alpha}} d\xi \\
=& \int_{\mathbb{R}} \mathcal{F}\{|\epsilon \cdot|^{-\beta}\}(\xi) e^{ix\xi} e^{-|\xi|^{\alpha}} d\xi \\
=& \int_{\mathbb{R}} \frac{1}{\epsilon} \mathcal{F}\{|\cdot|^{-\beta}\}(\frac{\xi}{\epsilon}) e^{ix\xi} e^{-|\xi|^{\alpha}} d\xi \\
=& \int_{\mathbb{R}} \frac{1}{\epsilon} |\frac{\xi}{\epsilon}|^{\beta-1} e^{ix\xi} e^{-|\xi|^{\alpha}} d\xi \\
\leq & \epsilon^{-\beta} \int_{\mathbb{R}} |\xi|^{\beta-1} e^{-|\xi|^{\alpha}} d\xi \\
\leq & C\epsilon^{-\beta}
\end{aligned}
$$

■

Lemma 7.4 *Suppose $\alpha \in (0,1)$. There exists a constant $C>0$, such that*

$$
\sup_{\epsilon,\epsilon'} \int_{\mathbb{R}^2} p_\epsilon(x_1+y_1) p_{\epsilon'}(x_2+y_2) |y_1-y_2|^{-\alpha} dy_1 dy_2 \leq C|x_1-x_2|^{-\alpha}.
$$

Proof We can write

$$
\int_{\mathbb{R}^2} p_\epsilon(x_1+y_1) p_{\epsilon'}(x_2+y_2) |y_1-y_2|^{-\alpha} dy_1 dy_2 = E\left(|\varepsilon X_1 - x_1 - \varepsilon' X_2 + x_2|^{-\alpha}\right).
$$

Thus Lemma 7.4 follows directly from Lemma 7.2. ■

Lemma 7.5 *Suppose $\alpha \in (0,1)$. There exists a constant $C>0$, such that*

$$
\sup_{\delta,\delta'} \int_0^t \int_0^t \varphi_\delta(t-s_1-r_1) \varphi_{\delta'}(t-s_2-r_2) |r_1-r_2|^{-\alpha} dr_1 dr_2 \leq C|s_1-s_2|^{-\alpha}
$$

Proof Since

$$
p_\delta(x) \geq p_\delta(x) I_{[0,\sqrt{\delta}]}(x) = \frac{1}{\sqrt{2\pi\delta}} e^{-\frac{x^2}{2\delta}} I_{[0,\sqrt{\delta}]}(x) \geq \frac{1}{\sqrt{2\pi e}} \varphi_{\sqrt{\delta}}(x),
$$

the lemma follows from Lemma 7.4. ■

Lemma 7.6 *Suppose that $2H_0 + \frac{1}{\alpha}\sum_{i=1}^d (2H_i - 2) > 1$. Let $X^1, \ldots, X^d$ be independent one-dimensional symmetric α-stable Lévy motion. Then we have*

$$
E\left(\int_0^t s^{2H_0-2} \prod_{i=1}^d |X_s^i|^{2H_i-2} ds\right)^2 < \infty.
$$

Proof We can write

$$E\left(\int_0^t s^{2H_0-2}\prod_{i=1}^d |X_s^i|^{2H_i-2}ds\right)^2 = 2\int_0^t\int_0^s (sr)^{2H_0-2}$$
$$\times\prod_{i=1}^d E(|X_s^i|^{2H_i-2}|X_r^i|^{2H_i-2})drds$$

Let Y be a standard symmetric α-stable distributed random variable. ¿From Lemma 7.3, taking into account that $2-2H_i<1$, we have when $r<s$,

$$\begin{aligned} E(|X_r^i|^{2H_i-2}|X_s^i|^{2H_i-2}) &= E[|X_r^i|^{2H_i-2}E[|(s-r)^{\frac{1}{\alpha}}Y+x|^{2H_i-2}|_{x=X_r^i}]] \\ &\le CE[|X_r^i|^{2H_i-2}(s-r)^{\frac{2H_i-2}{\alpha}}) \\ &\le Cr^{\frac{2H_i-1}{\alpha}}(s-r)^{\frac{2H_i-2}{\alpha}}. \end{aligned} \tag{7.17}$$

As a consequence, the conclusion of the lemma follows from the fact that

$$\int_0^t\int_0^s r^{2H_0-2+\frac{1}{\alpha}\sum_{i=1}^d(2H_i-2)}s^{2H_0-2}(s-r)^{\frac{1}{\alpha}\sum_{i=1}^d(2H_i-2)}drds<\infty,$$

because $2H_0-2+\frac{1}{\alpha}\sum_{i=1}^d(2H_i-2)>-1$ and $\frac{1}{\alpha}\sum_{i=1}^d(2H_i-2)>-1$. ■

Lemma 7.7 *For any* $0<\beta<1$,

$$E\big(|\xi|^{-\beta}-|y+\xi|^{-\beta}\big)\le C\min(1,y^2),$$

for some constant $C>0$*, where* $y>0$ *and* ξ *is a standard symmetric* α*-stable random variable.*

Proof Notice first that $E\big(|\xi|^{-\beta}-|y+\xi|^{-\beta}\big)<C$ where $C>0$ is a constant, since $\lim_{y\to\infty}E|y+\xi|^{-\alpha}=0$.
On the other hand,

$$\begin{aligned} &E\big(|\xi|^{-\beta}-|y+\xi|^{-\beta}\big) \\ =&\int_{\mathbb{R}}\mathcal{F}\left\{|\cdot|^{-\beta}-|y+\cdot|^{-\beta}\right\}(\xi)e^{-|\xi|^\alpha}d\xi \\ =&C\int_{\mathbb{R}}|\xi|^{\beta-1}e^{-|\xi|^\alpha}(1-e^{iy\xi})d\xi \\ =&C\int_{\mathbb{R}}|\xi|^{\beta-1}e^{-|\xi|^\alpha}(1-\cos(y\xi))d\xi \\ \le&C\int_{\mathbb{R}}|\xi|^{\beta-1}e^{-|\xi|^\alpha}y^2\xi^2d\xi \\ \le&Cy^2 \end{aligned}$$

■

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