Do all problems and give the **process** of your solution.

1. (20 points) Solve the initial value problem \( x \frac{dy}{dx} = y^3 \) and \( y(1) = 1 \)

**Solution.**
\[ y^3 dy = \frac{dx}{x}, \quad \frac{1}{4} y^4 = \ln x + C \]

Let \( x = 1 \): \( C = 1/4 \). The final answer: \( y^4 = 1 + 4 \ln x \).

2. (20 points) Solve the differential equation \( \frac{dy}{dx} + 4y = e^{-x} \).

**Solution.** This is a linear equation with \( P(x) = 4 \) and \( Q(x) = e^{-x} \).

\[
\int P(x)dx = \int 4dx = 4x \\
y = \exp \left\{ -\int P(x)dx \right\} \left[ \int Q(x) \exp \left\{ \int P(x)dx \right\} dx + C \right] \\
= e^{-4x} \left[ \int e^{4x} e^{-x} dx + C \right] = e^{-4x} \left[ \frac{1}{3} e^{3x} + C \right] = \frac{1}{3} e^{-x} + Ce^{-4x}
\]

3. (20 points) Solve the differential equation

\[ \frac{dy}{dx} = 1 + \frac{y}{x} + \left( \frac{y}{x} \right)^2 \]

**Solution.** Let \( v = \frac{y}{x} \). Then \( y = xv \) and \( \frac{dy}{dx} = v + x \frac{dv}{dx} \). Then

\[
v + x \frac{dv}{dx} = 1 + v + v^2, \quad x \frac{dv}{dx} = 1 + v^2, \quad \frac{dv}{1 + v^2} = \frac{dx}{x}
\]

Hence,

\[ \tan^{-1} v = \ln x + C, \quad v = \tan \left( \ln x + C \right) \]

So the solution is

\[ y = x \tan \left( \ln x + C \right) \]

4. Solve the equation \( (x^2 + y^2)dx + (1 + 2xy)dy = 0 \).

**Solution.** \( \partial M/\partial y = 2y = \partial N/\partial x \). Exact! We now solve \( \partial F/\partial x = x^2 + y^2 \) and \( \partial F/\partial y = 1 + 2xy \) for \( F(x, y) \).
From the first equation, \( F = \frac{1}{3}x^3 + y^2x + C(y) \). So \( \partial F/\partial y = 2yx + C'(y) \). Comparing this with the second equation, \( C'(y) = 1 \). So we take \( C(y) = y \). Thus, \( F(x, y) = \frac{1}{3}x^3 + y^2x + y \). The solution of the equation: \( \frac{1}{3}x^3 + xy^2 + y = C \).

5. (20 points) Solve the Bernoulli equation \( \frac{dy}{dx} + \frac{y}{x} = xy^2 \).

**Solution.** \( n = 2 \). Set \( v = y^{1-2} = y^{-1} \) or \( y = v^{-1} \). We have \( \frac{dy}{dx} = -v^{-2} \frac{dv}{dx} \). Bring it to the equation

\[
-v^{-2} \frac{dv}{dx} + v^{-1} = xv^{-2}, \quad \frac{dv}{dx} - \frac{v}{x} = -x
\]

This is a linear equation with \( P(x) = -\frac{1}{x} \) and \( Q(x) = -x \)

\[
\int P(x)dx = - \int \frac{1}{x} dx = - \ln x
\]

\[
v = \exp \left\{ - \int P(x)dx \right\} \left[ C + \int Q(x) \exp \left\{ \int P(x)dx \right\} dx \right]
\]

\[
= \exp \left\{ - \ln x \right\} \left[ C - \int x \exp \left\{ - \ln x \right\} dx \right] = x \left[ C - \int \frac{x}{x} dx \right] = x \left[ C - x \right]
\]

The final solution is \( y = \frac{1}{x(C - x)} \).