Test #1 (1:50-2:40pm) **Fall 2024** **Name:**

Do all problems and give the **process** of your solution.

1. (20 points) Given $2x - 3y + 4z = 5$ and $x + 6y + 4z = 3$,

(a). Find the angle between them.

Solution. $\mathbf{n_1} =< 2, -3, 4 >$ and $\mathbf{n_2} =< 1, 6, 4 >$

$$\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|} = \frac{2 \times 1 + (-3) \times 6 + 4 \times 4}{\sqrt{2^2 + (-3)^2 + 4^2}\sqrt{1^2 + 6^2 + 4^2}} = 0$$

$$\theta = \frac{\pi}{2}$$

Two planes are perpendicular.

(b). Find the parametric equation of the intersection line between them.

Solution. Solving $2x - 3y + 4z = 5$ and $x + 6y + 4z = 3$ together with $z = 0$ gets the point $\left(\frac{13}{5}, \frac{1}{15}, 0\right)$. The direction of the intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 4 \\ 1 & 6 & 4 \end{vmatrix} = -36\mathbf{i} - 4\mathbf{j} + 15\mathbf{k}$$

The parametric equation of the intersection line is

$$\mathbf{r}(t) = \langle \frac{13}{5} - 36t,\ \frac{1}{15} - 4t,\ 15t \rangle$$

2. (20 points) Given the three points $P(1, 1, 1)$, $Q(1, 0, 2)$, and $R(0, 0, 1)$ in space,

(a.) find the area of the triangle with the vertices P, Q, and R;
Solution. $\overrightarrow{PQ} =< 0, -1, 1 >$ and $\overrightarrow{PR} =< -1, -1, 0 >$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

The area of the triangle is

$$\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\sqrt{1^2 + 1^2 + 1^2} = \frac{1}{2}\sqrt{3}$$

(b.) find the equation of the plane that contains the points P, Q and R.

Solution.

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$(x-1)-(y-1)-(z-1)=0 \quad \text{or} \quad x-y-z=1$$

3. (15 points) Given the vector $\mathbf{a} =< 2, 2, -1 >$, find the unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$.

Solution.

$$\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{\sqrt{2^2+2^2+1^2}} < 2,2,-1 > = < \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} >$$

4. (15 points) Find parametric equation for the tangent line to the curve $\mathbf{r}(t) = \langle t^2-1, t^2+1, t+1\rangle$ at the point when $t=0$.

Solution. When $t=0$, $x=-1$, $y=1$ and $z=1$. $\mathbf{r}'(t) = \langle 2t, 2t, 1\rangle$. $\mathbf{r}'(0) = \langle 0,0,1\rangle$. The parametric equation of the tangent line is:

$$x=-1, \quad y=1, \quad z=1+t$$

Or

$$\mathbf{r}(t) = \langle -1, 1, 1+t\rangle$$

5. (15 points) Compute the length of the curve $\mathbf{r}(t) = \langle t^2, t^2+1, 1\rangle$ over $0 \le t \le 1$.

Solution. $\mathbf{r}'(t) = \langle 2t, 2t, 0\rangle$.

$$s = \int_0^1 \|\mathbf{r}'(t)\|dt = \int_0^1 \sqrt{4t^2+4t^2+0^2}dt = \int_0^1 2\sqrt{2}tdt = \sqrt{2}$$

6 (15 points). Let $w = \dfrac{1}{x+y^2+z^3}$. Find $\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}$ and $\dfrac{\partial w}{\partial z}$

Solution.

$$\frac{\partial w}{\partial x} = -\frac{1}{(x+y^2+z^3)^2}$$

$$\frac{\partial w}{\partial y} = -\frac{2y}{(x+y^2+z^3)^2}$$

$$\frac{\partial w}{\partial z} = -\frac{3z^2}{(x+y^2+z^3)^2}$$